RESOLUTION OF SURFACES REVISITED

0BH6

Contents

1. Introduction

0BH7 This chapter discusses resolution of singularities of Noetherian algebraic spaces of dimension 2. We have already discussed resolution of surfaces for schemes following Lipman [Lip78] in an earlier chapter. See Resolution of Surfaces, Section 1. Most of the results in this chapter are straightforward consequences of the results on schemes.

Unless specifically mentioned otherwise all geometric objects in this chapter will be algebraic spaces. Thus if we say "let $f : X \to Y$ be a modification" then this means that f is a morphism as in Spaces over Fields, Definition 8.1. Similarly for proper morphism, etc, etc.

2. Modifications

0BH8 Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring. We set $S = \mathrm{Spec}(A)$ and $U = S \setminus \{\mathfrak{m}\}$. In this section we will consider the category

0AE2 (2.0.1)
$$\left\{ f : X \longrightarrow S \ \middle| \ \begin{array}{c} X \text{ is an algebraic space} \\ f \text{ is a proper morphism} \\ f^{-1}(U) \to U \text{ is an isomorphism} \end{array} \right\}$$

A morphism from X/S to X'/S will be a morphism of algebraic spaces $X \to X'$ compatible with the structure morphisms over S. In Algebraization of Formal Spaces, Section 30 we have seen that this category only depends on the completion of A and we have proven some elementary properties of objects in this category. In

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this section we specifically study cases where $\dim(A) \leq 2$ or where the dimension of the closed fibre is at most 1.

0AE3 **Lemma 2.1.** *Let $(A, \mathfrak{m}, \kappa)$ be a 2-dimensional Noetherian local domain such that $U = \mathrm{Spec}(A) \setminus \{\mathfrak{m}\}$ is a normal scheme. Then any modification $f : X \to \mathrm{Spec}(A)$ is a morphism as in (2.0.1).*

Proof. Let $f : X \to S$ be a modification. We have to show that $f^{-1}(U) \to U$ is an isomorphism. Since every closed point u of U has codimension 1, this follows from Spaces over Fields, Lemma 3.3. $\square$

0AGM **Lemma 2.2.** *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Let $g : X \to Y$ be a morphism in the category (2.0.1). If the induced morphism $X_\kappa \to Y_\kappa$ of special fibres is a closed immersion, then g is a closed immersion.*

Proof. This is a special case of More on Morphisms of Spaces, Lemma 49.3. $\square$

0AYJ **Lemma 2.3.** *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local domain of dimension ≥ 1. Let $f : X \to \mathrm{Spec}(A)$ be a morphism of algebraic spaces. Assume at least one of the following conditions is satisfied*
  (1) *f is a modification (Spaces over Fields, Definition 8.1),*
  (2) *f is an alteration (Spaces over Fields, Definition 8.3),*
  (3) *f is locally of finite type, quasi-separated, X is integral, and there is exactly one point of $|X|$ mapping to the generic point of $\mathrm{Spec}(A)$,*
  (4) *f is locally of finite type, X is decent, and the points of $|X|$ mapping to the generic point of $\mathrm{Spec}(A)$ are the generic points of irreducible components of $|X|$,*
  (5) *add more here.*

Then $\dim(X_\kappa) \leq \dim(A) - 1$.

Proof. Cases (1), (2), and (3) are special cases of (4). Choose an affine scheme $U = \mathrm{Spec}(B)$ and an étale morphism $U \to X$. The ring map $A \to B$ is of finite type. We have to show that $\dim(U_\kappa) \leq \dim(A) - 1$. Since X is decent, the generic points of irreducible components of U are the points lying over generic points of irreducible components of $|X|$, see Decent Spaces, Lemma 20.1. Hence the fibre of $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ over (0) is the (finite) set of minimal primes $\mathfrak{q}_1, \ldots, \mathfrak{q}_r$ of B. Thus $A_f \to B_f$ is finite for some nonzero $f \in A$ (Algebra, Lemma 122.10). We conclude $\kappa(\mathfrak{q}_i)$ is a finite extension of the fraction field of A. Let $\mathfrak{q} \subset B$ be a prime lying over $\mathfrak{m}$. Then

$$\dim(B_\mathfrak{q}) = \max \dim((B/\mathfrak{q}_i)_\mathfrak{q}) \leq \dim(A)$$

the inequality by the dimension formula for $A \subset B/\mathfrak{q}_i$, see Algebra, Lemma 113.1. However, the dimension of $B_\mathfrak{q}/\mathfrak{m}B_\mathfrak{q}$ (which is the local ring of U_κ at the corresponding point) is at least one less because the minimal primes $\mathfrak{q}_i$ are not in $V(\mathfrak{m})$. We conclude by Properties, Lemma 10.2. $\square$

0AGN **Lemma 2.4.** *If $(A, \mathfrak{m}, \kappa)$ is a complete Noetherian local domain of dimension 2, then every modification of $\mathrm{Spec}(A)$ is projective over A.*

Proof. By More on Morphisms of Spaces, Lemma 43.6 it suffices to show that the special fibre of any modification X of $\mathrm{Spec}(A)$ has dimension ≤ 1. This follows from Lemma 2.3. $\square$

3. Strategy

0BH9 Let S be a scheme. Let X be a decent algebraic space over S. Let $x_1, \ldots, x_n \in |X|$ be pairwise distinct closed points. For each i we pick an elementary étale neighbourhood $(U_i, u_i) \to (X, x_i)$ as in Decent Spaces, Lemma 11.4. This means that U_i is an affine scheme, $U_i \to X$ is étale, u_i is the unique point of U_i lying over x_i, and $\mathrm{Spec}(\kappa(u_i)) \to X$ is a monomorphism representing x_i. After shrinking U_i we may and do assume that for $j \neq i$ there does not exist a point of U_i mapping to x_j. Observe that $u_i \in U_i$ is a closed point.

Denote $\mathcal{C}_{X, \{x_1, \ldots, x_n\}}$ the category of morphisms of algebraic spaces $f : Y \to X$ which induce an isomorphism $f^{-1}(X \setminus \{x_1, \ldots, x_n\}) \to X \setminus \{x_1, \ldots, x_n\}$. For each i denote $\mathcal{C}_{U_i, u_i}$ the category of morphisms of algebraic spaces $g_i : Y_i \to U_i$ which induce an isomorphism $g_i^{-1}(U_i \setminus \{u_i\}) \to U_i \setminus \{u_i\}$. Base change defines an functor

0BHA (3.0.1) $$F : \mathcal{C}_{X, \{x_1, \ldots, x_n\}} \longrightarrow \mathcal{C}_{U_1, u_1} \times \ldots \times \mathcal{C}_{U_n, u_n}$$

To reduce at least some of the problems in this chapter to the case of schemes we have the following lemma.

0BHB **Lemma 3.1.** *The functor F (3.0.1) is an equivalence.*

Proof. For $n = 1$ this is Limits of Spaces, Lemma 19.1. For $n > 1$ the lemma can be proved in exactly the same way or it can be deduced from it. For example, suppose that $g_i : Y_i \to U_i$ are objects of $\mathcal{C}_{U_i, u_i}$. Then by the case $n = 1$ we can find $f_i' : Y_i' \to X$ which are isomorphisms over $X \setminus \{x_i\}$ and whose base change to U_i is f_i. Then we can set

$$f : Y = Y_1' \times_X \ldots \times_X Y_n' \to X$$

This is an object of $\mathcal{C}_{X, \{x_1, \ldots, x_n\}}$ whose base change by $U_i \to X$ recovers g_i. Thus the functor is essentially surjective. We omit the proof of fully faithfulness. $\square$

0BHC **Lemma 3.2.** *Let $X, x_i, U_i \to X, u_i$ be as in (3.0.1). If $f : Y \to X$ corresponds to $g_i : Y_i \to U_i$ under F, then f is quasi-compact, quasi-separated, separated, locally of finite presentation, of finite presentation, locally of finite type, of finite type, proper, integral, finite, if and only if g_i is so for $i = 1, \ldots, n$.*

Proof. Follows from Limits of Spaces, Lemma 19.2. $\square$

0BHD **Lemma 3.3.** *Let $X, x_i, U_i \to X, u_i$ be as in (3.0.1). If $f : Y \to X$ corresponds to $g_i : Y_i \to U_i$ under F, then $Y_{x_i} \cong (Y_i)_{u_i}$ as algebraic spaces.*

Proof. This is clear because $u_i \to x_i$ is an isomorphism. $\square$

4. Dominating by quadratic transformations

0AHG We define the blowup of a space at a point only if X is decent.

0BHE **Definition 4.1.** Let S be a scheme. Let X be a decent algebraic space over S. Let $x \in |X|$ be a closed point. By Decent Spaces, Lemma 14.6 we can represent x by a closed immersion $i : \mathrm{Spec}(k) \to X$. The *blowing up $X' \to X$ of X at x* means the blowing up of X in the closed subspace $Z = i(\mathrm{Spec}(k)) \subset X$.

In this generality the blowing up of X at x is not necessarily proper. However, if X is locally Noetherian, then it follows from Divisors on Spaces, Lemma 17.11 that the blowing up is proper. Recall that a locally Noetherian algebraic space is

Noetherian if and only if it is quasi-compact and quasi-separated. Moreover, for a locally Noetherian algebraic space, being quasi-separated is equivalent to being decent (Decent Spaces, Lemma 14.1).

0BHF **Lemma 4.2.** *Let $X, x_i, U_i \to X, u_i$ be as in (3.0.1) and assume $f : Y \to X$ corresponds to $g_i : Y_i \to U_i$ under F. Then there exists a factorization*

$$Y = Z_m \to Z_{m-1} \to \ldots \to Z_1 \to Z_0 = X$$

of f where $Z_{j+1} \to Z_j$ is the blowing up of Z_j at a closed point z_j lying over $\{x_1, \ldots, x_n\}$ if and only if for each i there exists a factorization

$$Y_i = Z_{i,m_i} \to Z_{i,m_i-1} \to \ldots \to Z_{i,1} \to Z_{i,0} = U_i$$

of g_i where $Z_{i,j+1} \to Z_{i,j}$ is the blowing up of $Z_{i,j}$ at a closed point $z_{i,j}$ lying over u_i.

Proof. A blowing up is a representable morphism. Hence in either case we inductively see that $Z_j \to X$ or $Z_{i,j} \to U_i$ is representable. Whence each Z_j or $Z_{i,j}$ is a decent algebraic space by Decent Spaces, Lemma 6.5. This shows that the assertions make sense (since blowing up is only defined for decent spaces). To prove the equivalence, let's start with a sequence of blowups $Z_m \to Z_{m-1} \to \ldots \to Z_1 \to Z_0 = X$. The first morphism $Z_1 \to X$ is given by blowing up one of the x_i, say x_1. Applying F to $Z_1 \to X$ we find a blowup $Z_{1,1} \to U_1$ at u_1 is the blowing up at u_1 and otherwise $Z_{i,0} = U_i$ for $i > 1$. In the next step, we either blow up one of the x_i, $i \geq 2$ on Z_1 or we pick a closed point z_1 of the fibre of $Z_1 \to X$ over x_1. In the first case it is clear what to do and in the second case we use that $(Z_1)_{x_1} \cong (Z_{1,1})_{u_1}$ (Lemma 3.3) to get a closed point $z_{1,1} \in Z_{1,1}$ corresponding to z_1. Then we set $Z_{1,2} \to Z_{1,1}$ equal to the blowing up in $z_{1,1}$. Continuing in this manner we construct the factorizations of each g_i.

Conversely, given sequences of blowups $Z_{i,m_i} \to Z_{i,m_i-1} \to \ldots \to Z_{i,1} \to Z_{i,0} = U_i$ we construct the sequence of blowing ups of X in exactly the same manner. $\square$

0BHG **Lemma 4.3.** *Let S be a scheme. Let X be a Noetherian algebraic space over S. Let $T \subset |X|$ be a finite set of closed points x such that (1) X is regular at x and (2) the local ring of X at x has dimension 2. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals such that $\mathcal{O}_X/\mathcal{I}$ is supported on T. Then there exists a sequence*

$$X_m \to X_{m-1} \to \ldots \to X_1 \to X_0 = X$$

where $X_{j+1} \to X_j$ is the blowing up of X_j at a closed point x_j lying above a point of T such that $\mathcal{I}\mathcal{O}_{X_n}$ is an invertible ideal sheaf.

Proof. Say $T = \{x_1, \ldots, x_r\}$. Pick elementary étale neighbourhoods $(U_i, u_i) \to (X, x_i)$ as in Section 3. For each i the restriction $\mathcal{I}_i = \mathcal{I}|_{U_i} \subset \mathcal{O}_{U_i}$ is a quasi-coherent sheaf of ideals supported at u_i. The local ring of U_i at u_i is regular and has dimension 2. Thus we may apply Resolution of Surfaces, Lemma 4.1 to find a sequence

$$X_{i,m_i} \to X_{i,m_i-1} \to \ldots \to X_1 \to X_{i,0} = U_i$$

of blowing ups in closed points lying over u_i such that $\mathcal{I}_i\mathcal{O}_{X_{i,m_i}}$ is invertible. By Lemma 4.2 we find a sequence of blowing ups

$$X_m \to X_{m-1} \to \ldots \to X_1 \to X_0 = X$$

as in the statement of the lemma whose base change to our U_i produces the given sequences. It follows that $\mathcal{I}\mathcal{O}_{X_n}$ is an invertible ideal sheaf. Namely, we know this is true over $X \setminus \{x_1, \ldots, x_n\}$ and in an étale neighbourhood of the fibre of each x_i it is true by construction. $\square$

0BHH **Lemma 4.4.** *Let S be a scheme. Let X be a Noetherian algebraic space over S. Let $T \subset |X|$ be a finite set of closed points x such that (1) X is regular at x and (2) the local ring of X at x has dimension 2. Let $f : Y \to X$ be a proper morphism of algebraic spaces which is an isomorphism over $U = X \setminus T$. Then there exists a sequence*

$$X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 = X$$

where $X_{i+1} \to X_i$ is the blowing up of X_i at a closed point x_i lying above a point of T and a factorization $X_n \to Y \to X$ of the composition.

Proof. By More on Morphisms of Spaces, Lemma 39.5 there exists a U-admissible blowup $X' \to X$ which dominates $Y \to X$. Hence we may assume there exists an ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$ such that $\mathcal{O}_X/\mathcal{I}$ is supported on T and such that Y is the blowing up of X in $\mathcal{I}$. By Lemma 4.3 there exists a sequence

$$X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 = X$$

where $X_{i+1} \to X_i$ is the blowing up of X_i at a closed point x_i lying above a point of T such that $\mathcal{I}\mathcal{O}_{X_n}$ is an invertible ideal sheaf. By the universal property of blowing up (Divisors on Spaces, Lemma 17.5) we find the desired factorization. $\square$

5. Dominating by normalized blowups

0BHI In this section we prove that a modification of a surface can be dominated by a sequence of normalized blowups in points.

0BHJ **Definition 5.1.** Let S be a scheme. Let X be a decent algebraic space over S satisfying the equivalent conditions of Morphisms of Spaces, Lemma 49.1. Let $x \in |X|$ be a closed point. The *normalized blowup of X at x* is the composition $X'' \to X' \to X$ where $X' \to X$ is the blowup of X at x (Definition 4.1) and $X'' \to X'$ is the normalization of X'.

Here the normalization $X'' \to X'$ is defined as the algebraic space X' satisfies the equivalent conditions of Morphisms of Spaces, Lemma 49.1 by Divisors on Spaces, Lemma 17.8. See Morphisms of Spaces, Definition 49.6 for the definition of the normalization.

In general the normalized blowing up need not be proper even when X is Noetherian. Recall that an algebraic space is Nagata if it has an étale covering by affines which are spectra of Nagata rings (Properties of Spaces, Definition 7.2 and Remark 7.3 and Properties, Definition 13.1).

0BHK **Lemma 5.2.** *In Definition 5.1 if X is Nagata, then the normalized blowing up of X at x is a normal Nagata algebraic space proper over X.*

Proof. The blowup morphism $X' \to X$ is proper (as X is locally Noetherian we may apply Divisors on Spaces, Lemma 17.11). Thus X' is Nagata (Morphisms of Spaces, Lemma 26.1). Therefore the normalization $X'' \to X'$ is finite (Morphisms of Spaces, Lemma 49.9) and we conclude that $X'' \to X$ is proper as well (Morphisms of Spaces, Lemmas 45.9 and 40.4). It follows that the normalized blowing up is a normal (Morphisms of Spaces, Lemma 49.8) Nagata algebraic space. $\square$

Here is the analogue of Lemma 4.2 for normalized blowups.

0BHL **Lemma 5.3.** *Let $X, x_i, U_i \to X, u_i$ be as in (3.0.1) and assume $f : Y \to X$ corresponds to $g_i : Y_i \to U_i$ under F. Assume X satisfies the equivalent conditions of Morphisms of Spaces, Lemma 49.1. Then there exists a factorization*

$$Y = Z_m \to Z_{m-1} \to \ldots \to Z_1 \to Z_0 = X$$

of f where $Z_{j+1} \to Z_j$ is the normalized blowing up of Z_j at a closed point z_j lying over $\{x_1, \ldots, x_n\}$ if and only if for each i there exists a factorization

$$Y_i = Z_{i,m_i} \to Z_{i,m_i-1} \to \ldots \to Z_{i,1} \to Z_{i,0} = U_i$$

of g_i where $Z_{i,j+1} \to Z_{i,j}$ is the normalized blowing up of $Z_{i,j}$ at a closed point $z_{i,j}$ lying over u_i.

Proof. This follows by the exact same argument as used to prove Lemma 4.2. $\square$

A Nagata algebraic space is locally Noetherian.

0BHM **Lemma 5.4.** *Let S be a scheme. Let X be a Noetherian Nagata algebraic space over S with $\dim(X) = 2$. Let $f : Y \to X$ be a proper birational morphism. Then there exists a commutative diagram*

$$
\begin{array}{ccccccccc}
X_n & \longrightarrow & X_{n-1} & \longrightarrow & \ldots & \longrightarrow & X_1 & \longrightarrow & X_0 \\
\downarrow & & & & & & & & \downarrow \\
Y & & & & \longrightarrow & & & & X
\end{array}
$$

where $X_0 \to X$ is the normalization and where $X_{i+1} \to X_i$ is the normalized blowing up of X_i at a closed point.

Proof. Although one can prove this lemma directly for algebraic spaces, we will continue the approach used above to reduce it to the case of schemes.

We will use that Noetherian algebraic spaces are quasi-separated and hence points have well defined residue fields (for example by Decent Spaces, Lemma 11.4). We will use the results of Morphisms of Spaces, Sections 26, 35, and 49 without further mention. We may replace Y by its normalization. Let $X_0 \to X$ be the normalization. The morphism $Y \to X$ factors through X_0. Thus we may assume that both X and Y are normal.

Assume X and Y are normal. The morphism $f : Y \to X$ is an isomorphism over an open which contains every point of codimension 0 and 1 in Y and every point of Y over which the fibre is finite, see Spaces over Fields, Lemma 3.3. Hence we see that there is a finite set of closed points $T \subset |X|$ such that f is an isomorphism over $X \backslash T$. By More on Morphisms of Spaces, Lemma 39.5 there exists an $X \setminus T$-admissible blowup $Y' \to X$ which dominates Y. After replacing Y by the normalization of Y' we see that we may assume that $Y \to X$ is representable.

Say $T = \{x_1, \ldots, x_r\}$. Pick elementary étale neighbourhoods $(U_i, u_i) \to (X, x_i)$ as in Section 3. For each i the morphism $Y_i = Y \times_X U_i \to U_i$ is a proper birational morphism which is an isomorphism over $U_i \setminus \{u_i\}$. Thus we may apply Resolution of Surfaces, Lemma 5.3 to find a sequence

$$X_{i,m_i} \to X_{i,m_i-1} \to \ldots \to X_1 \to X_{i,0} = U_i$$

of normalized blowing ups in closed points lying over u_i such that X_{i,m_i} dominates Y_i. By Lemma 5.3 we find a sequence of normalized blowing ups

$$X_m \to X_{m-1} \to \ldots \to X_1 \to X_0 = X$$

as in the statement of the lemma whose base change to our U_i produces the given sequences. It follows that X_m dominates Y by the equivalence of categories of Lemma 3.1. $\square$

6. Base change to the completion

0BHN The following simple lemma will turn out to be a useful tool in what follows.

0BHP **Lemma 6.1.** *Let $(A, \mathfrak{m}, \kappa)$ be a local ring with finitely generated maximal ideal $\mathfrak{m}$. Let X be a decent algebraic space over A. Let $Y = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A^\wedge)$ where $A^\wedge$ is the $\mathfrak{m}$-adic completion of A. For a point $q \in |Y|$ with image $p \in |X|$ lying over the closed point of $\mathrm{Spec}(A)$ the map $\mathcal{O}_{X,p}^h \to \mathcal{O}_{Y,q}^h$ of henselian local rings induces an isomorphism on completions.*

Proof. This follows immediately from the case of schemes by choosing an elementary étale neighbourhood $(U, u) \to (X, p)$ as in Decent Spaces, Lemma 11.4, setting $V = U \times_X Y = U \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A^\wedge)$ and $v = (u, q)$. The case of schemes is Resolution of Surfaces, Lemma 11.1. $\square$

0BHQ **Lemma 6.2.** *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Let $X \to \mathrm{Spec}(A)$ be a morphism which is locally of finite type with X a decent algebraic space. Set $Y = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A^\wedge)$. Let $y \in |Y|$ with image $x \in |X|$. Then*

(1) *if $\mathcal{O}_{Y,y}^h$ is regular, then $\mathcal{O}_{X,x}^h$ is regular,*
(2) *if y is in the closed fibre, then $\mathcal{O}_{Y,y}^h$ is regular $\Leftrightarrow$ $\mathcal{O}_{X,x}^h$ is regular, and*
(3) *If X is proper over A, then X is regular if and only if Y is regular.*

Proof. By étale localization the first two statements follow immediately from the counter part to this lemma for schemes, see Resolution of Surfaces, Lemma 11.2. For part (3), since $Y \to X$ is surjective (as $A \to A^\wedge$ is faithfully flat) we see that Y regular implies X regular by part (1). Conversely, if X is regular, then the henselian local rings of Y are regular for all points of the special fibre. Let $y \in |Y|$ be a general point. Since $|Y| \to |\mathrm{Spec}(A^\wedge)|$ is closed in the proper case, we can find a specialization $y \rightsquigarrow y_0$ with y_0 in the closed fibre. Choose an elementary étale neighbourhood $(V, v_0) \to (Y, y_0)$ as in Decent Spaces, Lemma 11.4. Since Y is decent we can lift $y \rightsquigarrow y_0$ to a specialization $v \rightsquigarrow v_0$ in V (Decent Spaces, Lemma 12.2). Then we conclude that $\mathcal{O}_{V,v}$ is a localization of $\mathcal{O}_{V,v_0}$ hence regular and the proof is complete. $\square$

0BHR **Lemma 6.3.** *Let $(A, \mathfrak{m})$ be a local Noetherian ring. Let X be an algebraic space over A. Assume*

(1) *A is analytically unramified (Algebra, Definition 162.9),*
(2) *X is locally of finite type over A,*
(3) *$X \to \mathrm{Spec}(A)$ is étale at every point of codimension 0 in X.*

Then the normalization of X is finite over X.

Proof. Choose a scheme U and a surjective étale morphism $U \to X$. Then $U \to \mathrm{Spec}(A)$ satisfies the assumptions and hence the conclusions of Resolution of Surfaces, Lemma 11.5. $\square$

7. Implied properties

0BHS   In this section we prove that for a Noetherian integral algebraic space the existence of a regular alteration has quite a few consequences. This section should be skipped by those not interested in "bad" Noetherian algebraic spaces.

0BHT   **Lemma 7.1.** *Let S be a scheme. Let Y be a Noetherian integral algebraic space over S. Assume there exists an alteration $f : X \to Y$ with X regular. Then the normalization $Y^\nu \to Y$ is finite and Y has a dense open which is regular.*

Proof. By étale localization, it suffices to prove this when $Y = \mathrm{Spec}(A)$ where A is a Noetherian domain. Let B be the integral closure of A in its fraction field. Set $C = \Gamma(X, \mathcal{O}_X)$. By Cohomology of Spaces, Lemma 20.2 we see that C is a finite A-module. As X is normal (Properties of Spaces, Lemma 25.4) we see that C is normal domain (Spaces over Fields, Lemma 4.6). Thus $B \subset C$ and we conclude that B is finite over A as A is Noetherian.

There exists a nonempty open $V \subset Y$ such that $f^{-1}V \to V$ is finite, see Spaces over Fields, Definition 8.3. After shrinking V we may assume that $f^{-1}V \to V$ is flat (Morphisms of Spaces, Proposition 32.1). Thus $f^{-1}V \to V$ is faithfully flat. Then V is regular by Algebra, Lemma 164.4. $\square$

0BHU   **Lemma 7.2.** *Let $(A, \mathfrak{m}, \kappa)$ be a local Noetherian domain. Assume there exists an alteration $f : X \to \mathrm{Spec}(A)$ with X regular. Then*

(1) *there exists a nonzero $f \in A$ such that A_f is regular,*
(2) *the integral closure B of A in its fraction field is finite over A,*
(3) *the $\mathfrak{m}$-adic completion of B is a normal ring, i.e., the completions of B at its maximal ideals are normal domains, and*
(4) *the generic formal fibre of A is regular.*

Proof. Parts (1) and (2) follow from Lemma 7.1. We have to redo part of the proof of that lemma in order to set up notation for the proof of (3). Set $C = \Gamma(X, \mathcal{O}_X)$. By Cohomology of Spaces, Lemma 20.2 we see that C is a finite A-module. As X is normal (Properties of Spaces, Lemma 25.4) we see that C is normal domain (Spaces over Fields, Lemma 4.6). Thus $B \subset C$ and we conclude that B is finite over A as A is Noetherian. By Resolution of Surfaces, Lemma 13.2 in order to prove (3) it suffices to show that the $\mathfrak{m}$-adic completion $C^\wedge$ is normal.

By Algebra, Lemma 97.8 the completion $C^\wedge$ is the product of the completions of C at the prime ideals of C lying over $\mathfrak{m}$. There are finitely many of these and these are the maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ of C. (The corresponding result for B explains the final statement of the lemma.) Thus replacing A by $C_{\mathfrak{m}_i}$ and X by $X_i = X \times_{\mathrm{Spec}(C)} \mathrm{Spec}(C_{\mathfrak{m}_i})$ we reduce to the case discussed in the next paragraph. (Note that $\Gamma(X_i, \mathcal{O}) = C_{\mathfrak{m}_i}$ by Cohomology of Spaces, Lemma 11.2.)

Here A is a Noetherian local normal domain and $f : X \to \mathrm{Spec}(A)$ is a regular alteration with $\Gamma(X, \mathcal{O}_X) = A$. We have to show that the completion $A^\wedge$ of A is a normal domain. By Lemma 6.2 $Y = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A^\wedge)$ is regular. Since $\Gamma(Y, \mathcal{O}_Y) = A^\wedge$ by Cohomology of Spaces, Lemma 11.2. We conclude that $A^\wedge$ is normal as before. Namely, Y is normal by Properties of Spaces, Lemma 25.4. It is connected because $\Gamma(Y, \mathcal{O}_Y) = A^\wedge$ is local. Hence Y is normal and integral (as connected and normal implies integral for separated algebraic spaces). Thus

$\Gamma(Y, \mathcal{O}_Y) = A^\wedge$ is a normal domain by Spaces over Fields, Lemma 4.6. This proves (3).

Proof of (4). Let $\eta \in \operatorname{Spec}(A)$ denote the generic point and denote by a subscript η the base change to η. Since f is an alteration, the scheme X_η is finite and faithfully flat over η. Since $Y = X \times_{\operatorname{Spec}(A)} \operatorname{Spec}(A^\wedge)$ is regular by Lemma 6.2 we see that Y_η is regular (as a limit of opens in Y). Then $Y_\eta \to \operatorname{Spec}(A^\wedge \otimes_A \kappa(\eta))$ is finite faithfully flat onto the generic formal fibre. We conclude by Algebra, Lemma 164.4. $\square$

8. Resolution

0BHV   Here is a definition.

0BHW   **Definition 8.1.** Let S be a scheme. Let Y be a Noetherian integral algebraic space over S. A *resolution of singularities* of X is a modification $f : X \to Y$ such that X is regular.

In the case of surfaces we sometimes want a bit more information.

0BHX   **Definition 8.2.** Let S be a scheme. Let Y be a 2-dimensional Noetherian integral algebraic space over S. We say Y has a *resolution of singularities by normalized blowups* if there exists a sequence

$$Y_n \to X_{n-1} \to \ldots \to Y_1 \to Y_0 \to Y$$

where

(1) Y_i is proper over Y for $i = 0, \ldots, n$,
(2) $Y_0 \to Y$ is the normalization,
(3) $Y_i \to Y_{i-1}$ is a normalized blowup for $i = 1, \ldots, n$, and
(4) Y_n is regular.

Observe that condition (1) implies that the normalization Y_0 of Y is finite over Y and that the normalizations used in the normalized blowing ups are finite as well. We finally come to the main theorem of this chapter.

0BHY   **Theorem 8.3.** *Let S be a scheme. Let Y be a two dimensional integral Noetherian algebraic space over S. The following are equivalent*

(1) *there exists an alteration $X \to Y$ with X regular,*
(2) *there exists a resolution of singularities of Y,*
(3) *Y has a resolution of singularities by normalized blowups,*
(4) *the normalization $Y^\nu \to Y$ is finite, Y^ν has finitely many singular points $y_1, \ldots, y_m \in |Y|$, and for each i the completion of the henselian local ring $\mathcal{O}^h_{Y^\nu, y_i}$ is normal.*

Proof. The implications $(3) \Rightarrow (2) \Rightarrow (1)$ are immediate.

Let $X \to Y$ be an alteration with X regular. Then $Y^\nu \to Y$ is finite by Lemma 7.1. Consider the factorization $f : X \to Y^\nu$ from Morphisms of Spaces, Lemma 49.8. The morphism f is finite over an open $V \subset Y^\nu$ containing every point of codimension ≤ 1 in Y^ν by Spaces over Fields, Lemma 3.2. Then f is flat over V by Algebra, Lemma 128.1 and the fact that a normal local ring of dimension ≤ 2 is Cohen-Macaulay by Serre's criterion (Algebra, Lemma 157.4). Then V is regular by Algebra, Lemma 164.4. As Y^ν is Noetherian we conclude that $Y^\nu \backslash V = \{y_1, \ldots, y_m\}$ is finite. For each i let $\mathcal{O}^h_{Y^\nu, y_i}$ be the henselian local ring. Then $X \times_Y \operatorname{Spec}(\mathcal{O}^h_{Y^\nu, y_i})$

is a regular alteration of $\mathrm{Spec}(\mathcal{O}^h_{Y^\nu, y_i})$ (some details omitted). By Lemma 7.2 the completion of $\mathcal{O}^h_{Y^\nu, y_i}$ is normal. In this way we see that $(1) \Rightarrow (4)$.

Assume (4). We have to prove (3). We may immediately replace Y by its normalization. Let $y_1, \ldots, y_m \in |Y|$ be the singular points. Choose a collection of elementary étale neighbourhoods $(V_i, v_i) \to (Y, y_i)$ as in Section 3. For each i the henselian local ring $\mathcal{O}^h_{Y^\nu, y_i}$ is the henselization of $\mathcal{O}_{V_i, v_i}$. Hence these rings have isomorphic completions. Thus by the result for schemes (Resolution of Surfaces, Theorem 14.5) we see that there exist finite sequences of normalized blowups

$$X_{i,n_i} \to X_{i,n_i-1} \to \ldots \to V_i$$

blowing up only in points lying over v_i such that X_{i,n_i} is regular. By Lemma 5.3 there is a sequence of normalized blowing ups

$$X_n \to X_{n-1} \to \ldots \to X_1 \to Y$$

and of course X_n is regular too (look at the local rings). This completes the proof. $\qquad\square$

9. Examples

0AE8 Some examples related to the results earlier in this chapter.

0AE9 **Example 9.1.** Let k be a field. The ring $A = k[x, y, z]/(x^r + y^s + z^t)$ is a UFD for r, s, t pairwise coprime integers. Namely, since $x^r + y^s + z^t$ is irreducible A is a domain. The element z is a prime element, i.e., generates a prime ideal in A. On the other hand, if $t = 1 + ers$ for some e, then

$$A[1/z] \cong k[x', y', 1/z]$$

where $x' = x/z^{es}$, $y' = y/z^{er}$ and $z = (x')^r + (y')^s$. Thus $A[1/z]$ is a localization of a polynomial ring and hence a UFD. It follows from an argument of Nagata that A is a UFD. See Algebra, Lemma 120.7. A similar argument can be given if t is not congruent to 1 modulo rs.

[Sam68, 4(c)]

0AEA **Example 9.2.** The ring $A = \mathbf{C}[[x, y, z]]/(x^r + y^s + z^t)$ is not a UFD when $1 < r < s < t$ are pairwise coprime integers and not equal to $2, 3, 5$. For example consider the special case $A = \mathbf{C}[[x, y, z]]/(x^2 + y^5 + z^7)$. Consider the maps

$$\psi_\zeta : \mathbf{C}[[x, y, z]]/(x^2 + y^5 + z^7) \to \mathbf{C}[[t]]$$

given by

$$x \mapsto t^7, \quad y \mapsto t^3, \quad z \mapsto -\zeta t^2 (1 + t)^{1/7}$$

where ζ is a 7th root of unity. The kernel $\mathfrak{p}_\zeta$ of ψ_ζ is a height one prime, hence if A is a UFD, then it is principal, say given by $f_\zeta \in \mathbf{C}[[x, y, z]]$. Note that $V(x^3 - y^7) = \bigcup V(\mathfrak{p}_\zeta)$ and $A/(x^3 - y^7)$ is reduced away from the closed point. Hence, still assuming A is a UFD, we would obtain

$$\prod_\zeta f_\zeta = u(x^3 - y^7) + a(x^2 + y^5 + z^7) \quad \text{in} \quad \mathbf{C}[[x, y, z]]$$

for some unit $u \in \mathbf{C}[[x, y, z]]$ and some element $a \in \mathbf{C}[[x, y, z]]$. After scaling by a constant we may assume $u(0, 0, 0) = 1$. Note that the left hand side vanishes to order 7. Hence $a = -x \bmod \mathfrak{m}^2$. But then we get a term xy^5 on the right hand side which does not occur on the left hand side. A contradiction.

See [Bri68] and [Lip69] for nonvanishing of local Picard groups in general.

0AEB **Example 9.3.** There exists an excellent 2-dimensional Noetherian local ring and a modification $X \to S = \mathrm{Spec}(A)$ which is not a scheme. We sketch a construction. Let X be a normal surface over $\mathbf{C}$ with a unique singular point $x \in X$. Assume that there exists a resolution $\pi : X' \to X$ such that the exceptional fibre $C = \pi^{-1}(x)_{red}$ is a smooth projective curve. Furthermore, assume there exists a point $c \in C$ such that if $\mathcal{O}_C(nc)$ is in the image of $\mathrm{Pic}(X') \to \mathrm{Pic}(C)$, then $n = 0$. Then we let $X'' \to X'$ be the blowing up in the nonsingular point c. Let $C' \subset X''$ be the strict transform of C and let $E \subset X''$ be the exceptional fibre. By Artin's results ([Art70]; use for example [Mum61] to see that the normal bundle of C' is negative) we can blow down the curve C' in X'' to obtain an algebraic space X'''. Picture

$$
\begin{array}{ccc}
& X'' & \\
\swarrow & & \searrow \\
X' & & X''' \\
\searrow & & \swarrow \\
& X &
\end{array}
$$
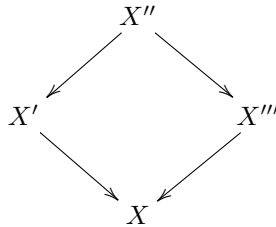
We claim that X''' is not a scheme. This provides us with our example because X''' is a scheme if and only if the base change of X''' to $A = \mathcal{O}_{X,x}$ is a scheme (details omitted). If X''' where a scheme, then the image of C' in X''' would have an affine neighbourhood. The complement of this neighbourhood would be an effective Cartier divisor on X''' (because X''' is nonsingular apart from 1 point). This effective Cartier divisor would correspond to an effective Cartier divisor on X'' meeting E and avoiding C'. Taking the image in X' we obtain an effective Cartier divisor meeting C (set theoretically) in c. This is impossible as no multiple of c is the restriction of a Cartier divisor by assumption.

To finish we have to find such a singular surface X. We can just take X to be the affine surface given by

$$x^3 + y^3 + z^3 + x^4 + y^4 + z^4 = 0$$

in $\mathbf{A}^3_{\mathbf{C}} = \mathrm{Spec}(\mathbf{C}[x, y, z])$ and singular point $(0, 0, 0)$. Then $(0, 0, 0)$ is the only singular point. Blowing up X in the maximal ideal corresponding to $(0, 0, 0)$ we find three charts each isomorphic to the smooth affine surface

$$1 + s^3 + t^3 + x(1 + s^4 + t^4) = 0$$

which is nonsingular with exceptional divisor C given by $x = 0$. The reader will recognize C as an elliptic curve. Finally, the surface X is rational as projection from $(0, 0, 0)$ shows, or because in the equation for the blowup we can solve for x. Finally, the Picard group of a nonsingular rational surface is countable, whereas the Picard group of an elliptic curve over the complex numbers is uncountable. Hence we can find a closed point c as indicated.

10. Other chapters

Preliminaries

(1) Introduction
(2) Conventions
(3) Set Theory

(4) Categories
(5) Topology
(6) Sheaves on Spaces
(7) Sites and Sheaves

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