PUSHOUTS OF ALGEBRAIC SPACES

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Contents

1. Introduction

0AHU The goal of this chapter is to discuss pushouts in the category of algebraic spaces. This can be done with varying assumptions. A fairly general pushout construction is given in [TT13]: one of the morphisms is affine and the other is a closed immersion. We discuss a particular case of this in Section 6 where we assume one of the morphisms is affine and the other is a thickening, a situation that often comes up in deformation theory.

In Sections 10 and 11 we discuss diagrams

$$
\begin{array}{ccc}
f^{-1}(X \setminus Z) & \longrightarrow & Y \\
\downarrow & & \downarrow{\scriptstyle f} \\
X \setminus Z & \longrightarrow & X
\end{array}
$$

where f is a quasi-compact and quasi-separated morphism of algebraic spaces, $Z \to X$ is a closed immersion of finite presentation, the map $f^{-1}(Z) \to Z$ is an isomorphism, and f is flat along $f^{-1}(Z)$. In this situation we glue quasi-coherent modules on $X \setminus Z$ and Y (in Section 10) to quasi-coherent modules on X and we glue algebraic spaces over $X \setminus Z$ and Y (in Section 11) to algebraic spaces over X.

In Section 13 we discuss how proper birational morphisms of Noetherian algebraic spaces give rise to coequalizer diagrams in algebraic spaces in some sense.

In Section 14 we use the construction of elementary distinguished squares in Section 9 to prove Nagata's theorem on compactifications in the setting of algebraic spaces.

2. Conventions

0GFM The standing assumption is that all schemes are contained in a big fppf site Sch_{fppf}. And all rings A considered have the property that $\mathrm{Spec}(A)$ is (isomorphic) to an object of this big site.

Let S be a scheme and let X be an algebraic space over S. In this chapter and the following we will write $X \times_S X$ for the product of X with itself (in the category of algebraic spaces over S), instead of $X \times X$.

3. Colimits of algebraic spaces

0GFN We briefly discuss colimits of algebraic spaces. Let S be a scheme. Let $\mathcal{I} \to (Sch/S)_{fppf}$, $i \mapsto X_i$ be a diagram (see Categories, Section 14). For each i we may consider the small étale site $X_{i,\acute{e}tale}$ whose objects are schemes étale over X_i, see Properties of Spaces, Section 18. For each morphism $i \to j$ of $\mathcal{I}$ we have the morphism $X_i \to X_j$ and hence a pullback functor $X_{j,\acute{e}tale} \to X_{i,\acute{e}tale}$. Hence we obtain a pseudo functor from $\mathcal{I}^{opp}$ into the 2-category of categories. Denote

$$\lim_i X_{i,\acute{e}tale}$$

the 2-limit (see insert future reference here). What does this mean concretely? An object of this limit is a system of étale morphisms $U_i \to X_i$ over $\mathcal{I}$ such that for each $i \to j$ in $\mathcal{I}$ the diagram

$$
\begin{array}{ccc}
U_i & \longrightarrow & U_j \\
\downarrow & & \downarrow \\
X_i & \longrightarrow & X_j
\end{array}
$$

is cartesian. Morphisms between objects are defined in the obvious manner. Suppose that $f_i : X_i \to T$ is a family of morphisms such that for each $i \to j$ the composition $X_i \to X_j \to T$ is equal to f_i. Then we get a functor $T_{\acute{e}tale} \to \lim X_{i,\acute{e}tale}$. With this notation in hand we can formulate our lemma.

07SX **Lemma 3.1.** *Let S be a scheme. Let $\mathcal{I} \to (Sch/S)_{fppf}$, $i \mapsto X_i$ be a diagram of schemes over S as above. Assume that*

(1) *$X = \mathrm{colim}\, X_i$ exists in the category of schemes,*
(2) *$\coprod X_i \to X$ is surjective,*
(3) *if $U \to X$ is étale and $U_i = X_i \times_X U$, then $U = \mathrm{colim}\, U_i$ in the category of schemes, and*
(4) *every object $(U_i \to X_i)$ of $\lim X_{i,\acute{e}tale}$ with $U_i \to X_i$ separated is in the essential image of the functor $X_{\acute{e}tale} \to \lim X_{i,\acute{e}tale}$.*

Then $X = \mathrm{colim}\, X_i$ in the category of algebraic spaces over S also.

Proof. Let Z be an algebraic space over S. Suppose that $f_i : X_i \to Z$ is a family of morphisms such that for each $i \to j$ the composition $X_i \to X_j \to Z$ is equal to f_i. We have to construct a morphism of algebraic spaces $f : X \to Z$ such that we can recover f_i as the composition $X_i \to X \to Z$. Let $W \to Z$ be a surjective

étale morphism of a scheme to Z. We may assume that W is a disjoint union of affines and in particular we may assume that $W \to Z$ is separated. For each i set $U_i = W \times_{Z,f_i} X_i$ and denote $h_i : U_i \to W$ the projection. Then $U_i \to X_i$ forms an object of $\lim X_{i,\text{étale}}$ with $U_i \to X_i$ separated. By assumption (4) we can find an étale morphism $U \to X$ and (functorial) isomorphisms $U_i = X_i \times_X U$. By assumption (3) there exists a morphism $h : U \to W$ such that the compositions $U_i \to U \to W$ are h_i. Let $g : U \to Z$ be the composition of h with the map $W \to Z$. To finish the proof we have to show that $g : U \to Z$ descends to a morphism $X \to Z$. To do this, consider the morphism $(h, h) : U \times_X U \to W \times_S W$. Composing with $U_i \times_{X_i} U_i \to U \times_X U$ we obtain (h_i, h_i) which factors through $W \times_Z W$. Since $U \times_X U$ is the colimit of the schemes $U_i \times_{X_i} U_i$ by (3) we see that (h, h) factors through $W \times_Z W$. Hence the two compositions $U \times_X U \to U \to W \to Z$ are equal. Because each $U_i \to X_i$ is surjective and assumption (2) we see that $U \to X$ is surjective. As Z is a sheaf for the étale topology, we conclude that $g : U \to Z$ descends to $f : X \to Z$ as desired. $\square$

We can check that a cocone is a colimit (fpqc) locally on the cocone.

0GFQ **Lemma 3.2.** *Let S be a scheme. Let B be an algebraic space over S. Let $\mathcal{I} \to (Sch/S)_{fppf}$, $i \mapsto X_i$ be a diagram of algebraic spaces over B. Let $(X, X_i \to X)$ be a cocone for the diagram in the category of algebraic spaces over B (Categories, Remark 14.5). If there exists a fpqc covering $\{U_a \to X\}_{a \in A}$ such that*

> (1) *for all $a \in A$ we have $U_a = \operatorname{colim} X_i \times_X U_a$ in the category of algebraic spaces over B, and*
> (2) *for all $a, b \in A$ we have $U_a \times_X U_b = \operatorname{colim} X_i \times_X U_a \times_X U_b$ in the category of algebraic spaces over B,*

then $X = \operatorname{colim} X_i$ in the category of algebraic spaces over B.

Proof. Namely, for an algebraic space Y over B a morphism $X \to Y$ over B is the same thing as a collection of morphism $U_a \to Y$ which agree on the overlaps $U_a \times_X U_b$ for all $a, b \in A$, see Descent on Spaces, Lemma 7.2. $\square$

We are going to find a common partial generalization of Lemmas 3.1 and 3.2 which can in particular be used to reduce a colimit construction to a subcategory of the category of all algebraic spaces.

Let S be a scheme and let B be an algebraic space over S. Let $\mathcal{I}$ be an index category and let $i \mapsto X_i$ be a diagram in the category of algebraic spaces over B, see Categories, Section 14. For each i we may consider the small étale site $X_{i,spaces,\text{étale}}$ whose objects are algebraic spaces étale over X_i, see Properties of Spaces, Section 18. For each morphism $i \to j$ of $\mathcal{I}$ we have the morphism $X_i \to X_j$ and hence a pullback functor $X_{j,spaces,\text{étale}} \to X_{i,spaces,\text{étale}}$. Hence we obtain a pseudo functor from $\mathcal{I}^{opp}$ into the 2-category of categories. Denote

$$\lim_i X_{i,spaces,\text{étale}}$$

the 2-limit (see insert future reference here). What does this mean concretely? An object of this limit is a diagram $i \mapsto (U_i \to X_i)$ in the category of arrows of

algebraic spaces over B such that for each $i \to j$ in $\mathcal{I}$ the diagram

$$\begin{array}{ccc} U_i & \longrightarrow & U_j \\ \downarrow & & \downarrow \\ X_i & \longrightarrow & X_j \end{array}$$

is cartesian. Morphisms between objects are defined in the obvious manner. Suppose that $f_i : X_i \to Z$ is a family of morphisms of algebraic spaces over B such that for each $i \to j$ the composition $X_i \to X_j \to Z$ is equal to f_i. Then we get a functor $Z_{spaces,\text{é}tale} \to \lim X_{i,spaces,\text{é}tale}$. With this notation in hand we can formulate our next lemma.

0GHL **Lemma 3.3.** *Let S be a scheme. Let B be an algebraic space over S. Let $\mathcal{I} \to (Sch/S)_{fppf}$, $i \mapsto X_i$ be a diagram of algebraic spaces over B. Let $(X, X_i \to X)$ be a cocone for the diagram in the category of algebraic spaces over B (Categories, Remark 14.5). Assume that*

(1) *the base change functor $X_{spaces,\text{é}tale} \to \lim X_{i,spaces,\text{é}tale}$, sending U to $U_i = X_i \times_X U$ is an equivalence,*

(2) *given*

    (a) *B' affine and étale over B,*

    (b) *Z an affine scheme over B',*

    (c) *$U \to X \times_B B'$ an étale morphism of algebraic spaces with U affine,*

    (d) *$f_i : U_i \to Z$ a cocone over B' of the diagram $i \mapsto U_i = U \times_X X_i$,*

    there exists a unique morphism $f : U \to Z$ over B' such that f_i equals the composition $U_i \to U \to Z$.

Then $X = \operatorname{colim} X_i$ in the category of all algebraic spaces over B.

Proof. In this paragraph we reduce to the case where B is an affine scheme. Let $B' \to B$ be an étale morphism of algebraic spaces. Observe that conditions (1) and (2) are preserved if we replace B, X_i, X by B', $X_i \times_B B'$, $X \times_B B'$. Let $\{B_a \to B\}_{a \in A}$ be an étale covering with B_a affine, see Properties of Spaces, Lemma 6.1. For $a \in A$ denote X_a, $X_{a,i}$ the base changes of X and the diagram to B_a. For $a, b \in A$ denote $X_{a,b}$ and $X_{a,b,i}$ the base changes of X and the diagram to $B_a \times_B B_b$. By Lemma 3.2 it suffices to prove that $X_a = \operatorname{colim} X_{a,i}$ and $X_{a,b} = \operatorname{colim} X_{a,b,i}$. This reduces us to the case where $B = B_a$ (an affine scheme) or $B = B_a \times_B B_b$ (a separated scheme). Repeating the argument once more, we conclude that we may assume B is an affine scheme (this uses that the intersection of affine opens in a separated scheme is affine).

Assume B is an affine scheme. Let Z be an algebraic space over B. We have to show

$$\operatorname{Mor}_B(X, Z) \longrightarrow \lim \operatorname{Mor}_B(X_i, Z)$$

is a bijection.

Proof of injectivity. Let $f, g : X \to Z$ be morphisms such that the compositions $f_i, g_i : X_i \to Z$ are the same for all i. Choose an affine scheme Z' and an étale morphism $Z' \to Z$. By Properties of Spaces, Lemma 6.1 we know we can cover Z by such affines. Set $U = X \times_{f,Z} Z'$ and $U' = X \times_{g,Z} Z'$ and denote $p : U \to X$ and $p' : U' \to X$ the projections. Since $f_i = g_i$ for all i, we see that

$$U_i = X_i \times_{f_i,Z} Z' = X_i \times_{g_i,Z} Z' = U'_i$$

compatible with transition morphisms. By (1) there is a unique isomorphism $\epsilon :$ $U \to U'$ as algebraic spaces over X, i.e., with $p = p' \circ \epsilon$ which is compatible with the displayed identifications. Choose an étale covering $\{h_a : U_a \to U\}$ with U_a affine. By (2) we see that $f \circ p \circ h_a = g \circ p' \circ \epsilon \circ h_a = g \circ p \circ h_a$. Since $\{h_a : U_a \to U\}$ is an étale covering we conclude $f \circ p = g \circ p$. Since the collection of morphisms $p : U \to X$ we obtain in this manner is an étale covering, we conclude that $f = g$.

Proof of surjectivity. Let $f_i : X_i \to Z$ be an element of the right hand side of the displayed arrow in the first paragraph of the proof. It suffices to find an étale covering $\{U_c \to X\}_{c \in C}$ such that the families $f_{c,i} \in \lim_i \operatorname{Mor}_B(X_i \times_X U_c, Z)$ come from morphisms $f_c : U_c \to Z$. Namely, by the uniqueness proved above the morphisms f_c will agree on $U_c \times_X U_b$ and hence will descend to give the desired morphism $f : X \to Z$. To find our covering, we first choose an étale covering $\{g_a : Z_a \to Z\}_{a \in A}$ where each Z_a is affine. Then we let $U_{a,i} = X_i \times_{f_i,Z} Z_a$. By (1) we find $U_{a,i} = X_i \times_X U_a$ for some algebraic spaces U_a étale over X. Then we choose étale coverings $\{U_{a,b} \to U_a\}_{b \in B_a}$ with $U_{a,b}$ affine and we consider the morphisms

$$U_{a,b,i} = X_i \times_X U_{a,b} \to X_i \times_X U_a = X_i \times_{f_i,Z} Z_a \to Z_a$$

By (2) we obtain morphisms $f_{a,b} : U_{a,b} \to Z_a$ compatible with these morphisms. Setting $C = \coprod_{a \in A} B_a$ and for $c \in C$ corresponding to $b \in B_a$ setting $U_c = U_{a,b}$ and $f_c = g_a \circ f_{a,b} : U_c \to Z$ we conclude. $\square$

Here is an application of these ideas to reduce the general case to the case of separated algebraic spaces.

0GFP **Lemma 3.4.** *Let S be a scheme. Let B be an algebraic space over S. Let $\mathcal{I} \to (Sch/S)_{fppf}$, $i \mapsto X_i$ be a diagram of algebraic spaces over B. Assume that*

(1) *each X_i is separated over B,*
(2) *$X = \operatorname{colim} X_i$ exists in the category of algebraic spaces separated over B,*
(3) *$\coprod X_i \to X$ is surjective,*
(4) *if $U \to X$ is an étale separated morphism of algebraic spaces and $U_i = X_i \times_X U$, then $U = \operatorname{colim} U_i$ in the category of algebraic spaces separated over B, and*
(5) *every object $(U_i \to X_i)$ of $\lim X_{i,spaces,\text{étale}}$ with $U_i \to X_i$ separated is of the form $U_i = X_i \times_X U$ for some étale separated morphism of algebraic spaces $U \to X$.*

Then $X = \operatorname{colim} X_i$ in the category of all algebraic spaces over B.

Proof. We encourage the reader to look instead at Lemma 3.3 and its proof.

Let Z be an algebraic space over B. Suppose that $f_i : X_i \to Z$ is a family of morphisms such that for each $i \to j$ the composition $X_i \to X_j \to Z$ is equal to f_i. We have to construct a morphism of algebraic spaces $f : X \to Z$ over B such that we can recover f_i as the composition $X_i \to X \to Z$. Let $W \to Z$ be a surjective étale morphism of a scheme to Z. We may assume that W is a disjoint union of affines and in particular we may assume that $W \to Z$ is separated and that W is separated over B. For each i set $U_i = W \times_{Z,f_i} X_i$ and denote $h_i : U_i \to W$ the projection. Then $U_i \to X_i$ forms an object of $\lim X_{i,spaces,\text{étale}}$ with $U_i \to X_i$ separated. By assumption (5) we can find a separated étale morphism $U \to X$ of algebraic spaces and (functorial) isomorphisms $U_i = X_i \times_X U$. By assumption (4) there exists a morphism $h : U \to W$ over B such that the compositions $U_i \to U \to W$ are h_i. Let

$g : U \to Z$ be the composition of h with the map $W \to Z$. To finish the proof we have to show that $g : U \to Z$ descends to a morphism $X \to Z$. To do this, consider the morphism $(h, h) : U \times_X U \to W \times_S W$. Composing with $U_i \times_{X_i} U_i \to U \times_X U$ we obtain (h_i, h_i) which factors through $W \times_Z W$. Since $U \times_X U$ is the colimit of the algebraic spaces $U_i \times_{X_i} U_i$ in the category of algebraic spaces separated over B by (4) we see that (h, h) factors through $W \times_Z W$. Hence the two compositions $U \times_X U \to U \to W \to Z$ are equal. Because each $U_i \to X_i$ is surjective and assumption (2) we see that $U \to X$ is surjective. As Z is a sheaf for the étale topology, we conclude that $g : U \to Z$ descends to $f : X \to Z$ as desired. $\qquad\square$
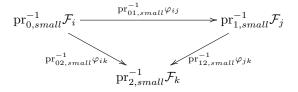
4. Descending étale sheaves

0GFR   This section is the analogue for algebraic spaces of Étale Cohomology, Section 104.

In order to conveniently express our results we need some notation. Let S be a scheme. Let $\mathcal{U} = \{f_i : X_i \to X\}$ be a family of morphisms of algebraic spaces over S with fixed target. A *descent datum* for étale sheaves with respect to $\mathcal{U}$ is a family $((\mathcal{F}_i)_{i \in I}, (\varphi_{ij})_{i,j \in I})$ where

(1) $\mathcal{F}_i$ is in $Sh(X_{i,\text{étale}})$, and
(2) $\varphi_{ij} : \mathrm{pr}_{0,small}^{-1}\mathcal{F}_i \longrightarrow \mathrm{pr}_{1,small}^{-1}\mathcal{F}_j$ is an isomorphism in $Sh((X_i \times_X X_j)_{\text{étale}})$

such that the *cocycle condition* holds: the diagrams

commute in $Sh((X_i \times_X X_j \times_X X_k)_{\text{étale}})$. There is an obvious notion of *morphisms of descent data* and we obtain a category of descent data. A descent datum $((\mathcal{F}_i)_{i \in I}, (\varphi_{ij})_{i,j \in I})$ is called *effective* if there exist a $\mathcal{F}$ in $Sh(X_{\text{étale}})$ and isomorphisms $\varphi_i : f_{i,small}^{-1}\mathcal{F} \to \mathcal{F}_i$ in $Sh(X_{i,\text{étale}})$ compatible with the φ_{ij}, i.e., such that

$$\varphi_{ij} = \mathrm{pr}_{1,small}^{-1}(\varphi_j) \circ \mathrm{pr}_{0,small}^{-1}(\varphi_i^{-1})$$

Another way to say this is the following. Given an object $\mathcal{F}$ of $Sh(X_{\text{étale}})$ we obtain the *canonical descent datum* $(f_{i,small}^{-1}\mathcal{F}_i, c_{ij})$ where c_{ij} is the canonical isomorphism
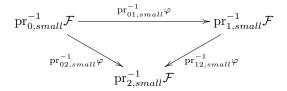
$$c_{ij} : \mathrm{pr}_{0,small}^{-1}f_{i,small}^{-1}\mathcal{F} \longrightarrow \mathrm{pr}_{1,small}^{-1}f_{j,small}^{-1}\mathcal{F}$$

The descent datum $((\mathcal{F}_i)_{i \in I}, (\varphi_{ij})_{i,j \in I})$ is effective if and only if it is isomorphic to the canonical descent datum associated to some $\mathcal{F}$ in $Sh(X_{\text{étale}})$.

If the family consists of a single morphism $\{X \to Y\}$, then we think of a descent datum as a pair $(\mathcal{F}, \varphi)$ where $\mathcal{F}$ is an object of $Sh(X_{\text{étale}})$ and φ is an isomorphism

$$\mathrm{pr}_{0,small}^{-1}\mathcal{F} \longrightarrow \mathrm{pr}_{1,small}^{-1}\mathcal{F}$$

in $Sh((X \times_Y X)_{étale})$ such that the cocycle condition holds:

$$\mathrm{pr}^{-1}_{0,small}\mathcal{F} \xrightarrow{\ \mathrm{pr}^{-1}_{01,small}\varphi\ } \mathrm{pr}^{-1}_{1,small}\mathcal{F}$$

with $\mathrm{pr}^{-1}_{02,small}\varphi$ and $\mathrm{pr}^{-1}_{12,small}\varphi$ mapping to $\mathrm{pr}^{-1}_{2,small}\mathcal{F}$

commutes in $Sh((X \times_Y X \times_Y X)_{étale})$. There is a notion of morphisms of descent data and effectivity exactly as before.

0GFS **Lemma 4.1.** *Let S be a scheme. Let $\{f_i : X_i \to X\}$ be an étale covering of algebraic spaces. The functor*

$$Sh(X_{étale}) \longrightarrow \text{descent data for étale sheaves wrt } \{f_i : X_i \to X\}$$

is an equivalence of categories.

Proof. In Properties of Spaces, Section 18 we have defined a site $X_{spaces,étale}$ whose objects are algebraic spaces étale over X with étale coverings. Moreover, we have a identifications $Sh(X_{étale}) = Sh(X_{spaces,étale})$ compatible with morphisms of algebraic spaces, i.e., compatible with pushforward and pullback. Hence the statement of the lemma follows from the much more general discussion in Sites, Section 26. $\square$

0GFT **Lemma 4.2.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\{Y_i \to Y\}_{i \in I}$ be an étale covering of algebraic spaces. If for each $i \in I$ the functor*

$$Sh(Y_{i,étale}) \longrightarrow \text{descent data for étale sheaves wrt } \{X \times_Y Y_i \to Y_i\}$$

is an equivalence of categories and for each $i, j \in I$ the functor

$$Sh((Y_i \times_Y Y_j)_{étale}) \longrightarrow \text{descent data for étale sheaves wrt } \{X \times_Y Y_i \times_Y Y_j \to Y_i \times_Y Y_j\}$$

is an equivalence of categories, then

$$Sh(Y_{étale}) \longrightarrow \text{descent data for étale sheaves wrt } \{X \to Y\}$$

is an equivalence of categories.

Proof. Formal consequence of Lemma 4.1 and the definitions. $\square$

0GFU **Lemma 4.3.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is representable (by schemes) and f has one of the following properties: surjective and integral, surjective and proper, or surjective and flat and locally of finite presentation Then*

$$Sh(Y_{étale}) \longrightarrow \text{descent data for étale sheaves wrt } \{X \to Y\}$$

is an equivalence of categories.

Proof. Each of the properties of morphisms of algebraic spaces mentioned in the statement of the lemma is preserved by arbitrary base change, see the lists in Spaces, Section 4. Thus we can apply Lemma 4.2 to see that we can work étale locally on Y. In this way we reduce to the case where Y is a scheme; some details omitted. In this case X is also a scheme and the result follows from Étale Cohomology, Lemma 104.2, 104.3, or 104.5. $\square$

0GFV **Lemma 4.4.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\pi : X' \to X$ be a morphism of algebraic spaces. Assume*

(1) *$f \circ \pi$ is representable (by schemes),*
(2) *$f \circ \pi$ has one of the following properties: surjective and integral, surjective and proper, or surjective and flat and locally of finite presentation.*

Then

$$Sh(Y_{\acute{e}tale}) \longrightarrow descent\ data\ for\ \acute{e}tale\ sheaves\ wrt\ \{X \to Y\}$$

is an equivalence of categories.

Proof. Formal consequence of Lemma 4.3 and Stacks, Lemma 3.7. $\square$

0GFW **Lemma 4.5.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which has one of the following properties: surjective and integral, surjective and proper, or surjective and flat and locally of finite presentation. Then the functor*

$$Sh(Y_{\acute{e}tale}) \longrightarrow descent\ data\ for\ \acute{e}tale\ sheaves\ wrt\ \{X \to Y\}$$

is an equivalence of categories.

Proof. Observe that the base change of a proper surjective morphism is proper and surjective, see Morphisms of Spaces, Lemmas 40.3 and 5.5. Hence by Lemma 4.2 we may work étale locally on Y. Hence we reduce to Y being an affine scheme; some details omitted.

Assume Y is affine. By Lemma 4.4 it suffices to find a morphism $X' \to X$ where X' is a scheme such that $X' \to Y$ is surjective and integral, surjective and proper, or surjective and flat and locally of finite presentation.

In case $X \to Y$ is integral and surjective, we can take $X = X'$ as an integral morphism is representable.

If f is proper and surjective, then the algebraic space X is quasi-compact and separated, see Morphisms of Spaces, Section 8 and Lemma 4.9. Choose a scheme X' and a surjective finite morphism $X' \to X$, see Limits of Spaces, Proposition 16.1. Then $X' \to Y$ is surjective and proper.

Finally, if $X \to Y$ is surjective and flat and locally of finite presentation then we can take an affine étale covering $\{U_i \to X\}$ and set X' equal to the disjoint $\coprod U_i$. $\square$

0GFX **Lemma 4.6.** *Let S be a scheme. Let $\{f_i : X_i \to X\}$ be an fppf covering of algebraic spaces over S. The functor*

$$Sh(X_{\acute{e}tale}) \longrightarrow descent\ data\ for\ \acute{e}tale\ sheaves\ wrt\ \{f_i : X_i \to X\}$$

is an equivalence of categories.

Proof. We have Lemma 4.5 for the morphism $f : \coprod X_i \to X$. Then a formal argument shows that descent data for f are the same thing as descent data for the covering, compare with Descent, Lemma 34.5. Details omitted. $\square$

0GFY **Lemma 4.7.** *Let S be a scheme. Let $f : Y' \to Y$ be a proper morphism of algebraic spaces over S. Let $i : Z \to Y$ be a closed immersion. Set $E = Z \times_Y Y'$.*

Picture

$$E \xrightarrow{\quad j \quad} Y'$$

$$\begin{array}{ccc} E & \xrightarrow{\ j\ } & Y' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{\ i\ } & Y \end{array}$$

If f is an isomorphism over $Y \setminus Z$, then the functor

$$Sh(Y_{\acute{e}tale}) \longrightarrow Sh(Y'_{\acute{e}tale}) \times_{Sh(E_{\acute{e}tale})} Sh(Z_{\acute{e}tale})$$

is an equivalence of categories.

Proof. Observe that $X = Y' \coprod Z \to Y$ is a proper surjective morphism. Thus it suffice to construct an equivalence of categories

$$Sh(Y'_{\acute{e}tale}) \times_{Sh(E_{\acute{e}tale})} Sh(Z_{\acute{e}tale}) \longrightarrow \text{descent data for étale sheaves wrt } \{X \to Y\}$$

compatible with pullback functors from Y because then we can use Lemma 4.5 to conclude. Thus let $(\mathcal{G}', \mathcal{G}, \alpha)$ be an object of $Sh(Y'_{\acute{e}tale}) \times_{Sh(E_{\acute{e}tale})} Sh(Z_{\acute{e}tale})$ with notation as in Categories, Example 31.3. Then we can consider the sheaf $\mathcal{F}$ on X defined by taking $\mathcal{G}'$ on the summand Y' and $\mathcal{G}$ on the summand Z. We have

$$X \times_Y X = Y' \times_Y Y' \amalg Y' \times_Y Z \amalg Z \times_Y Y' \amalg Z \times_Y Z = Y' \times_Y Y' \amalg E \amalg E \amalg Z$$

The isomorphisms of the two pullbacks of $\mathcal{F}$ to this algebraic space are obvious over the summands E, E, Z. The interesting part of the proof is to find an isomorphism $\mathrm{pr}_{0,small}^{-1}\mathcal{G}' \to \mathrm{pr}_{1,small}^{-1}\mathcal{G}'$ over $Y' \times_Y Y'$ satisfying the cocycle condition. However, our assumption that $Y' \to Y$ is an isomorphism over $Y \setminus Z$ implies that

$$h : Y \coprod E \times_Z E \longrightarrow Y' \times_Y Y'$$

is a surjective proper morphism. (It is in fact a finite morphism as it is the disjoint union of two closed immersions.) Hence it suffices to construct an isomorphism of the pullbacks of $\mathrm{pr}_{0,small}^{-1}\mathcal{G}'$ and $\mathrm{pr}_{1,small}^{-1}\mathcal{G}'$ by h_{small} satisfying a certain cocycle condition. For the diagonal, it is clear how to do this. And for the pullback to $E \times_Z E$ we use that both sheaves pull back to the pullback of $\mathcal{G}$ by the morphism $E \times_Z E \to Z$. We omit the details. $\qquad\square$

5. Descending étale morphisms of algebraic spaces

0GFZ   In this section we combine the glueing results for étale sheaves given in Section 4 with the flexibility of algebraic spaces to get some descent statements for étale morphisms of algebraic spaces.

0GG0   **Lemma 5.1.** *Let S be a scheme. Let $f : X \to Y$ be a proper surjective morphism of algebraic spaces over S. Any descent datum $(U/X, \varphi)$ relative to f (Descent on Spaces, Definition 22.1) with U étale over X is effective (Descent on Spaces, Definition 22.10). More precisely, there exists an étale morphism $V \to Y$ of algebraic spaces whose corresponding canonical descent datum is isomorphic to $(U/X, \varphi)$.*

Proof. Recall that U gives rise to a representable sheaf $\mathcal{F} = h_U$ in $Sh(X_{spaces,\acute{e}tale}) = Sh(X_{\acute{e}tale})$, see Properties of Spaces, Section 18. The descent datum on U relative to f exactly gives a descent datum $(\mathcal{F}, \varphi)$ for étale sheaves with respect to $\{X \to Y\}$. By Lemma 4.5 this descent datum is effective. Let $\mathcal{G}$ be the corresponding sheaf on $Y_{\acute{e}tale}$. By Properties of Spaces, Lemma 27.3 we obtain an étale morphism $V \to Y$ of algebraic spaces corresponding to $\mathcal{G}$; we omit the verification of the set theoretic

condition[1]. The given isomorphism $\mathcal{F} \to f_{small}^{-1}\mathcal{G}$ corresponds to an isomorphism $U \to V \times_Y X$ compatible with the descent datum. $\square$

0GG1 **Lemma 5.2.** *Let S be a scheme. Let $f : Y' \to Y$ be a proper morphism of algebraic spaces over S. Let $i : Z \to Y$ be a closed immersion. Set $E = Z \times_Y Y'$. Picture*

$$
\begin{array}{ccc}
E & \xrightarrow{\ j\ } & Y' \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\
Z & \xrightarrow{\ i\ } & Y
\end{array}
$$

If f is an isomorphism over $Y \setminus Z$, then the functor

$$Y_{spaces,\acute{e}tale} \longrightarrow Y'_{spaces,\acute{e}tale} \times_{E_{spaces,\acute{e}tale}} Z_{spaces,\acute{e}tale}$$
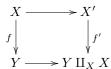
is an equivalence of categories.

Proof. Let $(V' \to Y', W \to Z, \alpha)$ be an object of the right hand side. Recall that V', resp. W gives rise to a representable sheaf $\mathcal{G}' = h_{V'}$ in $Sh(Y'_{spaces,\acute{e}tale}) = Sh(Y'_{\acute{e}tale})$, resp. $\mathcal{G} = h_W$ in $Sh(Z_{spaces,\acute{e}tale}) = Sh(Z_{\acute{e}tale})$, see Properties of Spaces, Section 18. The isomorphism $\alpha : V' \times_{Y'} E \to W \times_Z E$ determines an isomorphism $j_{small}^{-1}\mathcal{G}' \to g_{small}^{-1}\mathcal{G}$ of sheaves on E. By Lemma 4.7 we obtain a unique sheaf $\mathcal{F}$ on Y pulling pack to $\mathcal{G}'$ and $\mathcal{G}$ compatibly with the isomorphism. By Properties of Spaces, Lemma 27.3 we obtain an étale morphism $V \to Y$ of algebraic spaces corresponding to $\mathcal{F}$; we omit the verification of the set theoretic condition[2]. The given isomorphism $\mathcal{G}' \to f_{small}^{-1}\mathcal{F}$ and $\mathcal{G} \to i_{small}^{-1}\mathcal{F}$ corresponds to isomorphisms $V' \to V \times_Y Y'$ and $W \to V \times_Y Z$ compatible with α as desired. $\square$

6. Pushouts along thickenings and affine morphisms

07SW This section is analogue of More on Morphisms, Section 14.

07SY **Lemma 6.1.** *Let S be a scheme. Let $X \to X'$ be a thickening of schemes over S and let $X \to Y$ be an affine morphism of schemes over S. Let $Y' = Y \amalg_X X'$ be the pushout in the category of schemes (see More on Morphisms, Lemma 14.3). Then Y' is also a pushout in the category of algebraic spaces over S.*

Proof. This is an immediate consequence of Lemma 3.1 and More on Morphisms, Lemmas 14.3, 14.4, and 14.6. $\square$

07VX **Lemma 6.2.** *Let S be a scheme. Let $X \to X'$ be a thickening of algebraic spaces over S and let $X \to Y$ be an affine morphism of algebraic spaces over S. Then there exists a pushout*

$$
\begin{array}{ccc}
X & \longrightarrow & X' \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\
Y & \longrightarrow & Y \amalg_X X'
\end{array}
$$

in the category of algebraic spaces over S. Moreover $Y' = Y \amalg_X X'$ is a thickening of Y and

$$\mathcal{O}_{Y'} = \mathcal{O}_Y \times_{f_*\mathcal{O}_X} f'_*\mathcal{O}_{X'}$$

as sheaves on $Y_{\acute{e}tale} = (Y')_{\acute{e}tale}$.

---

[1]It follows from the fact that $\mathcal{F}$ satisfies the corresponding condition.
[2]It follows from the fact that $\mathcal{G}$ and $\mathcal{G}'$ satisfies the corresponding condition.

Proof. Choose a scheme V and a surjective étale morphism $V \to Y$. Set $U = V \times_Y X$. This is a scheme affine over V with a surjective étale morphism $U \to X$. By More on Morphisms of Spaces, Lemma 9.6 there exists a $U' \to X'$ surjective étale with $U = U' \times_{X'} X$. In particular the morphism of schemes $U \to U'$ is a thickening too. Apply More on Morphisms, Lemma 14.3 to obtain a pushout $V' = V \amalg_U U'$ in the category of schemes.

We repeat this procedure to construct a pushout

$$
\begin{array}{ccc}
U \times_X U & \longrightarrow & U' \times_{X'} U' \\
\downarrow & & \downarrow \\
V \times_Y V & \longrightarrow & R'
\end{array}
$$

in the category of schemes. Consider the morphisms

$$U \times_X U \to U \to V', \quad U' \times_{X'} U' \to U' \to V', \quad V \times_Y V \to V \to V'$$

where we use the first projection in each case. Clearly these glue to give a morphism $t' : R' \to V'$ which is étale by More on Morphisms, Lemma 14.6. Similarly, we obtain $s' : R' \to V'$ étale. The morphism $j' = (t', s') : R' \to V' \times_S V'$ is unramified (as t' is étale) and a monomorphism when restricted to the closed subscheme $V \times_Y V \subset R'$. As $V \times_Y V \subset R'$ is a thickening it follows that j' is a monomorphism too. Finally, j' is an equivalence relation as we can use the functoriality of pushouts of schemes to construct a morphism $c' : R' \times_{s',V',t'} R' \to R'$ (details omitted). At this point we set $Y' = U'/R'$, see Spaces, Theorem 10.5.

We have morphisms $X' = U'/U' \times_{X'} U' \to V'/R' = Y'$ and $Y = V/V \times_Y V \to V'/R' = Y'$. By construction these fit into the commutative diagram

$$
\begin{array}{ccc}
X & \longrightarrow & X' \\
f \downarrow & & \downarrow f' \\
Y & \longrightarrow & Y'
\end{array}
$$

Since $Y \to Y'$ is a thickening we have $Y_{\text{étale}} = (Y')_{\text{étale}}$, see More on Morphisms of Spaces, Lemma 9.6. The commutativity of the diagram gives a map of sheaves

$$\mathcal{O}_{Y'} \longrightarrow \mathcal{O}_Y \times_{f_* \mathcal{O}_X} f'_* \mathcal{O}_{X'}$$

on this set. By More on Morphisms, Lemma 14.3 this map is an isomorphism when we restrict to the scheme V', hence it is an isomorphism.

To finish the proof we show that the diagram above is a pushout in the category of algebraic spaces. To see this, let Z be an algebraic space and let $a' : X' \to Z$ and $b : Y \to Z$ be morphisms of algebraic spaces. By Lemma 6.1 we obtain a unique morphism $h : V' \to Z$ fitting into the commutative diagrams

$$
\begin{array}{ccc}
U' \longrightarrow V' \\
\downarrow \qquad \downarrow h \\
X' \xrightarrow{a'} Z
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
V \longrightarrow V' \\
\downarrow \qquad \downarrow h \\
Y \xrightarrow{b} Z
\end{array}
$$

The uniqueness shows that $h \circ t' = h \circ s'$. Hence h factors uniquely as $V' \to Y' \to Z$ and we win. $\qquad\square$

In the following lemma we use the fibre product of categories as defined in Categories, Example 31.3.

07VY **Lemma 6.3.** *Let S be a base scheme. Let $X \to X'$ be a thickening of algebraic spaces over S and let $X \to Y$ be an affine morphism of algebraic spaces over S. Let $Y' = Y \amalg_X X'$ be the pushout (see Lemma 6.2). Base change gives a functor*

$$F : (Spaces/Y') \longrightarrow (Spaces/Y) \times_{(Spaces/Y')} (Spaces/X')$$

given by $V' \longmapsto (V' \times_{Y'} Y, V' \times_{Y'} X', 1)$ which sends (Sch/Y') into $(Sch/Y) \times_{(Sch/Y')} (Sch/X')$. The functor F has a left adjoint

$$G : (Spaces/Y) \times_{(Spaces/Y')} (Spaces/X') \longrightarrow (Spaces/Y')$$

which sends the triple (V, U', φ) to the pushout $V \amalg_{(V \times_Y X)} U'$ in the category of algebraic spaces over S. The functor G sends $(Sch/Y) \times_{(Sch/Y')} (Sch/X')$ into (Sch/Y').

Proof. The proof is completely formal. Since the morphisms $X \to X'$ and $X \to Y$ are representable it is clear that F sends (Sch/Y') into $(Sch/Y) \times_{(Sch/Y')} (Sch/X')$.

Let us construct G. Let (V, U', φ) be an object of the fibre product category. Set $U = U' \times_{X'} X$. Note that $U \to U'$ is a thickening. Since $\varphi : V \times_Y X \to U' \times_{X'} X = U$ is an isomorphism we have a morphism $U \to V$ over $X \to Y$ which identifies U with the fibre product $X \times_Y V$. In particular $U \to V$ is affine, see Morphisms of Spaces, Lemma 20.5. Hence we can apply Lemma 6.2 to get a pushout $V' = V \amalg_U U'$. Denote $V' \to Y'$ the morphism we obtain in virtue of the fact that V' is a pushout and because we are given morphisms $V \to Y$ and $U' \to X'$ agreeing on U as morphisms into Y'. Setting $G(V, U', \varphi) = V'$ gives the functor G.
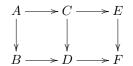
If (V, U', φ) is an object of $(Sch/Y) \times_{(Sch/Y')} (Sch/X')$ then $U = U' \times_{X'} X$ is a scheme too and we can form the pushout $V' = V \amalg_U U'$ in the category of schemes by More on Morphisms, Lemma 14.3. By Lemma 6.1 this is also a pushout in the category of schemes, hence G sends $(Sch/Y) \times_{(Sch/Y')} (Sch/X')$ into (Sch/Y').

Let us prove that G is a left adjoint to F. Let Z be an algebraic space over Y'. We have to show that

$$\mathrm{Mor}(V', Z) = \mathrm{Mor}((V, U', \varphi), F(Z))$$

where the morphism sets are taking in their respective categories. Let $g' : V' \to Z$ be a morphism. Denote $\tilde{g}$, resp. $\tilde{f}'$ the composition of g' with the morphism $V \to V'$, resp. $U' \to V'$. Base change $\tilde{g}$, resp. $\tilde{f}'$ by $Y \to Y'$, resp. $X' \to Y'$ to get a morphism $g : V \to Z \times_{Y'} Y$, resp. $f' : U' \to Z \times_{Y'} X'$. Then (g, f') is an element of the right hand side of the equation above (details omitted). Conversely, suppose that $(g, f') : (V, U', \varphi) \to F(Z)$ is an element of the right hand side. We may consider the composition $\tilde{g} : V \to Z$, resp. $\tilde{f}' : U' \to Z$ of g, resp. f by $Z \times_{Y'} X' \to Z$, resp. $Z \times_{Y'} Y \to Z$. Then $\tilde{g}$ and $\tilde{f}'$ agree as morphism from U to Z. By the universal property of pushout, we obtain a morphism $g' : V' \to Z$, i.e., an element of the left hand side. We omit the verification that these constructions are mutually inverse. $\square$

07VZ **Lemma 6.4.** *Let S be a scheme. Let*

$$
\begin{array}{ccc}
A \longrightarrow C \longrightarrow E \\
\downarrow \quad\quad \downarrow \quad\quad \downarrow \\
B \longrightarrow D \longrightarrow F
\end{array}
$$

be a commutative diagram of algebraic spaces over S. Assume that A, B, C, D and A, B, E, F form cartesian squares and that $B \to D$ is surjective étale. Then C, D, E, F is a cartesian square.
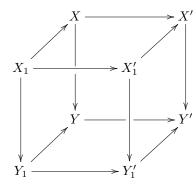
Proof. This is formal. $\square$

07W0 **Lemma 6.5.** *In the situation of Lemma 6.3 the functor $F \circ G$ is isomorphic to the identity functor.*

Proof. We will prove that $F \circ G$ is isomorphic to the identity by reducing this to the corresponding statement of More on Morphisms, Lemma 14.4.

Choose a scheme Y_1 and a surjective étale morphism $Y_1 \to Y$. Set $X_1 = Y_1 \times_Y X$. This is a scheme affine over Y_1 with a surjective étale morphism $X_1 \to X$. By More on Morphisms of Spaces, Lemma 9.6 there exists a $X_1' \to X'$ surjective étale with $X_1 = X_1' \times_{X'} X$. In particular the morphism of schemes $X_1 \to X_1'$ is a thickening too. Apply More on Morphisms, Lemma 14.3 to obtain a pushout $Y_1' = Y_1 \amalg_{X_1} X_1'$ in the category of schemes. In the proof of Lemma 6.2 we constructed Y' as a quotient of an étale equivalence relation on Y_1' such that we get a commutative diagram
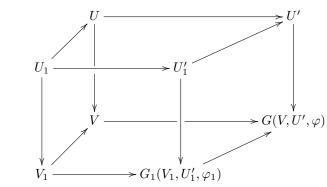
07W1 (6.5.1)

where all squares except the front and back squares are cartesian (the front and back squares are pushouts) and the northeast arrows are surjective étale. Denote F_1, G_1 the functors constructed in More on Morphisms, Lemma 14.4 for the front square. Then the diagram of categories

$$
\begin{array}{ccc}
(Sch/Y_1') \xrightarrow[F_1]{\overset{G_1}{\longleftarrow}} (Sch/Y_1) \times_{(Sch/Y_1')} (Sch/X_1') \\
\downarrow \quad\quad\quad\quad\quad\quad \downarrow \\
(Spaces/Y') \xrightarrow[F]{\overset{G}{\longleftarrow}} (Spaces/Y) \times_{(Spaces/Y')} (Spaces/X')
\end{array}
$$

is commutative by simple considerations regarding base change functors and the agreement of pushouts in schemes with pushouts in spaces of Lemma 6.1.

Let (V, U', φ) be an object of $(Spaces/Y) \times_{(Spaces/Y')} (Spaces/X')$. Denote $U = U' \times_{X'} X$ so that $G(V, U', \varphi) = V \amalg_U U'$. Choose a scheme V_1 and a surjective étale morphism $V_1 \to Y_1 \times_Y V$. Set $U_1 = V_1 \times_Y X$. Then

$$U_1 = V_1 \times_Y X \longrightarrow (Y_1 \times_Y V) \times_Y X = X_1 \times_Y V = X_1 \times_X X \times_Y V = X_1 \times_X U$$

is surjective étale too. By More on Morphisms of Spaces, Lemma 9.6 there exists a thickening $U_1 \to U_1'$ and a surjective étale morphism $U_1' \to X_1' \times_{X'} U'$ whose base change to $X_1 \times_X U$ is the displayed morphism. At this point (V_1, U_1', φ_1) is an object of $(Sch/Y_1) \times_{(Sch/Y_1')} (Sch/X_1')$. In the proof of Lemma 6.2 we constructed $G(V, U', \varphi) = V \amalg_U U'$ as a quotient of an étale equivalence relation on $G_1(V_1, U_1', \varphi_1) = V_1 \amalg_{U_1} U_1'$ such that we get a commutative diagram

07W2 (6.5.2)

where all squares except the front and back squares are cartesian (the front and back squares are pushouts) and the northeast arrows are surjective étale. In particular

$$G_1(V_1, U_1', \varphi_1) \to G(V, U', \varphi)$$

is surjective étale.

Finally, we come to the proof of the lemma. We have to show that the adjunction mapping $(V, U', \varphi) \to F(G(V, U', \varphi))$ is an isomorphism. We know $(V_1, U_1', \varphi_1) \to F_1(G_1(V_1, U_1', \varphi_1))$ is an isomorphism by More on Morphisms, Lemma 14.4. Recall that F and F_1 are given by base change. Using the properties of (6.5.2) and Lemma 6.4 we see that $V \to G(V, U', \varphi) \times_{Y'} Y$ and $U' \to G(V, U', \varphi) \times_{Y'} X'$ are isomorphisms, i.e., $(V, U', \varphi) \to F(G(V, U', \varphi))$ is an isomorphism. $\square$

08KV **Lemma 6.6.** *Let S be a base scheme. Let $X \to X'$ be a thickening of algebraic spaces over S and let $X \to Y$ be an affine morphism of algebraic spaces over S. Let $Y' = Y \amalg_X X'$ be the pushout (see Lemma 6.2). Let $V' \to Y'$ be a morphism of algebraic spaces over S. Set $V = Y \times_{Y'} V'$, $U' = X' \times_{Y'} V'$, and $U = X \times_{Y'} V'$. There is an equivalence of categories between*

(1) *quasi-coherent $\mathcal{O}_{V'}$-modules flat over Y', and*
(2) *the category of triples $(\mathcal{G}, \mathcal{F}', \varphi)$ where*
   (a) *$\mathcal{G}$ is a quasi-coherent $\mathcal{O}_V$-module flat over Y,*
   (b) *$\mathcal{F}'$ is a quasi-coherent $\mathcal{O}_{U'}$-module flat over X, and*
   (c) *$\varphi : (U \to V)^* \mathcal{G} \to (U \to U')^* \mathcal{F}'$ is an isomorphism of $\mathcal{O}_U$-modules.*

The equivalence maps $\mathcal{G}'$ to $((V \to V')^* \mathcal{G}', (U' \to V')^* \mathcal{G}', can)$. Suppose $\mathcal{G}'$ corresponds to the triple $(\mathcal{G}, \mathcal{F}', \varphi)$. Then

(a) *$\mathcal{G}'$ is a finite type $\mathcal{O}_{V'}$-module if and only if $\mathcal{G}$ and $\mathcal{F}'$ are finite type $\mathcal{O}_Y$ and $\mathcal{O}_{U'}$-modules.*

(b) *if $V' \to Y'$ is locally of finite presentation, then $\mathcal{G}'$ is an $\mathcal{O}_{V'}$-module of finite presentation if and only if $\mathcal{G}$ and $\mathcal{F}'$ are $\mathcal{O}_Y$ and $\mathcal{O}_{U'}$-modules of finite presentation.*

Proof. A quasi-inverse functor assigns to the triple $(\mathcal{G}, \mathcal{F}', \varphi)$ the fibre product

$$(V \to V')_* \mathcal{G} \times_{(U' \to V')_* \mathcal{F}} (U' \to V')_* \mathcal{F}'$$

where $\mathcal{F} = (U \to U')^* \mathcal{F}'$. This works, because on affines étale over V' and Y' we recover the equivalence of More on Algebra, Lemma 7.5. Details omitted.

Parts (a) and (b) reduce by étale localization (Properties of Spaces, Section 30) to the case where V' and Y' are affine in which case the result follows from More on Algebra, Lemmas 7.4 and 7.6. $\square$

07W3 **Lemma 6.7.** *In the situation of Lemma 6.5. If $V' = G(V, U', \varphi)$ for some triple (V, U', φ), then*

(1) *$V' \to Y'$ is locally of finite type if and only if $V \to Y$ and $U' \to X'$ are locally of finite type,*
(2) *$V' \to Y'$ is flat if and only if $V \to Y$ and $U' \to X'$ are flat,*
(3) *$V' \to Y'$ is flat and locally of finite presentation if and only if $V \to Y$ and $U' \to X'$ are flat and locally of finite presentation,*
(4) *$V' \to Y'$ is smooth if and only if $V \to Y$ and $U' \to X'$ are smooth,*
(5) *$V' \to Y'$ is étale if and only if $V \to Y$ and $U' \to X'$ are étale, and*
(6) *add more here as needed.*

If W' is flat over Y', then the adjunction mapping $G(F(W')) \to W'$ is an isomorphism. Hence F and G define mutually quasi-inverse functors between the category of spaces flat over Y' and the category of triples (V, U', φ) with $V \to Y$ and $U' \to X'$ flat.

Proof. Choose a diagram (6.5.1) as in the proof of Lemma 6.5.

Proof of (1) – (5). Let (V, U', φ) be an object of $(Spaces/Y) \times_{(Spaces/Y')} (Spaces/X')$. Construct a diagram (6.5.2) as in the proof of Lemma 6.5. Then the base change of $G(V, U', \varphi) \to Y'$ to Y_1' is $G_1(V_1, U_1', \varphi_1) \to Y_1'$. Hence (1) – (5) follow immediately from the corresponding statements of More on Morphisms, Lemma 14.6 for schemes.

Suppose that $W' \to Y'$ is flat. Choose a scheme W_1' and a surjective étale morphism $W_1' \to Y_1' \times_{Y'} W'$. Observe that $W_1' \to W'$ is surjective étale as a composition of surjective étale morphisms. We know that $G_1(F_1(W_1')) \to W_1'$ is an isomorphism by More on Morphisms, Lemma 14.6 applied to W_1' over Y_1' and the front of the diagram (with functors G_1 and F_1 as in the proof of Lemma 6.5). Then the construction of $G(F(W'))$ (as a pushout, i.e., as constructed in Lemma 6.2) shows that $G_1(F_1(W_1')) \to G(F(W))$ is surjective étale. Whereupon we conclude that $G(F(W)) \to W$ is étale, see for example Properties of Spaces, Lemma 16.3. But $G(F(W)) \to W$ is an isomorphism on underlying reduced algebraic spaces (by construction), hence it is an isomorphism. $\square$

7. Pushouts along closed immersions and integral morphisms

0GG2 This section is analogue of More on Morphisms, Section 67.

0EDP **Lemma 7.1.** *In More on Morphisms, Situation 67.1 let $Y \amalg_Z X$ be the pushout in the category of schemes (More on Morphisms, Proposition 67.3). Then $Y \amalg_Z X$ is also a pushout in the category of algebraic spaces over S.*

Proof. This is a consequence of Lemma 3.1, the proposition mentioned in the lemma and More on Morphisms, Lemmas 67.6 and 67.7. Conditions (1) and (2) of Lemma 3.1 follow immediately. To see (3) and (4) note that an étale morphism is locally quasi-finite and use that the equivalence of categories of More on Morphisms, Lemma 67.7 is constructed using the pushout construction of More on Morphisms, Lemmas 67.6. Minor details omitted. $\square$

8. Pushouts and derived categories

0DL6 In this section we discuss the behaviour of the derived category of modules under pushouts.

0DL7 **Lemma 8.1.** *Let S be a scheme. Consider a pushout*

$$
\begin{array}{ccc}
X & \xrightarrow{\ i\ } & X' \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\
Y & \xrightarrow{\ j\ } & Y'
\end{array}
$$

in the category of algebraic spaces over S as in Lemma 6.2. Assume i is a thickening. Then the essential image of the functor[3]

$$
D(\mathcal{O}_{Y'}) \longrightarrow D(\mathcal{O}_Y) \times_{D(\mathcal{O}_X)} D(\mathcal{O}_{X'})
$$

contains every triple (M, K', α) where $M \in D(\mathcal{O}_Y)$ and $K' \in D(\mathcal{O}_{X'})$ are pseudo-coherent.

Proof. Let (M, K', α) be an object of the target of the functor of the lemma. Here $\alpha : Lf^*M \to Li^*K'$ is an isomorphism which is adjoint to a map $\beta : M \to Rf_*Li^*K'$. Thus we obtain maps

$$
Rj_*M \xrightarrow{Rj_*\beta} Rj_*Rf_*Li^*K' = Rf'_*Ri_*Li^*K' \leftarrow Rf'_*K'
$$

where the arrow pointing left comes from $K' \to Ri_*Li^*K'$. Choose a distinguished triangle

$$
M' \to Rj_*M \oplus Rf'_*K' \to Rj_*Rf_*Li^*K' \to M'[1]
$$

in $D(\mathcal{O}_{Y'})$. The first arrow defines canonical maps $Lj^*M' \to M$ and $L(f')^*M' \to K'$ compatible with α. Thus it suffices to show that the maps $Lj^*M' \to M$ and $L(f')^*M' \to K$ are isomorphisms. This we may check étale locally on Y', hence we may assume Y' is étale.

Assume Y' affine and $M \in D(\mathcal{O}_Y)$ and $K' \in D(\mathcal{O}_{X'})$ are pseudo-coherent. Say our pushout corresponds to the fibre product

$$
\begin{array}{ccc}
B & \longleftarrow & B' \\
\uparrow & & \uparrow \\
A & \longleftarrow & A'
\end{array}
$$

---

[3]All functors given by derived pullback.

of rings where $B' \to B$ is surjective with locally nilpotent kernel I (and hence $A' \to A$ is surjective with locally nilpotent kernel I as well). The assumption on M and K' imply that M comes from a pseudo-coherent object of $D(A)$ and K' comes from a pseudo-coherent object of $D(B')$, see Derived Categories of Spaces, Lemmas 13.6, 4.2, and 13.2 and Derived Categories of Schemes, Lemma 3.5 and 10.2. Moreover, pushforward and derived pullback agree with the corresponding operations on derived categories of modules, see Derived Categories of Spaces, Remark 6.3 and Derived Categories of Schemes, Lemmas 3.7 and 3.8. This reduces us to the statement formulated in the next paragraph. (To be sure these references show the object M' lies $D_{QCoh}(\mathcal{O}_{Y'})$ as this is a triangulated subcategory of $D(\mathcal{O}_{Y'})$.)

Given a diagram of rings as above and a triple (M, K', α) where $M \in D(A)$, $K' \in D(B')$ are pseudo-coherent and $\alpha : M \otimes_A^{\mathbf{L}} B \to K' \otimes_{B'}^{\mathbf{L}} B$ is an isomorphism suppose we have distinguished triangle

$$M' \to M \oplus K' \to K' \otimes_{B'}^{\mathbf{L}} B \to M'[1]$$

in $D(A')$. Goal: show that the induced maps $M' \otimes_{A'}^{\mathbf{L}} A \to M$ and $M' \otimes_{A'}^{\mathbf{L}} B' \to K'$ are isomorphisms. To do this, choose a bounded above complex $E^\bullet$ of finite free A-modules representing M. Since (B', I) is a henselian pair (More on Algebra, Lemma 11.2) with $B = B'/I$ we may apply More on Algebra, Lemma 75.8 to see that there exists a bounded above complex $P^\bullet$ of free B'-modules such that α is represented by an isomorphism $E^\bullet \otimes_A B \cong P^\bullet \otimes_{B'} B$. Then we can consider the short exact sequence

$$0 \to L^\bullet \to E^\bullet \oplus P^\bullet \to P^\bullet \otimes_{B'} B \to 0$$

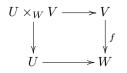of complexes of B'-modules. More on Algebra, Lemma 6.9 implies $L^\bullet$ is a bounded above complex of finite projective A'-modules (in fact it is rather easy to show directly that L^n is finite free in our case) and that we have $L^\bullet \otimes_{A'} A = E^\bullet$ and $L^\bullet \otimes_{A'} B' = P^\bullet$. The short exact sequence gives a distinguished triangle

$$L^\bullet \to M \oplus K' \to K' \otimes_{B'}^{\mathbf{L}} B \to (L^\bullet)[1]$$

in $D(A')$ (Derived Categories, Section 12) which is isomorphic to the given distinguished triangle by general properties of triangulated categories (Derived Categories, Section 4). In other words, $L^\bullet$ represents M' compatibly with the given maps. Thus the maps $M' \otimes_{A'}^{\mathbf{L}} A \to M$ and $M' \otimes_{A'}^{\mathbf{L}} B' \to K'$ are isomorphisms because we just saw that the corresponding thing is true for $L^\bullet$. $\square$

9. Constructing elementary distinguished squares

0DVH Elementary distinguished squares were defined in Derived Categories of Spaces, Section 9.

0DVI **Lemma 9.1.** *Let S be a scheme. Let $(U \subset W, f : V \to W)$ be an elementary distinguished square. Then*

$$
\begin{array}{ccc}
U \times_W V & \longrightarrow & V \\
\downarrow & & \downarrow f \\
U & \longrightarrow & W
\end{array}
$$

is a pushout in the category of algebraic spaces over S.

Proof. Observe that $U \amalg V \to W$ is a surjective étale morphism. The fibre product

$$(U \amalg V) \times_W (U \amalg V)$$

is the disjoint union of four pieces, namely $U = U \times_W U$, $U \times_W V$, $V \times_W U$, and $V \times_W V$. There is a surjective étale morphism

$$V \amalg (U \times_W V) \times_U (U \times_W V) \longrightarrow V \times_W V$$

because f induces an isomorphism over $W \setminus U$ (part of the definition of being an elementary distinguished square). Let B be an algebraic space over S and let $g : V \to B$ and $h : U \to B$ be morphisms over S which agree after restricting to $U \times_W V$. Then the description of $(U \amalg V) \times_W (U \amalg V)$ given above shows that $h \amalg g : U \amalg V \to B$ equalizes the two projections. Since B is a sheaf for the étale topology we obtain a unique factorization of $h \amalg g$ through W as desired. $\square$

0DVJ **Lemma 9.2.** *Let S be a scheme. Let V, U be algebraic spaces over S. Let $V' \subset V$ be an open subspace and let $f' : V' \to U$ be a separated étale morphism of algebraic spaces over S. Then there exists a pushout*

$$\begin{array}{ccc} V' & \longrightarrow & V \\ \downarrow & & \downarrow {\scriptstyle f} \\ U & \longrightarrow & W \end{array}$$

in the category of algebraic spaces over S and moreover $(U \subset W, f : V \to W)$ is an elementary distinguished square.

Proof. We are going to construct W as the quotient of an étale equivalence relation R on $U \amalg V$. Such a quotient is an algebraic space for example by Bootstrap, Theorem 10.1. Moreover, the proof of Lemma 9.1 tells us to take

$$R = U \amalg V' \amalg V' \amalg V \amalg (V' \times_U V' \setminus \Delta_{V'/U}(V'))$$

Since we assumed $V' \to U$ is separated, the image of $\Delta_{V'/U}$ is closed and hence the complement is an open subspace. The morphism $j : R \to (U \amalg V) \times_S (U \amalg V)$ is given by

$$u, \ v', \ v', \ v, \ (v'_1, v'_2) \mapsto (u, u), \ (f'(v'), v'), \ (v', f'(v')), \ (v, v), \ (v'_1, v'_2)$$

with obvious notation. It is immediately verified that this is a monomorphism, an equivalence relation, and that the induced morphisms $s, t : R \to U \amalg V$ are étale. Let $W = (U \amalg V)/R$ be the quotient algebraic space. We obtain a commutative diagram as in the statement of the lemma. To finish the proof it suffices to show that this diagram is an elementary distinguished square, since then Lemma 9.1 implies that it is a pushout. Thus we have to show that $U \to W$ is open and that f is étale and is an isomorphism over $W \setminus U$. This follows from the choice of R; we omit the details. $\square$

10. Formal glueing of quasi-coherent modules

0AEP This section is the analogue of More on Algebra, Section 89. In the case of morphisms of schemes, the result can be found in the paper by Joyet [Joy96]; this is a good place to start reading. For a discussion of applications to descent problems for stacks, see the paper by Moret-Bailly [MB96]. In the case of an affine morphism of schemes there is a statement in the appendix of the paper [FR70] but one needs

to add the hypothesis that the closed subscheme is cut out by a finitely generated ideal (as in the paper by Joyet) since otherwise the result does not hold. A generalization of this material to (higher) derived categories with potential applications to nonflat situations can be found in [Bha16, Section 5].

We start with a lemma on abelian sheaves supported on closed subsets.

0AEQ **Lemma 10.1.** *Let S be a scheme. Let $f : Y \to X$ be a morphism of algebraic spaces over S. Let $Z \subset X$ closed subspace such that $f^{-1}Z \to Z$ is integral and universally injective. Let $\overline{y}$ be a geometric point of Y and $\overline{x} = f(\overline{y})$. We have*

$$(Rf_*Q)_{\overline{x}} = Q_{\overline{y}}$$

in $D(Ab)$ for any object Q of $D(Y_{\text{étale}})$ supported on $|f^{-1}Z|$.

Proof. Consider the commutative diagram of algebraic spaces

$$
\begin{array}{ccc}
f^{-1}Z & \xrightarrow{\ i'\ } & Y \\
{\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\
Z & \xrightarrow{\ i\ } & X
\end{array}
$$

By Cohomology of Spaces, Lemma 9.4 we can write $Q = Ri'_*K'$ for some object K' of $D(f^{-1}Z_{\text{étale}})$. By Morphisms of Spaces, Lemma 53.7 we have $K' = (f')^{-1}K$ with $K = Rf'_*K'$. Then we have $Rf_*Q = Rf_*Ri'_*K' = Ri_*Rf'_*K' = Ri_*K$. Let $\overline{z}$ be the geometric point of Z corresponding to $\overline{x}$ and let $\overline{z}'$ be the geometric point of $f^{-1}Z$ corresponding to $\overline{y}$. We obtain the result of the lemma as follows

$$Q_{\overline{y}} = (Ri'_*K')_{\overline{y}} = K'_{\overline{z}'} = (f')^{-1}K_{\overline{z}'} = K_{\overline{z}} = Ri_*K_{\overline{x}} = Rf_*Q_{\overline{x}}$$

The middle equality holds because of the description of the stalk of a pullback given in Properties of Spaces, Lemma 19.9. $\qquad\square$

0AER **Lemma 10.2.** *Let S be a scheme. Let $f : Y \to X$ be a morphism of algebraic spaces over S. Let $Z \subset X$ closed subspace such that $f^{-1}Z \to Z$ is integral and universally injective. Let $\overline{y}$ be a geometric point of Y and $\overline{x} = f(\overline{y})$. Let $\mathcal{G}$ be an abelian sheaf on Y. Then the map of two term complexes*

$$(f_*\mathcal{G}_{\overline{x}} \to (f \circ j')_*(\mathcal{G}|_V)_{\overline{x}}) \longrightarrow (\mathcal{G}_{\overline{y}} \to j'_*(\mathcal{G}|_V)_{\overline{y}})$$

induces an isomorphism on kernels and an injection on cokernels. Here $V = Y \setminus f^{-1}Z$ and $j' : V \to Y$ is the inclusion.

Proof. Choose a distinguished triangle

$$\mathcal{G} \to Rj'_*\mathcal{G}|_V \to Q \to \mathcal{G}[1]$$

n $D(Y_{\text{étale}})$. The cohomology sheaves of Q are supported on $|f^{-1}Z|$. We apply Rf_* and we obtain

$$Rf_*\mathcal{G} \to Rf_*Rj'_*\mathcal{G}|_V \to Rf_*Q \to Rf_*\mathcal{G}[1]$$

Taking stalks at $\overline{x}$ we obtain an exact sequence

$$0 \to (R^{-1}f_*Q)_{\overline{x}} \to f_*\mathcal{G}_{\overline{x}} \to (f \circ j')_*(\mathcal{G}|_V)_{\overline{x}} \to (R^0f_*Q)_{\overline{x}}$$

We can compare this with the exact sequence

$$0 \to H^{-1}(Q)_{\overline{y}} \to \mathcal{G}_{\overline{y}} \to j'_*(\mathcal{G}|_V)_{\overline{y}} \to H^0(Q)_{\overline{y}}$$

Thus we see that the lemma follows because $Q_{\overline{y}} = Rf_*Q_{\overline{x}}$ by Lemma 10.1. $\qquad\square$

0AES **Lemma 10.3.** *Let S be a scheme. Let X be an algebraic space over S. Let $f : Y \to X$ be a quasi-compact and quasi-separated morphism. Let $\overline{x}$ be a geometric point of X and let $\mathrm{Spec}(\mathcal{O}_{X,\overline{x}}) \to X$ be the canonical morphism. For a quasi-coherent module $\mathcal{G}$ on Y we have*

$$f_* \mathcal{G}_{\overline{x}} = \Gamma(Y \times_X \mathrm{Spec}(\mathcal{O}_{X,\overline{x}}), p^* \mathcal{F})$$

where $p : Y \times_X \mathrm{Spec}(\mathcal{O}_{X,\overline{x}}) \to Y$ is the projection.

Proof. Observe that $f_* \mathcal{G}_{\overline{x}} = \Gamma(\mathrm{Spec}(\mathcal{O}_{X,\overline{x}}), h^* f_* \mathcal{G})$ where $h : \mathrm{Spec}(\mathcal{O}_{X,\overline{x}}) \to X$. Hence the result is true because h is flat so that Cohomology of Spaces, Lemma 11.2 applies. $\square$

0AET **Lemma 10.4.** *Let S be a scheme. Let X be an algebraic space over S. Let $i : Z \to X$ be a closed immersion of finite presentation. Let $Q \in D_{QCoh}(\mathcal{O}_X)$ be supported on $|Z|$. Let $\overline{x}$ be a geometric point of X and let $I_{\overline{x}} \subset \mathcal{O}_{X,\overline{x}}$ be the stalk of the ideal sheaf of Z. Then the cohomology modules $H^n(Q_{\overline{x}})$ are $I_{\overline{x}}$-power torsion (see More on Algebra, Definition 88.1).*

Proof. Choose an affine scheme U and an étale morphism $U \to X$ such that $\overline{x}$ lifts to a geometric point $\overline{u}$ of U. Then we can replace X by U, Z by $U \times_X Z$, Q by the restriction $Q|_U$, and $\overline{x}$ by $\overline{u}$. Thus we may assume that $X = \mathrm{Spec}(A)$ is affine. Let $I \subset A$ be the ideal defining Z. Since $i : Z \to X$ is of finite presentation, the ideal $I = (f_1, \ldots, f_r)$ is finitely generated. The object Q comes from a complex of A-modules $M^\bullet$, see Derived Categories of Spaces, Lemma 4.2 and Derived Categories of Schemes, Lemma 3.5. Since the cohomology sheaves of Q are supported on Z we see that the localization $M_f^\bullet$ is acyclic for each $f \in I$. Take $x \in H^p(M^\bullet)$. By the above we can find n_i such that $f_i^{n_i} x = 0$ in $H^p(M^\bullet)$ for each i. Then with $n = \sum n_i$ we see that I^n annihilates x. Thus $H^p(M^\bullet)$ is I-power torsion. Since the ring map $A \to \mathcal{O}_{X,\overline{x}}$ is flat and since $I_{\overline{x}} = I \mathcal{O}_{X,\overline{x}}$ we conclude. $\square$

0AEU **Lemma 10.5.** *Let S be a scheme. Let $f : Y \to X$ be a morphism of algebraic spaces over S. Let $Z \subset X$ be a closed subspace. Assume $f^{-1}Z \to Z$ is an isomorphism and that f is flat in every point of $f^{-1}Z$. For any Q in $D_{QCoh}(\mathcal{O}_Y)$ supported on $|f^{-1}Z|$ we have $Lf^* Rf_* Q = Q$.*

Proof. We show the canonical map $Lf^* Rf_* Q \to Q$ is an isomorphism by checking on stalks at $\overline{y}$. If $\overline{y}$ is not in $f^{-1}Z$, then both sides are zero and the result is true. Assume the image $\overline{x}$ of $\overline{y}$ is in Z. By Lemma 10.1 we have $Rf_* Q_{\overline{x}} = Q_{\overline{y}}$ and since f is flat at $\overline{y}$ we see that

$$(Lf^* Rf_* Q)_{\overline{y}} = (Rf_* Q)_{\overline{x}} \otimes_{\mathcal{O}_{X,\overline{x}}} \mathcal{O}_{Y,\overline{y}} = Q_{\overline{y}} \otimes_{\mathcal{O}_{X,\overline{x}}} \mathcal{O}_{Y,\overline{y}}$$

Thus we have to check that the canonical map

$$Q_{\overline{y}} \otimes_{\mathcal{O}_{X,\overline{x}}} \mathcal{O}_{Y,\overline{y}} \longrightarrow Q_{\overline{y}}$$

is an isomorphism in the derived category. Let $I_{\overline{x}} \subset \mathcal{O}_{X,\overline{x}}$ be the stalk of the ideal sheaf defining Z. Since $Z \to X$ is locally of finite presentation this ideal is finitely generated and the cohomology groups of $Q_{\overline{y}}$ are $I_{\overline{y}} = I_{\overline{x}} \mathcal{O}_{Y,\overline{y}}$-power torsion by Lemma 10.4 applied to Q on Y. It follows that they are also $I_{\overline{x}}$-power torsion. The ring map $\mathcal{O}_{X,\overline{x}} \to \mathcal{O}_{Y,\overline{y}}$ is flat and induces an isomorphism after dividing by $I_{\overline{x}}$ and $I_{\overline{y}}$ because we assumed that $f^{-1}Z \to Z$ is an isomorphism. Hence we see that the cohomology modules of $Q_{\overline{y}} \otimes_{\mathcal{O}_{X,\overline{x}}} \mathcal{O}_{Y,\overline{y}}$ are equal to the cohomology modules of $Q_{\overline{y}}$ by More on Algebra, Lemma 89.2 which finishes the proof. $\square$

0AEV **Situation 10.6.** Here S is a base scheme, $f : Y \to X$ is a quasi-compact and quasi-separated morphism of algebraic spaces over S, and $Z \to X$ is a closed immersion of finite presentation. We assume that $f^{-1}(Z) \to Z$ is an isomorphism and that f is flat in every point $x \in |f^{-1}Z|$. We set $U = X \setminus Z$ and $V = Y \setminus f^{-1}(Z)$. Picture

$$
\begin{array}{ccc}
V & \xrightarrow{\;j'\;} & Y \\
{\scriptstyle f|_V}\downarrow & & \downarrow{\scriptstyle f} \\
U & \xrightarrow{\;j\;} & X
\end{array}
$$

In Situation 10.6 we define $QCoh(Y \to X, Z)$ as the category of triples $(\mathcal{H}, \mathcal{G}, \varphi)$ where $\mathcal{H}$ is a quasi-coherent sheaf of $\mathcal{O}_U$-modules, $\mathcal{G}$ is a quasi-coherent sheaf of $\mathcal{O}_Y$-modules, and $\varphi : f^*\mathcal{H} \to \mathcal{G}|_V$ is an isomorphism of $\mathcal{O}_V$-modules. There is a canonical functor

0AEW (10.6.1) $$QCoh(\mathcal{O}_X) \longrightarrow QCoh(Y \to X, Z)$$

which maps $\mathcal{F}$ to the system $(\mathcal{F}|_U, f^*\mathcal{F}, can)$. By analogy with the proof given in the affine case, we construct a functor in the opposite direction. To an object $(\mathcal{H}, \mathcal{G}, \varphi)$ we assign the $\mathcal{O}_X$-module

0AEX (10.6.2) $$\mathrm{Ker}(j_*\mathcal{H} \oplus f_*\mathcal{G} \to (f \circ j')_*\mathcal{G}|_V)$$

Observe that j and j' are quasi-compact morphisms as $Z \to X$ is of finite presentation. Hence f_*, j_*, and $(f \circ j')_*$ transform quasi-coherent modules into quasi-coherent modules (Morphisms of Spaces, Lemma 11.2). Thus the module (10.6.2) is quasi-coherent.

0AEY **Lemma 10.7.** *In Situation 10.6. The functor (10.6.2) is right adjoint to the functor (10.6.1).*

Proof. This follows easily from the adjointness of f^* to f_* and j^* to j_*. Details omitted. $\square$

0AEZ **Lemma 10.8.** *In Situation 10.6. Let $X' \to X$ be a flat morphism of algebraic spaces. Set $Z' = X' \times_X Z$ and $Y' = X' \times_X Y$. The pullbacks $QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_{X'})$ and $QCoh(Y \to X, Z) \to QCoh(Y' \to X', Z')$ are compatible with the functors (10.6.2) and 10.6.1).*

Proof. This is true because pullback commutes with pullback and because flat pullback commutes with pushforward along quasi-compact and quasi-separated morphisms, see Cohomology of Spaces, Lemma 11.2. $\square$

0AF0 **Proposition 10.9.** *In Situation 10.6 the functor (10.6.1) is an equivalence with quasi-inverse given by (10.6.2).*

Proof. We first treat the special case where X and Y are affine schemes and where the morphism f is flat. Say $X = \mathrm{Spec}(R)$ and $Y = \mathrm{Spec}(S)$. Then f corresponds to a flat ring map $R \to S$. Moreover, $Z \subset X$ is cut out by a finitely generated ideal $I \subset R$. Choose generators $f_1, \ldots, f_t \in I$. By the description of quasi-coherent modules in terms of modules (Schemes, Section 7), we see that the category $QCoh(Y \to X, Z)$ is canonically equivalent to the category $\mathrm{Glue}(R \to S, f_1, \ldots, f_t)$ of More on Algebra, Remark 89.10 such that the functors (10.6.1) and (10.6.2) correspond to the functors Can and H^0. Hence the result follows from More on Algebra, Proposition 89.15 in this case.

We return to the general case. Let $\mathcal{F}$ be a quasi-coherent module on X. We will show that

$$\alpha : \mathcal{F} \longrightarrow \mathrm{Ker}\,(j_*\mathcal{F}|_U \oplus f_*f^*\mathcal{F} \to (f \circ j')_*f^*\mathcal{F}|_V)$$

is an isomorphism. Let $(\mathcal{H}, \mathcal{G}, \varphi)$ be an object of $QCoh(Y \to X, Z)$. We will show that

$$\beta : f^* \mathrm{Ker}\,(j_*\mathcal{H} \oplus f_*\mathcal{G} \to (f \circ j')_*\mathcal{G}|_V) \longrightarrow \mathcal{G}$$

and

$$\gamma : j^* \mathrm{Ker}\,(j_*\mathcal{H} \oplus f_*\mathcal{G} \to (f \circ j')_*\mathcal{G}|_V) \longrightarrow \mathcal{H}$$

are isomorphisms. To see these statements are true it suffices to look at stalks. Let $\overline{y}$ be a geometric point of Y mapping to the geometric point $\overline{x}$ of X.

Fix an object $(\mathcal{H}, \mathcal{G}, \varphi)$ of $QCoh(Y \to X, Z)$. By Lemma 10.2 and a diagram chase (omitted) the canonical map

$$\mathrm{Ker}(j_*\mathcal{H} \oplus f_*\mathcal{G} \to (f \circ j')_*\mathcal{G}|_V)_{\overline{x}} \longrightarrow \mathrm{Ker}(j_*\mathcal{H}_{\overline{x}} \oplus \mathcal{G}_{\overline{y}} \to j'_*\mathcal{G}_{\overline{y}})$$

is an isomorphism.

In particular, if $\overline{y}$ is a geometric point of V, then we see that $j'_*\mathcal{G}_{\overline{y}} = \mathcal{G}_{\overline{y}}$ and hence that this kernel is equal to $\mathcal{H}_{\overline{x}}$. This easily implies that $\alpha_{\overline{x}}$, $\beta_{\overline{x}}$, and $\beta_{\overline{y}}$ are isomorphisms in this case.

Next, assume that $\overline{y}$ is a point of $f^{-1}Z$. Let $I_{\overline{x}} \subset \mathcal{O}_{X,\overline{x}}$, resp. $I_{\overline{y}} \subset \mathcal{O}_{Y,\overline{y}}$ be the stalk of the ideal cutting out Z, resp. $f^{-1}Z$. Then $I_{\overline{x}}$ is a finitely generated ideal, $I_{\overline{y}} = I_{\overline{x}}\mathcal{O}_{Y,\overline{y}}$, and $\mathcal{O}_{X,\overline{x}} \to \mathcal{O}_{Y,\overline{y}}$ is a flat local homomorphism inducing an isomorphism $\mathcal{O}_{X,\overline{x}}/I_{\overline{x}} = \mathcal{O}_{Y,\overline{y}}/I_{\overline{y}}$. At this point we can bootstrap using the diagram of categories

Namely, as in the first paragraph of the proof we identify

$$\mathrm{Glue}(\mathcal{O}_{X,\overline{x}} \to \mathcal{O}_{Y,\overline{y}}, f_1, \ldots, f_t) = QCoh(\mathrm{Spec}(\mathcal{O}_{Y,\overline{y}}) \to \mathrm{Spec}(\mathcal{O}_{X,\overline{x}}), V(I_{\overline{x}}))$$

The right vertical functor is given by pullback, and it is clear that the inner square is commutative. Our computation of the stalk of the kernel in the third paragraph of the proof combined with Lemma 10.3 implies that the outer square (using the curved arrows) commutes. Thus we conclude using the case of a flat morphism of affine schemes which we handled in the first paragraph of the proof. $\square$

0AFJ   **Lemma 10.10.** *In Situation 10.6 the functor Rf_* induces an equivalence between $D_{QCoh,|f^{-1}Z|}(\mathcal{O}_Y)$ and $D_{QCoh,|Z|}(\mathcal{O}_X)$ with quasi-inverse given by Lf^*.*
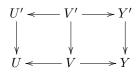
Proof. Since f is quasi-compact and quasi-separated we see that Rf_* defines a functor from $D_{QCoh,|f^{-1}Z|}(\mathcal{O}_Y)$ to $D_{QCoh,|Z|}(\mathcal{O}_X)$, see Derived Categories of Spaces, Lemma 6.1. By Derived Categories of Spaces, Lemma 5.5 we see that Lf^* maps $D_{QCoh,|Z|}(\mathcal{O}_X)$ into $D_{QCoh,|f^{-1}Z|}(\mathcal{O}_Y)$. In Lemma 10.5 we have seen that $Lf^*Rf_*Q = Q$ for Q in $D_{QCoh,|f^{-1}Z|}(\mathcal{O}_Y)$. By the dual of Derived Categories,

Lemma 7.2 to finish the proof it suffices to show that $Lf^*K = 0$ implies $K = 0$ for K in $D_{QCoh,|Z|}(\mathcal{O}_X)$. This follows from the fact that f is flat at all points of $f^{-1}Z$ and the fact that $f^{-1}Z \to Z$ is surjective. $\square$

0AF1 **Lemma 10.11.** *In Situation 10.6 there exists an fpqc covering $\{X_i \to X\}_{i \in I}$ refining the family $\{U \to X, Y \to X\}$.*

Proof. For the definition and general properties of fpqc coverings we refer to Topologies, Section 9. In particular, we can first choose an étale covering $\{X_i \to X\}$ with X_i affine and by base changing Y, Z, and U to each X_i we reduce to the case where X is affine. In this case U is quasi-compact and hence a finite union $U = U_1 \cup \ldots \cup U_n$ of affine opens. Then Z is quasi-compact hence also $f^{-1}Z$ is quasi-compact. Thus we can choose an affine scheme W and an étale morphism $h : W \to Y$ such that $h^{-1}f^{-1}Z \to f^{-1}Z$ is surjective. Say $W = \mathrm{Spec}(B)$ and $h^{-1}f^{-1}Z = V(J)$ where $J \subset B$ is an ideal of finite type. By Pro-étale Cohomology, Lemma 5.1 there exists a localization $B \to B'$ such that points of $\mathrm{Spec}(B')$ correspond exactly to points of $W = \mathrm{Spec}(B)$ specializing to $h^{-1}f^{-1}Z = V(J)$. It follows that the composition $\mathrm{Spec}(B') \to \mathrm{Spec}(B) = W \to Y \to X$ is flat as by assumption $f : Y \to X$ is flat at all the points of $f^{-1}Z$. Then $\{\mathrm{Spec}(B') \to X, U_1 \to X, \ldots, U_n \to X\}$ is an fpqc covering by Topologies, Lemma 9.2. $\square$

11. Formal glueing of algebraic spaces

0AF2 In Situation 10.6 we consider the category $Spaces(Y \to X, Z)$ of commutative diagrams of algebraic spaces over S of the form

$$
\begin{array}{ccccc}
U' & \longleftarrow & V' & \longrightarrow & Y' \\
\downarrow & & \downarrow & & \downarrow \\
U & \longleftarrow & V & \longrightarrow & Y
\end{array}
$$

where both squares are cartesian. There is a canonical functor

0AF3 (11.0.1) $\qquad\qquad Spaces/X \longrightarrow Spaces(Y \to X, Z)$

which maps $X' \to X$ to the morphisms $U \times_X X' \leftarrow V \times_X X' \to Y \times_X X'$.

0AF4 **Lemma 11.1.** *In Situation 10.6 the functor (11.0.1) restricts to an equivalence*

(1) *from the category of algebraic spaces affine over X to the full subcategory of $Spaces(Y \to X, Z)$ consisting of $(U' \leftarrow V' \to Y')$ with $U' \to U$, $V' \to V$, and $Y' \to Y$ affine,*

(2) *from the category of closed immersions $X' \to X$ to the full subcategory of $Spaces(Y \to X, Z)$ consisting of $(U' \leftarrow V' \to Y')$ with $U' \to U$, $V' \to V$, and $Y' \to Y$ closed immersions, and*

(3) *same statement as in (2) for finite morphisms.*

Proof. The category of algebraic spaces affine over X is equivalent to the category of quasi-coherent sheaves $\mathcal{A}$ of $\mathcal{O}_X$-algebras. The full subcategory of $Spaces(Y \to X, Z)$ consisting of $(U' \leftarrow V' \to Y')$ with $U' \to U$, $V' \to V$, and $Y' \to Y$ affine is equivalent to the category of algebra objects of $QCoh(Y \to X, Z)$. In both cases this follows from Morphisms of Spaces, Lemma 20.7 with quasi-inverse given by the relative spectrum construction (Morphisms of Spaces, Definition 20.8) which commutes with arbitrary base change. Thus part (1) of the lemma follows from Proposition 10.9.

Fully faithfulness in part (2) follows from part (1). For essential surjectivity, we reduce by part (1) to proving that $X' \to X$ is a closed immersion if and only if both $U \times_X X' \to U$ and $Y \times_X X' \to Y$ are closed immersions. By Lemma 10.11 $\{U \to X, Y \to X\}$ can be refined by an fpqc covering. Hence the result follows from Descent on Spaces, Lemma 11.17.

For (3) use the argument proving (2) and Descent on Spaces, Lemma 11.23. $\qquad\square$

0AF5 **Lemma 11.2.** *In Situation 10.6 the functor (11.0.1) reflects isomorphisms.*

Proof. By a formal argument with base change, this reduces to the following question: A morphism $a : X' \to X$ of algebraic spaces such that $U \times_X X' \to U$ and $Y \times_X X' \to Y$ are isomorphisms, is an isomorphism. The family $\{U \to X, Y \to X\}$ can be refined by an fpqc covering by Lemma 10.11. Hence the result follows from Descent on Spaces, Lemma 11.15. $\qquad\square$

0AF6 **Lemma 11.3.** *In Situation 10.6 the functor (11.0.1) is fully faithful on algebraic spaces separated over X. More precisely, it induces a bijection*

$$\mathrm{Mor}_X(X'_1, X'_2) \longrightarrow \mathrm{Mor}_{Spaces(Y \to X, Z)}(F(X'_1), F(X'_2))$$

whenever $X'_2 \to X$ is separated.

Proof. Since $X'_2 \to X$ is separated, the graph $i : X'_1 \to X'_1 \times_X X'_2$ of a morphism $X'_1 \to X'_2$ over X is a closed immersion, see Morphisms of Spaces, Lemma 4.6. Moreover a closed immersion $i : T \to X'_1 \times_X X'_2$ is the graph of a morphism if and only if $\mathrm{pr}_1 \circ i$ is an isomorphism. The same is true for

(1) the graph of a morphism $U \times_X X'_1 \to U \times_X X'_2$ over U,
(2) the graph of a morphism $V \times_X X'_1 \to V \times_X X'_2$ over V, and
(3) the graph of a morphism $Y \times_X X'_1 \to Y \times_X X'_2$ over Y.

Moreover, if morphisms as in (1), (2), (3) fit together to form a morphism in the category $Spaces(Y \to X, Z)$, then these graphs fit together to give an object of $Spaces(Y \times_X (X'_1 \times_X X'_2) \to X'_1 \times_X X'_2, Z \times_X (X'_1 \times_X X'_2))$ whose triple of morphisms are closed immersions. The proof is finished by applying Lemmas 11.1 and 11.2. $\qquad\square$

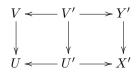## 12. Glueing and the Beauville-Laszlo theorem

0F9M Let $R \to R'$ be a ring homomorphism and let $f \in R$ be an element such that

$$0 \to R \to R_f \oplus R' \to R'_f \to 0$$

is a short exact sequence. This implies that $R/f^n R \cong R'/f^n R'$ for all n and $(R \to R', f)$ is a glueing pair in the sense of More on Algebra, Section 90. Set $X = \mathrm{Spec}(R)$, $U = \mathrm{Spec}(R_f)$, $X' = \mathrm{Spec}(R')$ and $U' = \mathrm{Spec}(R'_f)$. Picture

$$\begin{array}{ccc} U' & \longrightarrow & X' \\ \downarrow & & \downarrow \\ U & \longrightarrow & X \end{array}$$

In this situation we can consider the category $Spaces(U \leftarrow U' \rightarrow X')$ whose objects are commutative diagrams

$$
\begin{array}{ccccc}
V & \longleftarrow & V' & \longrightarrow & Y' \\
\downarrow & & \downarrow & & \downarrow \\
U & \longleftarrow & U' & \longrightarrow & X'
\end{array}
$$

of algebraic spaces with both squares cartesian and whose morphism are defined in the obvious manner. An object of this category will be denoted (V, V', Y') with arrows surpressed from the notation. There is a functor

0F9N $\qquad$ (12.0.1) $\qquad\qquad Spaces/X \longrightarrow Spaces(U \leftarrow U' \rightarrow X')$

given by base change: $Y \mapsto (U \times_X Y, U' \times_X Y, X' \times_X Y)$.

We have seen in More on Algebra, Section 90 that not every R-module M can be recovered from its gluing data. Similarly, the functor (12.0.1) won't be fully faithful on the category of all spaces over X. In order to single out a suitable subcategory of algebraic spaces over X we need a lemma.

0F9P $\quad$ **Lemma 12.1.** *Let $(R \rightarrow R', f)$ be a glueing pair, see above. Let Y be an algebraic space over X. The following are equivalent*

(1) *there exists an étale covering $\{Y_i \rightarrow Y\}_{i \in I}$ with Y_i affine and $\Gamma(Y_i, \mathcal{O}_{Y_i})$ glueable as an R-module,*

(2) *for every étale morphism $W \rightarrow Y$ with W affine $\Gamma(W, \mathcal{O}_W)$ is a glueable R-module.*

Proof. It is immediate that (2) implies (1). Assume $\{Y_i \rightarrow Y\}$ is as in (1) and let $W \rightarrow Y$ be as in (2). Then $\{Y_i \times_Y W \rightarrow W\}_{i \in I}$ is an étale covering, which we may refine by an étale covering $\{W_j \rightarrow W\}_{j=1,\ldots,m}$ with W_j affine (Topologies, Lemma 4.4). Thus to finish the proof it suffices to show the following three algebraic statements:

(1) if $R \rightarrow A \rightarrow B$ are ring maps with $A \rightarrow B$ étale and A glueable as an R-module, then B is glueable as an R-module,

(2) finite products of glueable R-modules are glueable,

(3) if $R \rightarrow A \rightarrow B$ are ring maps with $A \rightarrow B$ faithfully étale and B glueable as an R-module, then A is glueable as an R-module.

Namely, the first of these will imply that $\Gamma(W_j, \mathcal{O}_{W_j})$ is a glueable R-module, the second will imply that $\prod \Gamma(W_j, \mathcal{O}_{W_j})$ is a glueable R-module, and the third will imply that $\Gamma(W, \mathcal{O}_W)$ is a glueable R-module.

Consider an étale R-algebra homomorphism $A \rightarrow B$. Set $A' = A \otimes_R R'$ and $B' = B \otimes_R R' = A' \otimes_A B$. Statements (1) and (3) then follow from the following facts: (a) A, resp. B is glueable if and only if the sequence

$$
0 \rightarrow A \rightarrow A_f \oplus A' \rightarrow A'_f \rightarrow 0, \quad \text{resp.} \quad 0 \rightarrow B \rightarrow B_f \oplus B' \rightarrow B'_f \rightarrow 0,
$$

is exact, (b) the second sequence is equal to the functor $- \otimes_A B$ applied to the first and (c) (faithful) flatness of $A \rightarrow B$. We omit the proof of (2). $\qquad\square$

Let $(R \rightarrow R', f)$ be a glueing pair, see above. We will say an algebraic space Y over $X = \mathrm{Spec}(R)$ is *glueable for $(R \rightarrow R', f)$* if the equivalent conditions of Lemma 12.1 are satisfied.

0F9Q **Lemma 12.2.** *Let $(R \to R', f)$ be a glueing pair, see above. The functor (12.0.1) restricts to an equivalence between the category of affine Y/X which are glueable for $(R \to R', f)$ and the full subcategory of objects (V, V', Y') of $Spaces(U \leftarrow U' \to X')$ with V, V', Y' affine.*

Proof. Let (V, V', Y') be an object of $Spaces(U \leftarrow U' \to X')$ with V, V', Y' affine. Write $V = \mathrm{Spec}(A_1)$ and $Y' = \mathrm{Spec}(A')$. By our definition of the category $Spaces(U \leftarrow U' \to X')$ we find that V' is the spectrum of $A_1 \otimes_{R_f} R'_f = A_1 \otimes_R R'$ and the spectrum of A'_f. Hence we get an isomorphism $\varphi : A'_f \to A_1 \otimes_R R'$ of R'_f-algebras. By More on Algebra, Theorem 90.17 there exists a unique glueable R-module A and isomorphisms $A_f \to A_1$ and $A \otimes_R R' \to A'$ of modules compatible with φ. Since the sequence

$$0 \to A \to A_1 \oplus A' \to A'_f \to 0$$

is short exact, the multiplications on A_1 and A' define a unique R-algebra structure on A such that the maps $A \to A_1$ and $A \to A'$ are ring homomorphisms. We omit the verification that this construction defines a quasi-inverse to the functor (12.0.1) restricted to the subcategories mentioned in the statement of the lemma. $\square$

0F9R **Lemma 12.3.** *Let P be one of the following properties of morphisms: "finite", "closed immersion", "flat", "finite type", "flat and finite presentation", "étale". Under the equivalence of Lemma 12.2 the morphisms having P correspond to morphisms of triples whose components have P.*

Proof. Let P' be one of the following properties of homomorphisms of rings: "finite", "surjective", "flat", "finite type", "flat and of finite presentation", "étale". Translated into algebra, the statement means the following: If $A \to B$ is an R-algebra homomorphism and A and B are glueable for $(R \to R', f)$, then $A_f \to B_f$ and $A \otimes_R R' \to B \otimes_R R'$ have P' if and only if $A \to B$ has P'.

By More on Algebra, Lemmas 90.5 and 90.19 the algebraic statement is true for P' equal to "finite" or "flat".

If $A_f \to B_f$ and $A \otimes_R R' \to B \otimes_R R'$ are surjective, then $N = B/A$ is an R-module with $N_f = 0$ and $N \otimes_R R' = 0$ and hence vanishes by More on Algebra, Lemma 90.3. Thus $A \to B$ is surjective.

If $A_f \to B_f$ and $A \otimes_R R' \to B \otimes_R R'$ are finite type, then we can choose an A-algebra homomorphism $A[x_1, \ldots, x_n] \to B$ such that $A_f[x_1, \ldots, x_n] \to B_f$ and $(A \otimes_R R')[x_1, \ldots, x_n] \to B \otimes_R R'$ are surjective (small detail omitted). We conclude that $A[x_1, \ldots, x_n] \to B$ is surjective by the previous result. Thus $A \to B$ is of finite type.

If $A_f \to B_f$ and $A \otimes_R R' \to B \otimes_R R'$ are flat and of finite presentation, then we know that $A \to B$ is flat and of finite type by what we have already shown. Choose a surjection $A[x_1, \ldots, x_n] \to B$ and denote I the kernel. By flatness of B over A we see that I_f is the kernel of $A_f[x_1, \ldots, x_n] \to B_f$ and $I \otimes_R R'$ is the kernel of $A \otimes_R R'[x_1, \ldots, x_n] \to B \otimes_R R'$. Thus I_f is a finite $A_f[x_1, \ldots, x_n]$-module and $I \otimes_R R'$ is a finite $(A \otimes_R R')[x_1, \ldots, x_n]$-module. By More on Algebra, Lemma 90.5 applied to I viewed as a module over $A[x_1, \ldots, x_n]$ we conclude that I is a finitely generated ideal and we conclude $A \to B$ is flat and of finite presentation.

If $A_f \to B_f$ and $A \otimes_R R' \to B \otimes_R R'$ are étale, then we know that $A \to B$ is flat and of finite presentation by what we have already shown. Since the fibres of $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ are isomorphic to fibres of $\mathrm{Spec}(B_f) \to \mathrm{Spec}(A_f)$ or $\mathrm{Spec}(B/fB) \to \mathrm{Spec}(A/fA)$, we conclude that $A \to B$ is unramified, see Morphisms, Lemmas 35.11 and 35.12. We conclude that $A \to B$ is étale by Morphisms, Lemma 36.16 for example. $\square$

0F9S **Lemma 12.4.** *Let $(R \to R', f)$ be a glueing pair, see above. The functor (12.0.1) is faithful on the full subcategory of algebraic spaces Y/X glueable for $(R \to R', f)$.*

Proof. Let $f, g : Y \to Z$ be two morphisms of algebraic spaces over X with Y and Z glueable for $(R \to R', f)$ such that f and g are mapped to the same morphism in the category $Spaces(U \leftarrow U' \to X')$. We have to show the equalizer $E \to Y$ of f and g is an isomorphism. Working étale locally on Y we may assume Y is an affine scheme. Then E is a scheme and the morphism $E \to Y$ is a monomorphism and locally quasi-finite, see Morphisms of Spaces, Lemma 4.1. Moreover, the base change of $E \to Y$ to U and to X' is an isomorphism. As Y is the disjoint union of the affine open $V = U \times_X Y$ and the affine closed $V(f) \times_X Y$, we conclude E is the disjoint union of their isomorphic inverse images. It follows in particular that E is quasi-compact. By Zariski's main theorem (More on Morphisms, Lemma 43.3) we conclude that E is quasi-affine. Set $B = \Gamma(E, \mathcal{O}_E)$ and $A = \Gamma(Y, \mathcal{O}_Y)$ so that we have an R-algebra homomorphism $A \to B$. Since $E \to Y$ becomes an isomorphism after base change to U and X' we obtain ring maps $B \to A_f$ and $B \to A \otimes_R R'$ agreeing as maps into $A \otimes_R R'_f$. Since A is glueable for $(R \to R', f)$ we get a ring map $B \to A$ which is left inverse to the map $A \to B$. The corresponding morphism $Y = \mathrm{Spec}(A) \to \mathrm{Spec}(B)$ maps into the open subscheme $E \subset \mathrm{Spec}(B)$ pointwise because this is true after base change to U and X'. Hence we get a morphism $Y \to E$ over Y. Since $E \to Y$ is a monomorhism we conclude $Y \to E$ is an isomorphism as desired. $\square$

0F9T **Lemma 12.5.** *Let $(R \to R', f)$ be a glueing pair, see above. The functor (12.0.1) is fully faithful on the full subcategory of algebraic spaces Y/X which are (a) glueable for $(R \to R', f)$ and (b) have affine diagonal $Y \to Y \times_X Y$.*

Proof. Let Y, Z be two algebraic spaces over X which are both glueable for $(R \to R', f)$ and assume the diagonal of Z is affine. Let $a : U \times_X Y \to U \times_X Z$ over U and $b : X' \times_X Y \to X' \times_X Z$ over X' be two morphisms of algebraic spaces which induce the same morphism $c : U' \times_X Y \to U' \times_X Z$ over U'. We want to construct a morphism $f : Y \to Z$ over X which produces the morphisms a, b on base change to U, X'. By the faithfulness of Lemma 12.4, it suffices to construct the morphism f étale locally on Y (details omitted). Thus we may and do assume Y is affine.

Let $y \in |Y|$ be a point. If y maps into the open $U \subset X$, then $U \times_X Y$ is an open of Y on which the morphism f is defined (we can just take a). Thus we may assume y maps into the closed subset $V(f)$ of X. Since $R/fR = R'/fR'$ there is a unique point $y' \in |X' \times_X Y|$ mapping to y. Denote $z' = b(y') \in |X' \times_X Z|$ and $z \in |Z|$ the images of y'. Choose an étale neighbourhood $(W, w) \to (Z, z)$ with W affine. Observe that

$$(U \times_X W) \times_{U \times_X Z, a} (U \times_X Y), \quad (U' \times_X W) \times_{U' \times_X Z, c} (U' \times_X Y),$$

and

$$(X' \times_X W) \times_{X' \times_X Z, b} (X' \times_X Y)$$

form an object of $Spaces(U \leftarrow U' \rightarrow X')$ with affine parts (this is where we use that Z has affine diagonal). Hence by Lemma 12.2 there exists a unique affine scheme V glueable for $(R \rightarrow R', f)$ such that
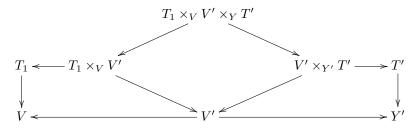
$$(U \times_X V, U' \times_X V, X' \times_X V)$$

is the triple displayed above. By fully faithfulness for the affine case (Lemma 12.2) we get a unique morphisms $V \rightarrow W$ and $V \rightarrow Y$ agreeing with the first and second projection morphisms over U and X' in the construction above. By Lemma 12.3 the morphism $V \rightarrow Y$ is étale. To finish the proof, it suffices to show that there is a point $v \in |V|$ mapping to y (because then f is defined on an étale neighbourhood of y, namely V). There is a unique point $w' \in |X' \times_X W|$ mapping to w. By uniqueness w' is mapped to z' under the map $|X' \times_X W| \rightarrow |X' \times_X Z|$. Then we consider the cartesian diagram

$$
\begin{array}{ccc}
X' \times_X V & \longrightarrow & X' \times_X W \\
\downarrow & & \downarrow \\
X' \times_X Y & \longrightarrow & X' \times_X Z
\end{array}
$$

to see that there is a point $v' \in |X' \times_X V|$ mapping to y' and w', see Properties of Spaces, Lemma 4.3. Of course the image v of v' in $|V|$ maps to y and the proof is complete. $\square$

0F9U **Lemma 12.6.** *Let $(R \rightarrow R', f)$ be a glueing pair, see above. Any object (V, V', Y') of $Spaces(U \leftarrow U' \rightarrow X')$ with V, V', Y' quasi-affine is isomorphic to the image under the functor (12.0.1) of a separated algebraic space Y over X.*

Proof. Choose n', $T' \rightarrow Y'$ and n_1, $T_1 \rightarrow V$ as in Properties, Lemma 18.6. Picture

$$
\begin{array}{ccccc}
& & T_1 \times_V V' \times_Y T' & & \\
& \swarrow & & \searrow & \\
T_1 \longleftarrow T_1 \times_V V' & & & V' \times_{Y'} T' \longrightarrow T' \\
\downarrow & & & & \downarrow \\
V \longleftarrow & & V' & & \longrightarrow Y'
\end{array}
$$

Observe that $T_1 \times_V V'$ and $V' \times_{Y'} T'$ are affine (namely the morphisms $V' \rightarrow V$ and $V' \rightarrow Y'$ are affine as base changes of the affine morphisms $U' \rightarrow U$ and $U' \rightarrow X'$). By construction we see that

$$\mathbf{A}^{n'}_{T_1 \times_V V'} \cong T_1 \times_V V' \times_{Y'} T' \cong \mathbf{A}^{n_1}_{V' \times_{Y'} T'}$$

In other words, the affine schemes $\mathbf{A}^{n'}_{T_1}$ and $\mathbf{A}^{n_1}_{T'}$ are part of a triple making an affine object of $Spaces(U \leftarrow U' \rightarrow X')$. By Lemma 12.2 there exists a morphism of affine schemes $T \rightarrow X$ and isomorphisms $U \times_X T \cong \mathbf{A}^{n'}_{T_1}$ and $X' \times_X T \cong \mathbf{A}^{n_1}_{T'}$ compatible with the isomorphisms displayed above. These isomorphisms produce morphisms

$$U \times_X T \longrightarrow V \quad \text{and} \quad X' \times_X T \longrightarrow Y'$$

satisfying the property of Properties, Lemma 18.6 with $n = n' + n_1$ and moreover define a morphism from the triple $(U \times_X T, U' \times_X T, X' \times_X T)$ to our triple (V, V', Y') in the category $Spaces(U \leftarrow U' \rightarrow X')$.

By Lemma 12.2 there is an affine scheme W whose image in $Spaces(U \leftarrow U' \to X')$ is isomorphic to the triple

$$((U \times_X T) \times_V (U \times_X T), (U' \times_X T) \times_{V'} (U' \times_X T), (X' \times_X T) \times_{Y'} (X' \times_X T))$$

By fully faithfulness of this construction, we obtain two maps $p_0, p_1 : W \to T$ whose base changes to U, U', X' are the projection morphisms. By Lemma 12.3 the morphisms p_0, p_1 are flat and of finite presentation and the morphism $(p_0, p_1) : W \to T \times_X T$ is a closed immersion. In fact, $W \to T \times_X T$ is an equivalence relation: by the lemmas used above we may check symmetry, reflexivity, and transitivity after base change to U and X', where these are obvious (details omitted). Thus the quotient sheaf

$$Y = T/W$$

is an algebraic space for example by Bootstrap, Theorem 10.1. Since it is clear that Y/X is sent to the triple (V, V', Y'). The base change of the diagonal $\Delta : Y \to Y \times_X Y$ by the quasi-compact surjective flat morphism $T \times_X T \to Y \times_X Y$ is the closed immersion $W \to T \times_X T$. Thus Δ is a closed immersion by Descent on Spaces, Lemma 11.17. Thus the algebraic space Y is separated and the proof is complete. $\square$

13. Coequalizers and glueing

0AGF   Let X be a Noetherian algebraic space and $Z \to X$ a closed subspace. Let $X' \to X$ be the blowing up in Z. In this section we show that X can be recovered from X', Z_n and glueing data where Z_n is the nth infinitesimal neighbourhood of Z in X.

0AGG   **Lemma 13.1.** *Let S be a scheme. Let*

$$g : Y \longrightarrow X$$

be a morphism of algebraic spaces over S. Assume X is locally Noetherian, and g is proper. Let $R = Y \times_X Y$ with projection morphisms $t, s : R \to Y$. There exists a coequalizer X' of $s, t : R \to Y$ in the category of algebraic spaces over S. Moreover

1. *The morphism $X' \to X$ is finite.*
2. *The morphism $Y \to X'$ is proper.*
3. *The morphism $Y \to X'$ is surjective.*
4. *The morphism $X' \to X$ is universally injective.*
5. *If g is surjective, the morphism $X' \to X$ is a universal homeomorphism.*

Proof. Denote $h : R \to X$ denote the composition of either s or t with g. Then h is proper by Morphisms of Spaces, Lemmas 40.3 and 40.4. The sheaves

$$g_* \mathcal{O}_Y \quad \text{and} \quad h_* \mathcal{O}_R$$

are coherent $\mathcal{O}_X$-algebras by Cohomology of Spaces, Lemma 20.2. The X-morphisms s, t induce $\mathcal{O}_X$-algebra maps $s^\sharp, t^\sharp$ from the first to the second. Set

$$\mathcal{A} = \text{Equalizer} \left(s^\sharp, t^\sharp : g_* \mathcal{O}_Y \longrightarrow h_* \mathcal{O}_R \right)$$

Then $\mathcal{A}$ is a coherent $\mathcal{O}_X$-algebra and we can define

$$X' = \underline{\text{Spec}}_X(\mathcal{A})$$

as in Morphisms of Spaces, Definition 20.8. By Morphisms of Spaces, Remark 20.9 and functoriality of the $\underline{\text{Spec}}$ construction there is a factorization

$$Y \longrightarrow X' \longrightarrow X$$

and the morphism $g' : Y \to X'$ equalizes s and t.

Before we show that X' is the coequalizer of s and t, we show that $Y \to X'$ and $X' \to X$ have the desired properties. Since $\mathcal{A}$ is a coherent $\mathcal{O}_X$-module it is clear that $X' \to X$ is a finite morphism of algebraic spaces. This proves (1). The morphism $Y \to X'$ is proper by Morphisms of Spaces, Lemma 40.6. This proves (2). Denote $Y \to Y' \to X$ with $Y' = \underline{\operatorname{Spec}}_X(g_*\mathcal{O}_Y)$ the Stein factorization of g, see More on Morphisms of Spaces, Theorem 36.4. Of course we obtain morphisms $Y \to Y' \to X' \to X$ fitting with the morphisms studied above. Since $\mathcal{O}_{X'} \subset g_*\mathcal{O}_Y$ is a finite extension we see that $Y' \to X'$ is finite and surjective. Some details omitted; hint: use Algebra, Lemma 36.17 and reduce to the affine case by étale localization. Since $Y \to Y'$ is surjective (with geometrically connected fibres) we conclude that $Y \to X'$ is surjective. This proves (3). To show that $X' \to X$ is universally injective, we have to show that $X' \to X' \times_X X'$ is surjective, see Morphisms of Spaces, Definition 19.3 and Lemma 19.2. Since $Y \to X'$ is surjective (see above) and since base changes and compositions of surjective morphisms are surjective by Morphisms of Spaces, Lemmas 5.5 and 5.4 we see that $Y \times_X Y \to X' \times_X X'$ is surjective. However, since $Y \to X'$ equalizes s and t, we see that $Y \times_X Y \to X' \times_X X'$ factors through $X' \to X' \times_X X'$ and we conclude this latter map is surjective. This proves (4). Finally, if g is surjective, then since g factors through $X' \to X$ we see that $X' \to X$ is surjective. Since a surjective, universally injective, finite morphism is a universal homeomorphism (because it is universally bijective and universally closed), this proves (5).

In the rest of the proof we show that $Y \to X'$ is the coequalizer of s and t in the category of algebraic spaces over S. Observe that X' is locally Noetherian (Morphisms of Spaces, Lemma 23.5). Moreover, observe that $Y \times_{X'} Y \to Y \times_X Y$ is an isomorphism as $Y \to X'$ equalizes s and t (this is a categorical statement). Hence in order to prove the statement that $Y \to X'$ is the coequalizer of s and t, we may and do assume $X = X'$. In other words, $\mathcal{O}_X$ is the equalizer of the maps $s^\sharp, t^\sharp : g_*\mathcal{O}_Y \to h_*\mathcal{O}_R$.

Let $X_1 \to X$ be a flat morphism of algebraic spaces over S with X_1 locally Noetherian. Denote $g_1 : Y_1 \to X_1$, $h_1 : R_1 \to X_1$ and $s_1, t_1 : R_1 \to Y_1$ the base changes of g, h, s, t to X_1. Of course g_1 is proper and $R_1 = Y_1 \times_{X_1} Y_1$. Since we have flat base change for pushforward of quasi-coherent modules, Cohomology of Spaces, Lemma 11.2, we see that $\mathcal{O}_{X_1}$ is the equalizer of the maps $s_1^\sharp, t_1^\sharp : g_{1,*}\mathcal{O}_{Y_1} \to h_{1,*}\mathcal{O}_{R_1}$. Hence all the assumptions we have are preserved by this base change.

At this point we are going to check conditions (1) and (2) of Lemma 3.3. Condition (1) follows from Lemma 5.1 and the fact that g is proper and surjective (because $X = X'$). To check condition (2), by the remarks on base change above, we reduce to the statement discussed and proved in the next paragraph.

Assume $S = \operatorname{Spec}(A)$ is an affine scheme, $X = X'$ is an affine scheme, and Z is an affine scheme over S. We have to show that

$$\operatorname{Mor}_S(X, Z) \longrightarrow \operatorname{Equalizer}(s, t : \operatorname{Mor}_S(Y, Z) \to \operatorname{Mor}_S(R, Z))$$

is bijective. However, this is clear from the fact that $X = X'$ which implies $\mathcal{O}_X$ is the equalizer of the maps $s^\sharp, t^\sharp : g_*\mathcal{O}_Y \to h_*\mathcal{O}_R$ which in turn implies

$$\Gamma(X, \mathcal{O}_X) = \operatorname{Equalizer}\left(s^\sharp, t^\sharp : \Gamma(Y, \mathcal{O}_Y) \to \Gamma(R, \mathcal{O}_R)\right)$$

Namely, we have

$$\mathrm{Mor}_S(X, Z) = \mathrm{Hom}_A(\Gamma(Z, \mathcal{O}_Z), \Gamma(X, \mathcal{O}_X))$$

and similarly for Y and R, see Properties of Spaces, Lemma 33.1. $\square$

We will work in the following situation.

0AGH **Situation 13.2.** Let S be a scheme. Let X be a locally Noetherian algebraic space over S. Let $Z \to X$ be a closed immersion and let $U \subset X$ be the complementary open subspace. Finally, let $f : X' \to X$ be a proper morphism of algebraic spaces such that $f^{-1}(U) \to U$ is an isomorphism.

0AGI **Lemma 13.3.** *In Situation 13.2 let $Y = X' \amalg Z$ and $R = Y \times_X Y$ with projections $t, s : R \to Y$. There exists a coequalizer X_1 of $s, t : R \to Y$ in the category of algebraic spaces over S. The morphism $X_1 \to X$ is a finite universal homeomorphism, an isomorphism over U, and $Z \to X$ lifts to X_1.*

Proof. Existence of X_1 and the fact that $X_1 \to X$ is a finite universal homeomorphism is a special case of Lemma 13.1. The formation of X_1 commutes with étale localization on X (see proof of Lemma 13.1). Thus the morphism $X_1 \to X$ is an isomorphism over U. It is immediate from the construction that $Z \to X$ lifts to X_1. $\square$

In Situation 13.2 for $n \geq 1$ let $Z_n \subset X$ be the nth order infinitesimal neighbourhood of Z in X, i.e., the closed subscheme defined by the nth power of the sheaf of ideals cutting out Z. Consider $Y_n = X' \amalg Z_n$ and $R_n = Y_n \times_X Y_n$ and the coequalizer

$$R_n \underset{\longrightarrow}{\overset{\longrightarrow}{\longrightarrow}} Y_n \longrightarrow X_n \longrightarrow X$$

as in Lemma 13.3. The maps $Y_n \to Y_{n+1}$ and $R_n \to R_{n+1}$ induce morphisms

0AGJ (13.3.1) $$X_1 \to X_2 \to X_3 \to \ldots \to X$$

Each of these morphisms is a universal homeomorphism as the morphisms $X_n \to X$ are universal homeomorphisms.

0AGK **Lemma 13.4.** *In Situation 13.2 assume X quasi-compact. In (13.3.1) for all n large enough, there exists an m such that $X_n \to X_{n+m}$ factors through a closed immersion $X \to X_{n+m}$.*

Proof. Let's look a bit more closely at the construction of X_n and how it changes as we increase n. We have $X_n = \underline{\mathrm{Spec}}(\mathcal{A}_n)$ where $\mathcal{A}_n$ is the equalizer of $s_n^\sharp$ and $t_n^\sharp$ going from $g_{n,*}\mathcal{O}_{Y_n}$ to $h_{n,*}\mathcal{O}_{R_n}$. Here $g_n : Y_n = X' \amalg Z_n \to X$ and $h_n : R_n = Y_n \times_X Y_n \to X$ are the given morphisms. Let $\mathcal{I} \subset \mathcal{O}_X$ be the coherent sheaf of ideals corresponding to Z. Then

$$g_{n,*}\mathcal{O}_{Y_n} = f_*\mathcal{O}_{X'} \times \mathcal{O}_X/\mathcal{I}^n$$

Similarly, we have a decomposition

$$R_n = X' \times_X X' \amalg X' \times_X Z_n \amalg Z_n \times_X X' \amalg Z_n \times_X Z_n$$

As $Z_n \to X$ is a monomorphism, we see that $X' \times_X Z_n = Z_n \times_X X'$ and that this identification is compatible with the two morphisms to X, with the two morphisms to X', and with the two morphisms to Z_n. Denote $f_n : X' \times_X Z_n \to X$ the morphism to X. Denote

$$\mathcal{A} = \mathrm{Equalizer}(\ f_*\mathcal{O}_{X'} \underset{\longrightarrow}{\overset{\longrightarrow}{\longrightarrow}} (f \times f)_*\mathcal{O}_{X' \times_X X'}\ )$$

By the remarks above we find that

$$\mathcal{A}_n = \text{Equalizer}(\ \mathcal{A} \times \mathcal{O}_X/\mathcal{I}^n \rightrightarrows f_{n,*}\mathcal{O}_{X' \times_X Z_n}\ )$$

We have canonical maps

$$\mathcal{O}_X \to \ldots \to \mathcal{A}_3 \to \mathcal{A}_2 \to \mathcal{A}_1$$

of coherent $\mathcal{O}_X$-algebras. The statement of the lemma means that for n large enough there exists an $m \geq 0$ such that the image of $\mathcal{A}_{n+m} \to \mathcal{A}_n$ is isomorphic to $\mathcal{O}_X$. This we may check étale locally on X. Hence by Properties of Spaces, Lemma 6.3 we may assume X is an affine Noetherian scheme.

Since $X_n \to X$ is an isomorphism over U we see that the kernel of $\mathcal{O}_X \to \mathcal{A}_n$ is supported on $|Z|$. Since X is Noetherian, the sequence of kernels $\mathcal{J}_n = \text{Ker}(\mathcal{O}_X \to \mathcal{A}_n)$ stabilizes (Cohomology of Spaces, Lemma 13.1). Say $\mathcal{J}_{n_0} = \mathcal{J}_{n_0+1} = \ldots = \mathcal{J}$. By Cohomology of Spaces, Lemma 13.2 we find that $\mathcal{I}^t\mathcal{J} = 0$ for some $t \geq 0$. On the other hand, there is an $\mathcal{O}_X$-algebra map $\mathcal{A}_n \to \mathcal{O}_X/\mathcal{I}^n$ and hence $\mathcal{J} \subset \mathcal{I}^n$ for all n. By Artin-Rees (Cohomology of Spaces, Lemma 13.3) we find that $\mathcal{J} \cap \mathcal{I}^n \subset \mathcal{I}^{n-c}\mathcal{J}$ for some $c \geq 0$ and all $n \gg 0$. We conclude that $\mathcal{J} = 0$.

Pick $n \geq n_0$ as in the previous paragraph. Then $\mathcal{O}_X \to \mathcal{A}_n$ is injective. Hence it now suffices to find $m \geq 0$ such that the image of $\mathcal{A}_{n+m} \to \mathcal{A}_n$ is equal to the image of $\mathcal{O}_X$. Observe that $\mathcal{A}_n$ sits in a short exact sequence

$$0 \to \text{Ker}(\mathcal{A} \to f_{n,*}\mathcal{O}_{X' \times_X Z_n}) \to \mathcal{A}_n \to \mathcal{O}_X/\mathcal{I}^n \to 0$$

and similarly for $\mathcal{A}_{n+m}$. Hence it suffices to show

$$\text{Ker}(\mathcal{A} \to f_{n+m,*}\mathcal{O}_{X' \times_X Z_{n+m}}) \subset \text{Im}(\mathcal{I}^n \to \mathcal{A})$$

for some $m \geq 0$. To do this we may work étale locally on X and since X is Noetherian we may assume that X is a Noetherian affine scheme. Say $X = \text{Spec}(R)$ and $\mathcal{I}$ corresponds to the ideal $I \subset R$. Let $\mathcal{A} = \widetilde{A}$ for a finite R-algebra A. Let $f_*\mathcal{O}_{X'} = \widetilde{B}$ for a finite R-algebra B. Then $R \to A \subset B$ and these maps become isomorphisms on inverting any element of I.

Note that $f_{n,*}\mathcal{O}_{X' \times_X Z_n}$ is equal to $f_*(\mathcal{O}_{X'}/I^n\mathcal{O}_{X'})$ in the notation used in Cohomology of Spaces, Section 22. By Cohomology of Spaces, Lemma 22.4 we see that there exists a $c \geq 0$ such that

$$\text{Ker}(B \to \Gamma(X, f_*(\mathcal{O}_{X'}/I^{n+m+c}\mathcal{O}_{X'})))$$

is contained in $I^{n+m}B$. On the other hand, as $R \to B$ is finite and an isomorphism after inverting any element of I we see that $I^{n+m}B \subset \text{Im}(I^n \to B)$ for m large enough (can be chosen independent of n). This finishes the proof as $A \subset B$. $\qquad\square$

0AGL **Remark 13.5.** The meaning of Lemma 13.4 is the system $X_1 \to X_2 \to X_3 \to \ldots$ is essentially constant with value X. See Categories, Definition 22.1.

14. Compactifications

0F44 This section is the analogue of More on Flatness, Section 33. The theorem in this section is the main theorem in [CLO12].

Let B be a quasi-compact and quasi-separated algebraic space over some base scheme S. We will say an algebraic space X over B *has a compactification over B* or *is compactifyable over B* if there exists a quasi-compact open immersion $X \to \overline{X}$

into an algebraic space $\overline{X}$ proper over B. If X has a compactification over B, then $X \to B$ is separated and of finite type. The main theorem of this section is that the converse is true as well.

0F45 **Lemma 14.1.** *Let S be a scheme. Let $X \to Y$ be a morphism of algebraic spaces over S. If $(U \subset X, f : V \to X)$ is an elementary distinguished square such that $U \to Y$ and $V \to Y$ are separated and $U \times_X V \to U \times_Y V$ is closed, then $X \to Y$ is separated.*

Proof. We have to check that $\Delta : X \to X \times_Y X$ is a closed immersion. There is an étale covering of $X \times_Y X$ given by the four parts $U \times_Y U$, $U \times_Y V$, $V \times_Y U$, and $V \times_Y V$. Observe that $(U \times_Y U) \times_{(X \times_Y X), \Delta} X = U$, $(U \times_Y V) \times_{(X \times_Y X), \Delta} X = U \times_X V$, $(V \times_Y U) \times_{(X \times_Y X), \Delta} X = V \times_X U$, and $(V \times_Y V) \times_{(X \times_Y X), \Delta} X = V$. Thus the assumptions of the lemma exactly tell us that Δ is a closed immersion. $\square$

0F46 **Lemma 14.2.** *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $U \subset X$ be a quasi-compact open.*

(1) *If $Z_1, Z_2 \subset X$ are closed subspaces of finite presentation such that $Z_1 \cap Z_2 \cap U = \emptyset$, then there exists a U-admissible blowing up $X' \to X$ such that the strict transforms of Z_1 and Z_2 are disjoint.*

(2) *If $T_1, T_2 \subset |U|$ are disjoint constructible closed subsets, then there is a U-admissible blowing up $X' \to X$ such that the closures of T_1 and T_2 are disjoint.*

Proof. Proof of (1). The assumption that $Z_i \to X$ is of finite presentation signifies that the quasi-coherent ideal sheaf $\mathcal{I}_i$ of Z_i is of finite type, see Morphisms of Spaces, Lemma 28.12. Denote $Z \subset X$ the closed subspace cut out by the product $\mathcal{I}_1 \mathcal{I}_2$. Observe that $Z \cap U$ is the disjoint union of $Z_1 \cap U$ and $Z_2 \cap U$. By Divisors on Spaces, Lemma 19.5 there is a $U \cap Z$-admissible blowup $Z' \to Z$ such that the strict transforms of Z_1 and Z_2 are disjoint. Denote $Y \subset Z$ the center of this blowing up. Then $Y \to X$ is a closed immersion of finite presentation as the composition of $Y \to Z$ and $Z \to X$ (Divisors on Spaces, Definition 19.1 and Morphisms of Spaces, Lemma 28.2). Thus the blowing up $X' \to X$ of Y is a U-admissible blowing up. By general properties of strict transforms, the strict transform of Z_1, Z_2 with respect to $X' \to X$ is the same as the strict transform of Z_1, Z_2 with respect to $Z' \to Z$, see Divisors on Spaces, Lemma 18.3. Thus (1) is proved.
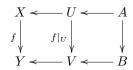
Proof of (2). By Limits of Spaces, Lemma 14.1 there exists a finite type quasi-coherent sheaf of ideals $\mathcal{J}_i \subset \mathcal{O}_U$ such that $T_i = V(\mathcal{J}_i)$ (set theoretically). By Limits of Spaces, Lemma 9.8 there exists a finite type quasi-coherent sheaf of ideals $\mathcal{I}_i \subset \mathcal{O}_X$ whose restriction to U is $\mathcal{J}_i$. Apply the result of part (1) to the closed subspaces $Z_i = V(\mathcal{I}_i)$ to conclude. $\square$

0F47 **Lemma 14.3.** *Let S be a scheme. Let $f : X \to Y$ be a proper morphism of quasi-compact and quasi-separated algebraic spaces over S. Let $V \subset Y$ be a quasi-compact open and $U = f^{-1}(V)$. Let $T \subset |V|$ be a closed subset such that $f|_U : U \to V$ is an isomorphism over an open neighbourhood of T in V. Then there exists a V-admissible blowing up $Y' \to Y$ such that the strict transform $f' : X' \to Y'$ of f is an isomorphism over an open neighbourhood of the closure of T in $|Y'|$.*

Proof. Let $T' \subset |V|$ be the complement of the maximal open over which $f|_U$ is an isomorphism. Then T', T are closed in $|V|$ and $T \cap T' = \emptyset$. Since $|V|$ is a spectral

topological space (Properties of Spaces, Lemma 15.2) we can find constructible closed subsets T_c, T'_c of $|V|$ with $T \subset T_c$, $T' \subset T'_c$ such that $T_c \cap T'_c = \emptyset$ (choose a quasi-compact open W of $|V|$ containing T' not meeting T and set $T_c = |V| \setminus W$, then choose a quasi-compact open W' of $|V|$ containing T_c not meeting T' and set $T'_c = |V| \setminus W'$). By Lemma 14.2 we may, after replacing Y by a V-admissible blowing up, assume that T_c and T'_c have disjoint closures in $|Y|$. Let Y_0 be the open subspace of Y corresponding to the open $|Y| \setminus \overline{T}'_c$ and set $V_0 = V \cap Y_0$, $U_0 = U \times_V V_0$, and $X_0 = X \times_Y Y_0$. Since $U_0 \to V_0$ is an isomorphism, we can find a V_0-admissible blowing up $Y'_0 \to Y_0$ such that the strict transform X'_0 of X_0 maps isomorphically to Y'_0, see More on Morphisms of Spaces, Lemma 39.4. By Divisors on Spaces, Lemma 19.3 there exists a V-admissible blow up $Y' \to Y$ whose restriction to Y_0 is $Y'_0 \to Y_0$. If $f' : X' \to Y'$ denotes the strict transform of f, then we see what we want is true because f' restricts to an isomorphism over Y'_0. $\qquad\square$

0F48 **Lemma 14.4.** *Let S be a scheme. Consider a diagram*

$$
\begin{array}{ccccc}
X & \longleftarrow & U & \longleftarrow & A \\
{\scriptstyle f}\downarrow & & {\scriptstyle f|_U}\downarrow & & \downarrow \\
Y & \longleftarrow & V & \longleftarrow & B
\end{array}
$$

of quasi-compact and quasi-separated algebraic spaces over S. Assume

(1) *f is proper,*
(2) *V is a quasi-compact open of Y, $U = f^{-1}(V)$,*
(3) *$B \subset V$ and $A \subset U$ are closed subspaces,*
(4) *$f|_A : A \to B$ is an isomorphism, and f is étale at every point of A.*

Then there exists a V-admissible blowing up $Y' \to Y$ such that the strict transform $f' : X' \to Y'$ satisfies: for every geometric point $\overline{a}$ of the closure of $|A|$ in $|X'|$ there exists a quotient $\mathcal{O}_{X', \overline{a}} \to \mathcal{O}$ such that $\mathcal{O}_{Y', f'(\overline{a})} \to \mathcal{O}$ is finite flat.

As you can see from the proof, more is true, but the statement is already long enough and this will be sufficient later on.

Proof. Let $T' \subset |U|$ be the complement of the maximal open on which $f|_U$ is étale. Then T' is closed in $|U|$ and disjoint from $|A|$. Since $|U|$ is a spectral topological space (Properties of Spaces, Lemma 15.2) we can find constructible closed subsets T_c, T'_c of $|U|$ with $|A| \subset T_c$, $T' \subset T'_c$ such that $T_c \cap T'_c = \emptyset$ (see proof of Lemma 14.3). By Lemma 14.2 there is a U-admissible blowing up $X_1 \to X$ such that T_c and T'_c have disjoint closures in $|X_1|$. Let $X_{1,0}$ be the open subspace of X_1 corresponding to the open $|X_1| \setminus \overline{T}'_c$ and set $U_0 = U \cap X_{1,0}$. Observe that the scheme theoretic image $\overline{A}_1 \subset X_1$ of A is contained in $X_{1,0}$ by construction.

After replacing Y by a V-admissible blowing up and taking strict transforms, we may assume $X_{1,0} \to Y$ is flat, quasi-finite, and of finite presentation, see More on Morphisms of Spaces, Lemmas 39.1 and 37.3. Consider the commutative diagram

$$
\begin{array}{ccc}
X_1 & \longrightarrow & X \\
& \searrow \quad \swarrow & \\
& Y &
\end{array}
\qquad \text{and the diagram} \qquad
\begin{array}{ccc}
\overline{A}_1 & \longrightarrow & \overline{A} \\
& \searrow \quad \swarrow & \\
& \overline{B} &
\end{array}
$$

of scheme theoretic images. The morphism $\overline{A}_1 \to \overline{A}$ is surjective because it is proper and hence the scheme theoretic image of $\overline{A}_1 \to \overline{A}$ must be equal to $\overline{A}$ and then we can use Morphisms of Spaces, Lemma 40.8. The statement on étale local rings follows by choosing a lift of the geometric point $\overline{a}$ to a geometric point $\overline{a}_1$ of $\overline{A}_1$ and setting $\mathcal{O} = \mathcal{O}_{X_1, \overline{a}_1}$. Namely, since $X_1 \to Y$ is flat and quasi-finite on $X_{1,0} \supset \overline{A}_1$, the map $\mathcal{O}_{Y', f'(\overline{a})} \to \mathcal{O}_{X_1, \overline{a}_1}$ is finite flat, see Algebra, Lemmas 156.3 and 153.3. $\qquad\square$

0F49   **Lemma 14.5.** *Let S be a scheme. Let $X \to B$ and $Y \to B$ be morphisms of algebraic spaces over S. Let $U \subset X$ be an open subspace. Let $V \to X \times_B Y$ be a quasi-compact morphism whose composition with the first projection maps into U. Let $Z \subset X \times_B Y$ be the scheme theoretic image of $V \to X \times_B Y$. Let $X' \to X$ be a U-admissible blowup. Then the scheme theoretic image of $V \to X' \times_B Y$ is the strict transform of Z with respect to the blowing up.*

   Proof. Denote $Z' \to Z$ the strict transform. The morphism $Z' \to X'$ induces a morphism $Z' \to X' \times_B Y$ which is a closed immersion (as Z' is a closed subspace of $X' \times_X Z$ by definition). Thus to finish the proof it suffices to show that the scheme theoretic image Z'' of $V \to Z'$ is Z'. Observe that $Z'' \subset Z'$ is a closed subspace such that $V \to Z'$ factors through Z''. Since both $V \to X \times_B Y$ and $V \to X' \times_B Y$ are quasi-compact (for the latter this follows from Morphisms of Spaces, Lemma 8.9 and the fact that $X' \times_B Y \to X \times_B Y$ is separated as a base change of a proper morphism), by Morphisms of Spaces, Lemma 16.3 we see that $Z \cap (U \times_B Y) = Z'' \cap (U \times_B Y)$. Thus the inclusion morphism $Z'' \to Z'$ is an isomorphism away from the exceptional divisor E of $Z' \to Z$. However, the structure sheaf of Z' does not have any nonzero sections supported on E (by definition of strict transforms) and we conclude that the surjection $\mathcal{O}_{Z'} \to \mathcal{O}_{Z''}$ must be an isomorphism. $\qquad\square$

0F4A   **Lemma 14.6.** *Let S be a scheme. Let B be a quasi-compact and quasi-separated algebraic space over S. Let U be an algebraic space of finite type and separated over B. Let $V \to U$ be an étale morphism. If V has a compactification $V \subset Y$ over B, then there exists a V-admissible blowing up $Y' \to Y$ and an open $V \subset V' \subset Y'$ such that $V \to U$ extends to a proper morphism $V' \to U$.*

   Proof. Consider the scheme theoretic image $Z \subset Y \times_B U$ of the "diagonal" morphism $V \to Y \times_B U$. If we replace Y by a V-admissible blowing up, then Z is replaced by the strict transform with respect to this blowing up, see Lemma 14.5. Hence by More on Morphisms of Spaces, Lemma 39.4 we may assume $Z \to Y$ is an open immersion. If $V' \subset Y$ denotes the image, then we see that the induced morphism $V' \to U$ is proper because the projection $Y \times_B U \to U$ is proper and $V' \cong Z$ is a closed subspace of $Y \times_B U$. $\qquad\square$

The following lemma is formulated for finite type separated algebraic spaces over a finite type algebraic space over $\mathbf{Z}$. The version for quasi-compact and quasi-separated algebraic spaces is true as well (with essentially the same proof), but will be trivially implied by the main theorem in this section. We strongly urge the reader to read the proof of this lemma in the case of schemes first.

0F4B   **Lemma 14.7.** *Let B be an algebraic space of finite type over $\mathbf{Z}$. Let U be an algebraic space of finite type and separated over B. Let $(U_2 \subset U, f : U_1 \to U)$ be an*

elementary distinguished square. Assume U_1 and U_2 have compactifications over B and $U_1 \times_U U_2 \to U$ has dense image. Then U has a compactification over B.

Proof. Choose a compactification $U_i \subset X_i$ over B for $i = 1, 2$. We may assume U_i is scheme theoretically dense in X_i. We may assume there is an open $V_i \subset X_i$ and a proper morphism $\psi_i : V_i \to U$ extending $U_i \to U$, see Lemma 14.6. Picture

$$
\begin{array}{ccc}
U_i & \longrightarrow V_i \longrightarrow X_i \\
\downarrow & \swarrow \psi_i \\
U
\end{array}
$$

Denote $Z_1 \subset U$ the reduced closed subspace corresponding to the closed subset $|U| \setminus |U_2|$. Recall that $f^{-1}Z_1$ is a closed subspace of U_1 mapping isomorphically to Z_1. Denote $Z_2 \subset U$ the reduced closed subspace corresponding to the closed subset $|U| \setminus \mathrm{Im}(|f|) = |U_2| \setminus \mathrm{Im}(|U_1 \times_U U_2| \to |U_2|)$. Thus we have

$$U = U_2 \amalg Z_1 = Z_2 \amalg \mathrm{Im}(f) = Z_2 \amalg \mathrm{Im}(U_1 \times_U U_2 \to U_2) \amalg Z_1$$

set theoretically. Denote $Z_{i,i} \subset V_i$ the inverse image of Z_i under ψ_i. Observe that ψ_2 is an isomorphism over an open neighbourhood of Z_2. Observe that $Z_{1,1} = \psi_1^{-1}Z_1 = f^{-1}Z_1 \amalg T$ for some closed subspace $T \subset V_1$ disjoint from $f^{-1}Z_1$ and furthermore ψ_1 is étale along $f^{-1}Z_1$. Denote $Z_{i,j} \subset V_i$ the inverse image of Z_j under ψ_i. Observe that $\psi_i : Z_{i,j} \to Z_j$ is a proper morphism. Since Z_i and Z_j are disjoint closed subspaces of U, we see that $Z_{i,i}$ and $Z_{i,j}$ are disjoint closed subspaces of V_i.

Denote $\overline{Z}_{i,i}$ and $\overline{Z}_{i,j}$ the scheme theoretic images of $Z_{i,i}$ and $Z_{i,j}$ in X_i. We recall that $|Z_{i,j}|$ is dense in $|\overline{Z}_{i,j}|$, see Morphisms of Spaces, Lemma 17.7. After replacing X_i by a V_i-admissible blowup we may assume that $\overline{Z}_{i,i}$ and $\overline{Z}_{i,j}$ are disjoint, see Lemma 14.2. We assume this holds for both X_1 and X_2. Observe that this property is preserved if we replace X_i by a further V_i-admissible blowup. Hence we may replace X_1 by another V_1-admissible blowup and assume $|\overline{Z}_{1,1}|$ is the disjoint union of the closures of $|T|$ and $|f^{-1}Z_1|$ in $|X_1|$.

Set $V_{12} = V_1 \times_U V_2$. We have an immersion $V_{12} \to X_1 \times_B X_2$ which is the composition of the closed immersion $V_{12} = V_1 \times_U V_2 \to V_1 \times_B V_2$ (Morphisms of Spaces, Lemma 4.5) and the open immersion $V_1 \times_B V_2 \to X_1 \times_B X_2$. Let $X_{12} \subset X_1 \times_B X_2$ be the scheme theoretic image of $V_{12} \to X_1 \times_B X_2$. The projection morphisms

$$p_1 : X_{12} \to X_1 \quad \text{and} \quad p_2 : X_{12} \to X_2$$

are proper as X_1 and X_2 are proper over B. If we replace X_1 by a V_1-admissible blowing up, then X_{12} is replaced by the strict transform with respect to this blowing up, see Lemma 14.5.

Denote $\psi : V_{12} \to U$ the compositions $\psi = \psi_1 \circ p_1|_{V_{12}} = \psi_2 \circ p_2|_{V_{12}}$. Consider the closed subspace

$$Z_{12,2} = (p_1|_{V_{12}})^{-1}Z_{1,2} = (p_2|_{V_{12}})^{-1}Z_{2,2} = \psi^{-1}Z_2 \subset V_{12}$$

The morphism $p_1|_{V_{12}} : V_{12} \to V_1$ is an isomorphism over an open neighbourhood of $Z_{1,2}$ because $\psi_2 : V_2 \to U$ is an isomorphism over an open neighbourhood of Z_2 and $V_{12} = V_1 \times_U V_2$. By Lemma 14.3 there exists a V_1-admissible blowing up $X_1' \to X_1$ such that the strict tranform $p_1' : X_{12}' \to X_1'$ of p_1 is an isomorphism over an open neighbourhood of the closure of $|Z_{1,2}|$ in $|X_1'|$. After replacing X_1 by X_1' and X_{12}

by X'_{12} we may assume that p_1 is an isomorphism over an open neighbourhood of $|\overline{Z}_{1,2}|$.

The result of the previous paragraph tells us that

$$X_{12} \cap (\overline{Z}_{1,2} \times_B \overline{Z}_{2,1}) = \emptyset$$

where the intersection taken in $X_1 \times_B X_2$. Namely, the inverse image $p_1^{-1}\overline{Z}_{1,2}$ in X_{12} maps isomorphically to $\overline{Z}_{1,2}$. In particular, we see that $|Z_{12,2}|$ is dense in $|p_1^{-1}\overline{Z}_{1,2}|$. Thus p_2 maps $|p_1^{-1}\overline{Z}_{1,2}|$ into $|\overline{Z}_{2,2}|$. Since $|\overline{Z}_{2,2}| \cap |\overline{Z}_{2,1}| = \emptyset$ we conclude.

It turns out that we need to do one additional blowing up before we can conclude the argument. Namely, let $V_2 \subset W_2 \subset X_2$ be the open subspace with underlying topological space

$$|W_2| = |V_2| \cup (|X_2| \setminus |\overline{Z}_{2,1}|) = |X_2| \setminus (|\overline{Z}_{2,1}| \setminus |Z_{2,1}|)$$

Since $p_2(p_1^{-1}\overline{Z}_{1,2})$ is contained in W_2 (see above) we see that replacing X_2 by a W_2-admissible blowup and X_{21} by the corresponding strict transform will preserve the property of p_1 being an isomorphism over an open neighbourhood of $\overline{Z}_{1,2}$. Since $\overline{Z}_{2,1} \cap W_2 = \overline{Z}_{2,1} \cap V_2 = Z_{2,1}$ we see that $Z_{2,1}$ is a closed subspace of W_2 and V_2. Observe that $V_{12} = V_1 \times_U V_2 = p_1^{-1}(V_1) = p_2^{-1}(V_2)$ as open subspaces of X_{12} as it is the largest open subspace of X_{12} over which the morphism $\psi : V_{12} \to U$ extends; details omitted[4]. We have the following equalities of closed subspaces of V_{12}:

$$p_2^{-1}Z_{2,1} = p_2^{-1}\psi_2^{-1}Z_1 = p_1^{-1}\psi_1^{-1}Z_1 = p_1^{-1}Z_{1,1} = p_1^{-1}f^{-1}Z_1 \amalg p_1^{-1}T$$

Here and below we use the slight abuse of notation of writing p_2 in stead of the restriction of p_2 to V_{12}, etc. Since $p_2^{-1}(Z_{2,1})$ is a closed subspace of $p_2^{-1}(W_2)$ as $Z_{2,1}$ is a closed subspace of W_2 we conclude that also $p_1^{-1}f^{-1}Z_1$ is a closed subspace of $p_2^{-1}(W_2)$. Finally, the morphism $p_2 : X_{12} \to X_2$ is étale at points of $p_1^{-1}f^{-1}Z_1$ as ψ_1 is étale along $f^{-1}Z_1$ and $V_{12} = V_1 \times_U V_2$. Thus we may apply Lemma 14.4 to the morphism $p_2 : X_{12} \to X_2$, the open W_2, the closed subspace $Z_{2,1} \subset W_2$, and the closed subspace $p_1^{-1}f^{-1}Z_1 \subset p_2^{-1}(W_2)$. Hence after replacing X_2 by a W_2-admissible blowup and X_{12} by the corresponding strict transform, we obtain for every geometric point $\overline{y}$ of the closure of $|p_1^{-1}f^{-1}Z_1|$ a local ring map $\mathcal{O}_{X_{12},\overline{y}} \to \mathcal{O}$ such that $\mathcal{O}_{X_2,p_2(\overline{y})} \to \mathcal{O}$ is finite flat.

Consider the algebraic space

$$W_2 = U \coprod_{U_2} (X_2 \setminus \overline{Z}_{2,1}),$$

and with $T \subset V_1$ as in the first paragraph the algebraic space

$$W_1 = U \coprod_{U_1} (X_1 \setminus \overline{Z}_{1,2} \cup \overline{T}),$$

obtained by pushout, see Lemma 9.2. Let us apply Lemma 14.1 to see that $W_i \to B$ is separated. First, $U \to B$ and $X_i \to B$ are separated. Let us check the quasi-compact immersion $U_i \to U \times_B (X_i \setminus \overline{Z}_{i,j})$ is closed using the valuative criterion, see Morphisms of Spaces, Lemma 42.1. Choose a valuation ring A over B with fraction field K and compatible morphisms $(u, x_i) : \mathrm{Spec}(A) \to U \times_B X_i$ and $u_i : \mathrm{Spec}(K) \to U_i$. Since ψ_i is proper, we can find a unique $v_i : \mathrm{Spec}(A) \to V_i$
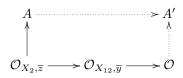
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[4]Namely, $V_1 \times_U V_2$ is proper over U so if ψ extends to a larger open of X_{12}, then $V_1 \times_U V_2$ would be closed in this open by Morphisms of Spaces, Lemma 40.6. Then we get equality as $V_{12} \subset X_{12}$ is dense.

compatible with u and u_i. Since X_i is proper over B we see that $x_i = v_i$. If v_i does not factor through $U_i \subset V_i$, then we conclude that x_i maps the closed point of $\operatorname{Spec}(A)$ into $Z_{i,j}$ or T when $i = 1$. This finishes the proof because we removed $\overline{Z}_{i,j}$ and $\overline{T}$ in the construction of W_i.

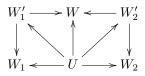On the other hand, for any valuation ring A over B with fraction field K and any morphism
$$\gamma : \operatorname{Spec}(K) \to \operatorname{Im}(U_1 \times_U U_2 \to U)$$
over B, we claim that after replacing A by an extension of valuation rings, there is an i and an extension of γ to a morphism $h_i : \operatorname{Spec}(A) \to W_i$. Namely, we first extend γ to a morphism $g_2 : \operatorname{Spec}(A) \to X_2$ using the valuative criterion of properness. If the image of g_2 does not meet $\overline{Z}_{2,1}$, then we obtain our morphism into W_2. Otherwise, denote $\overline{z} \in \overline{Z}_{2,1}$ a geometric point lying over the image of the closed point under g_2. We may lift this to a geometric point $\overline{y}$ of X_{12} in the closure of $|p_1^{-1} f^{-1} Z_1|$ because the map of spaces $|p_1^{-1} f^{-1} Z_1| \to |\overline{Z}_{2,1}|$ is closed with image containing the dense open $|Z_{2,1}|$. After replacing A by its strict henselization (More on Algebra, Lemma 123.6) we get the following diagram

$$
\begin{array}{ccc}
A & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\!\!\!\!\!> & A' \\
\uparrow & & \uparrow \\
\mathcal{O}_{X_2,\overline{z}} \longrightarrow \mathcal{O}_{X_{12},\overline{y}} \longrightarrow & & \mathcal{O}
\end{array}
$$

where $\mathcal{O}_{X_{12},\overline{y}} \to \mathcal{O}$ is the map we found in the 5th paragraph of the proof. Since the horizontal composition is finite and flat we can find an extension of valuation rings A'/A and dotted arrow making the diagram commute. After replacing A by A' this means that we obtain a lift $g_{12} : \operatorname{Spec}(A) \to X_{12}$ whose closed point maps into the closure of $|p_1^{-1} f^{-1} Z_1|$. Then $g_1 = p_1 \circ g_{12} : \operatorname{Spec}(A) \to X_1$ is a morphism whose closed point maps into the closure of $|f^{-1} Z_1|$. Since the closure of $|f^{-1} Z_1|$ is disjoint from the closure of $|T|$ and contained in $|\overline{Z}_{1,1}|$ which is disjoint from $|\overline{Z}_{1,2}|$ we conclude that g_1 defines a morphism $h_1 : \operatorname{Spec}(A) \to W_1$ as desired.

Consider a diagram

$$
\begin{array}{ccccc}
W_1' & \longrightarrow & W & \longleftarrow & W_2' \\
\downarrow & \searrow \quad \uparrow \quad \nearrow & & & \downarrow \\
W_1 & \longleftarrow & U & \longrightarrow & W_2
\end{array}
$$

as in More on Morphisms of Spaces, Lemma 40.1. By the previous paragraph for every solid diagram

$$
\begin{array}{ccc}
\operatorname{Spec}(K) & \xrightarrow{\ \gamma\ } & W \\
\downarrow & \nearrow & \downarrow \\
\operatorname{Spec}(A) & \longrightarrow & B
\end{array}
$$

where $\operatorname{Im}(\gamma) \subset \operatorname{Im}(U_1 \times_U U_2 \to U)$ there is an i and an extension $h_i : \operatorname{Spec}(A) \to W_i$ of γ after possibly replacing A by an extension of valuation rings. Using the valuative criterion of properness for $W_i' \to W_i$, we can then lift h_i to $h_i' : \operatorname{Spec}(A) \to W_i'$. Hence the dotted arrow in the diagram exists after possibly extending A. Since W is separated over B, we see that the choice of extension isn't needed and the arrow is unique as well, see Morphisms of Spaces, Lemmas 41.5 and 43.1. Then

finally the existence of the dotted arrow implies that $W \to B$ is universally closed by Morphisms of Spaces, Lemma 42.5. As $W \to B$ is already of finite type and separated, we win. $\square$

0F4C **Lemma 14.8.** *Let S be a scheme. Let X be a Noetherian algebraic space over S. Let $U \subset X$ be a proper dense open subspace. Then there exists an affine scheme V and an étale morphism $V \to X$ such that*

(1) *the open subspace $W = U \cup \operatorname{Im}(V \to X)$ is strictly larger than U,*
(2) *$(U \subset W, V \to W)$ is a distinguished square, and*
(3) *$U \times_W V \to U$ has dense image.*

Proof. Choose a stratification

$$\emptyset = U_{n+1} \subset U_n \subset U_{n-1} \subset \ldots \subset U_1 = X$$

and morphisms $f_p : V_p \to U_p$ as in Decent Spaces, Lemma 8.6. Let p be the smallest integer such that $U_p \not\subset U$ (this is possible as $U \neq X$). Choose an affine open $V \subset V_p$ such that the étale morphism $f_p|_V : V \to X$ does not factor through U. Consider the open $W = U \cup \operatorname{Im}(V \to X)$ and the reduced closed subspace $Z \subset W$ with $|Z| = |W| \setminus |U|$. Then $f^{-1}Z \to Z$ is an isomorphism because we have the corresponding property for the morphism f_p, see the lemma cited above. Thus $(U \subset W, f : V \to W)$ is a distinguished square. It may not be true that the open $I = \operatorname{Im}(U \times_W V \to U)$ is dense in U. The algebraic space $U' \subset U$ whose underlying set is $|U| \setminus \overline{|I|}$ is Noetherian and hence we can find a dense open subscheme $U'' \subset U'$, see for example Properties of Spaces, Proposition 13.3. Then we can find a dense open affine $U''' \subset U''$, see Properties, Lemmas 5.7 and 29.1. After we replace f by $V \amalg U''' \to X$ everything is clear. $\square$

0F4D **Theorem 14.9.** *Let S be a scheme. Let B be a quasi-compact and quasi-separated* [CLO12] *algebraic space over S. Let $X \to B$ be a separated, finite type morphism. Then X has a compactification over B.*

Proof. We first reduce to the Noetherian case. We strongly urge the reader to skip this paragraph. First, we may replace S by $\operatorname{Spec}(\mathbf{Z})$. See Spaces, Section 16 and Properties of Spaces, Definition 3.1. There exists a closed immersion $X \to X'$ with $X' \to B$ of finite presentation and separated. See Limits of Spaces, Proposition 11.7. If we find a compactification of X' over B, then taking the scheme theoretic closure of X in this will give a compactification of X over B. Thus we may assume $X \to B$ is separated and of finite presentation. We may write $B = \lim B_i$ as a directed limit of a system of Noetherian algebraic spaces of finite type over $\operatorname{Spec}(\mathbf{Z})$ with affine transition morphisms. See Limits of Spaces, Proposition 8.1. We can choose an i and a morphism $X_i \to B_i$ of finite presentation whose base change to B is $X \to B$, see Limits of Spaces, Lemma 7.1. After increasing i we may assume $X_i \to B_i$ is separated, see Limits of Spaces, Lemma 6.9. If we can find a compactification of X_i over B_i, then the base change of this to B will be a compactification of X over B. This reduces us to the case discussed in the next paragraph.

Assume B is of finite type over $\mathbf{Z}$ in addition to being quasi-compact and quasi-separated. Let $U \to X$ be an étale morphism of algebraic spaces such that U has a compactification Y over $\operatorname{Spec}(\mathbf{Z})$. The morphism

$$U \longrightarrow B \times_{\operatorname{Spec}(\mathbf{Z})} Y$$

is separated and quasi-finite by Morphisms of Spaces, Lemma 27.10 (the displayed morphism factors into an immersion hence is a monomorphism). Hence by Zariski's main theorem (More on Morphisms of Spaces, Lemma 34.3) there is an open immersion of U into an algebraic space Y' finite over $B \times_{\mathrm{Spec}(\mathbf{Z})} Y$. Then $Y' \to B$ is proper as the composition $Y' \to B \times_{\mathrm{Spec}(\mathbf{Z})} Y \to B$ of two proper morphisms (use Morphisms of Spaces, Lemmas 45.9, 40.4, and 40.3). We conclude that U has a compactification over B.

There is a dense open subspace $U \subset X$ which is a scheme. (Properties of Spaces, Proposition 13.3). In fact, we may choose U to be an affine scheme (Properties, Lemmas 5.7 and 29.1). Thus U has a compactification over $\mathrm{Spec}(\mathbf{Z})$; this is easily shown directly but also follows from the theorem for schemes, see More on Flatness, Theorem 33.8. By the previous paragraph U has a compactification over B. By Noetherian induction we can find a maximal dense open subspace $U \subset X$ which has a compactification over B. We will show that the assumption that $U \neq X$ leads to a contradiction. Namely, by Lemma 14.8 we can find a strictly larger open $U \subset W \subset X$ and a distinguished square $(U \subset W, f : V \to W)$ with V affine and $U \times_W V$ dense image in U. Since V is affine, as before it has a compactification over B. Hence Lemma 14.7 applies to show that W has a compactification over B which is the desired contradiction. $\square$

15. Other chapters

Preliminaries

Schemes

Topics in Scheme Theory

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