MORPHISMS OF ALGEBRAIC SPACES

03H8

Contents

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1. Introduction

03H9 In this chapter we introduce some types of morphisms of algebraic spaces. A reference is [Knu71].

The goal is to extend the definition of each of the types of morphisms of schemes defined in the chapters on schemes, and on morphisms of schemes to the category of algebraic spaces. Each case is slightly different and it seems best to treat them all separately.

2. Conventions

040V The standing assumption is that all schemes are contained in a big fppf site Sch_{fppf}. And all rings A considered have the property that $\mathrm{Spec}(A)$ is (isomorphic) to an object of this big site.

Let S be a scheme and let X be an algebraic space over S. In this chapter and the following we will write $X \times_S X$ for the product of X with itself (in the category of algebraic spaces over S), instead of $X \times X$.

3. Properties of representable morphisms

03HA Let S be a scheme. Let $f : X \to Y$ be a representable morphism of algebraic spaces. In Spaces, Section 5 we defined what it means for f to have property $\mathcal{P}$ in case $\mathcal{P}$ is a property of morphisms of schemes which

(1) is preserved under any base change, see Schemes, Definition 18.3, and
(2) is fppf local on the base, see Descent, Definition 22.1.

Namely, in this case we say f has property $\mathcal{P}$ if and only if for every scheme U and any morphism $U \to Y$ the morphism of schemes $X \times_Y U \to U$ has property $\mathcal{P}$.

According to the lists in Spaces, Section 4 this applies to the following properties: (1)(a) closed immersions, (1)(b) open immersions, (1)(c) quasi-compact

immersions, (2) quasi-compact, (3) universally-closed, (4) (quasi-)separated, (5) monomorphism, (6) surjective, (7) universally injective, (8) affine, (9) quasi-affine, (10) (locally) of finite type, (11) (locally) quasi-finite, (12) (locally) of finite presentation, (13) locally of finite type of relative dimension d, (14) universally open, (15) flat, (16) syntomic, (17) smooth, (18) unramified (resp. G-unramified), (19) étale, (20) proper, (21) finite or integral, (22) finite locally free, (23) universally submersive, (24) universal homeomorphism, and (25) immersion.

In this chapter we will redefine these notions for not necessarily representable morphisms of algebraic spaces. Whenever we do this we will make sure that the new definition agrees with the old one, in order to avoid ambiguity.

Note that the definition above applies whenever X is a scheme, since a morphism from a scheme to an algebraic space is representable. And in particular it applies when both X and Y are schemes. In Spaces, Lemma 5.3 we have seen that in this case the definitions match, and no ambiguity arise.

Furthermore, in Spaces, Lemma 5.5 we have seen that the property of representable morphisms of algebraic spaces so defined is stable under arbitrary base change by a morphism of algebraic spaces. And finally, in Spaces, Lemmas 5.4 and 5.7 we have seen that if $\mathcal{P}$ is stable under compositions, which holds for the properties (1)(a), (1)(b), (1)(c), (2) – (25), except (13) above, then taking products of representable morphisms preserves property $\mathcal{P}$ and compositions of representable morphisms preserves property $\mathcal{P}$.

We will use these facts below, and whenever we do we will simply refer to this section as a reference.

4. Separation axioms

03HJ It makes sense to list some a priori properties of the diagonal of a morphism of algebraic spaces.

03HK **Lemma 4.1.** *Let S be a scheme contained in Sch_{fppf}. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\Delta_{X/Y} : X \to X \times_Y X$ be the diagonal morphism. Then*

(1) *$\Delta_{X/Y}$ is representable,*
(2) *$\Delta_{X/Y}$ is locally of finite type,*
(3) *$\Delta_{X/Y}$ is a monomorphism,*
(4) *$\Delta_{X/Y}$ is separated, and*
(5) *$\Delta_{X/Y}$ is locally quasi-finite.*

Proof. We are going to use the fact that $\Delta_{X/S}$ is representable (by definition of an algebraic space) and that it satisfies properties (2) – (5), see Spaces, Lemma 13.1. Note that we have a factorization

$$X \longrightarrow X \times_Y X \longrightarrow X \times_S X$$

of the diagonal $\Delta_{X/S} : X \to X \times_S X$. Since $X \times_Y X \to X \times_S X$ is a monomorphism, and since $\Delta_{X/S}$ is representable, it follows formally that $\Delta_{X/Y}$ is representable. In particular, the rest of the statements now make sense, see Section 3.

Choose a surjective étale morphism $U \to X$, with U a scheme. Consider the diagram

$$\begin{array}{ccccc} R = U \times_X U & \longrightarrow & U \times_Y U & \longrightarrow & U \times_S U \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & X \times_Y X & \longrightarrow & X \times_S X \end{array}$$

Both squares are cartesian, hence so is the outer rectangle. The top row consists of schemes, and the vertical arrows are surjective étale morphisms. By Spaces, Lemma 11.4 the properties (2) – (5) for $\Delta_{X/Y}$ are equivalent to those of $R \to U \times_Y U$. In the proof of Spaces, Lemma 13.1 we have seen that $R \to U \times_S U$ has properties (2) – (5). The morphism $U \times_Y U \to U \times_S U$ is a monomorphism of schemes. These facts imply that $R \to U \times_Y U$ have properties (2) – (5).

Namely: For (3), note that $R \to U \times_Y U$ is a monomorphism as the composition $R \to U \times_S U$ is a monomorphism. For (2), note that $R \to U \times_Y U$ is locally of finite type, as the composition $R \to U \times_S U$ is locally of finite type (Morphisms, Lemma 15.8). A monomorphism which is locally of finite type is locally quasi-finite because it has finite fibres (Morphisms, Lemma 20.7), hence (5). A monomorphism is separated (Schemes, Lemma 23.3), hence (4). □

03HL **Definition 4.2.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\Delta_{X/Y} : X \to X \times_Y X$ be the diagonal morphism.

(1) We say f is *separated* if $\Delta_{X/Y}$ is a closed immersion.
(2) We say f is *locally separated*[1] if $\Delta_{X/Y}$ is an immersion.
(3) We say f is *quasi-separated* if $\Delta_{X/Y}$ is quasi-compact.

This definition makes sense since $\Delta_{X/Y}$ is representable, and hence we know what it means for it to have one of the properties described in the definition. We will see below (Lemma 4.13) that this definition matches the ones we already have for morphisms of schemes and representable morphisms.

03KK **Lemma 4.3.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is separated, then f is locally separated and f is quasi-separated.*

Proof. This is true, via the general principle Spaces, Lemma 5.8, because a closed immersion of schemes is an immersion and is quasi-compact. □

03KL **Lemma 4.4.** *All of the separation axioms listed in Definition 4.2 are stable under base change.*

Proof. Let $f : X \to Y$ and $Y' \to Y$ be morphisms of algebraic spaces. Let $f' : X' \to Y'$ be the base change of f by $Y' \to Y$. Then $\Delta_{X'/Y'}$ is the base change of $\Delta_{X/Y}$ by the morphism $X' \times_{Y'} X' \to X \times_Y X$. By the results of Section 3 each of the properties of the diagonal used in Definition 4.2 is stable under base change. Hence the lemma is true. □

03KN **Lemma 4.5.** *Let S be a scheme. Let $f : X \to Z$, $g : Y \to Z$ and $Z \to T$ be morphisms of algebraic spaces over S. Consider the induced morphism $i : X \times_Z Y \to X \times_T Y$. Then*

[1] In the literature this term often refers to quasi-separated and locally separated morphisms.

(1) *i is representable, locally of finite type, locally quasi-finite, separated and a monomorphism,*
(2) *if $Z \to T$ is locally separated, then i is an immersion,*
(3) *if $Z \to T$ is separated, then i is a closed immersion, and*
(4) *if $Z \to T$ is quasi-separated, then i is quasi-compact.*

Proof. By general category theory the following diagram

$$\begin{array}{ccc} X \times_Z Y & \xrightarrow[i]{} & X \times_T Y \\ \downarrow & & \downarrow \\ Z & \xrightarrow{\Delta_{Z/T}} & Z \times_T Z \end{array}$$

is a fibre product diagram. Hence i is the base change of the diagonal morphism $\Delta_{Z/T}$. Thus the lemma follows from Lemma 4.1, and the material in Section 3. □

03KO **Lemma 4.6.** *Let S be a scheme. Let T be an algebraic space over S. Let $g : X \to Y$ be a morphism of algebraic spaces over T. Consider the graph $i : X \to X \times_T Y$ of g. Then*

(1) *i is representable, locally of finite type, locally quasi-finite, separated and a monomorphism,*
(2) *if $Y \to T$ is locally separated, then i is an immersion,*
(3) *if $Y \to T$ is separated, then i is a closed immersion, and*
(4) *if $Y \to T$ is quasi-separated, then i is quasi-compact.*

Proof. This is a special case of Lemma 4.5 applied to the morphism $X = X \times_Y Y \to X \times_T Y$. □

03KP **Lemma 4.7.** *Let S be a scheme. Let $f : X \to T$ be a morphism of algebraic spaces over S. Let $s : T \to X$ be a section of f (in a formula $f \circ s = id_T$). Then*

(1) *s is representable, locally of finite type, locally quasi-finite, separated and a monomorphism,*
(2) *if f is locally separated, then s is an immersion,*
(3) *if f is separated, then s is a closed immersion, and*
(4) *if f is quasi-separated, then s is quasi-compact.*

Proof. This is a special case of Lemma 4.6 applied to $g = s$ so the morphism $i = s : T \to T \times_T X$. □

03KQ **Lemma 4.8.** *All of the separation axioms listed in Definition 4.2 are stable under composition of morphisms.*

Proof. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of algebraic spaces to which the axiom in question applies. The diagonal $\Delta_{X/Z}$ is the composition

$$X \longrightarrow X \times_Y X \longrightarrow X \times_Z X.$$

Our separation axiom is defined by requiring the diagonal to have some property $\mathcal{P}$. By Lemma 4.5 above we see that the second arrow also has this property. Hence the lemma follows since the composition of (representable) morphisms with property $\mathcal{P}$ also is a morphism with property $\mathcal{P}$, see Section 3. □

04ZH **Lemma 4.9.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.*

(1) *If Y is separated and f is separated, then X is separated.*
(2) *If Y is quasi-separated and f is quasi-separated, then X is quasi-separated.*
(3) *If Y is locally separated and f is locally separated, then X is locally separated.*
(4) *If Y is separated over S and f is separated, then X is separated over S.*
(5) *If Y is quasi-separated over S and f is quasi-separated, then X is quasi-separated over S.*
(6) *If Y is locally separated over S and f is locally separated, then X is locally separated over S.*

Proof. Parts (4), (5), and (6) follow immediately from Lemma 4.8 and Spaces, Definition 13.2. Parts (1), (2), and (3) reduce to parts (4), (5), and (6) by thinking of X and Y as algebraic spaces over $\mathrm{Spec}(\mathbf{Z})$, see Properties of Spaces, Definition 3.1. □

03KR **Lemma 4.10.** *Let S be a scheme. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of algebraic spaces over S.*

(1) *If $g \circ f$ is separated then so is f.*
(2) *If $g \circ f$ is locally separated then so is f.*
(3) *If $g \circ f$ is quasi-separated then so is f.*

Proof. Consider the factorization

$$X \to X \times_Y X \to X \times_Z X$$

of the diagonal morphism of $g \circ f$. In any case the last morphism is a monomorphism. Hence for any scheme T and morphism $T \to X \times_Y X$ we have the equality

$$X \times_{(X \times_Y X)} T = X \times_{(X \times_Z X)} T.$$

Hence the result is clear. □

04ZI **Lemma 4.11.** *Let S be a scheme. Let X be an algebraic space over S.*

(1) *If X is separated then X is separated over S.*
(2) *If X is locally separated then X is locally separated over S.*
(3) *If X is quasi-separated then X is quasi-separated over S.*

Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(4) *If X is separated over S then f is separated.*
(5) *If X is locally separated over S then f is locally separated.*
(6) *If X is quasi-separated over S then f is quasi-separated.*

Proof. Parts (4), (5), and (6) follow immediately from Lemma 4.10 and Spaces, Definition 13.2. Parts (1), (2), and (3) follow from parts (4), (5), and (6) by thinking of X and Y as algebraic spaces over $\mathrm{Spec}(\mathbf{Z})$, see Properties of Spaces, Definition 3.1. □

03KM **Lemma 4.12.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{P}$ be any of the separation axioms of Definition 4.2. The following are equivalent*

(1) *f is $\mathcal{P}$,*
(2) *for every scheme Z and morphism $Z \to Y$ the base change $Z \times_Y X \to Z$ of f is $\mathcal{P}$,*

(3) *for every affine scheme Z and every morphism $Z \to Y$ the base change $Z \times_Y X \to Z$ of f is $\mathcal{P}$,*
(4) *for every affine scheme Z and every morphism $Z \to Y$ the algebraic space $Z \times_Y X$ is $\mathcal{P}$ (see Properties of Spaces, Definition 3.1),*
(5) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that the base change $V \times_Y X \to V$ has $\mathcal{P}$, and*
(6) *there exists a Zariski covering $Y = \bigcup Y_i$ such that each of the morphisms $f^{-1}(Y_i) \to Y_i$ has $\mathcal{P}$.*

Proof. We will repeatedly use Lemma 4.4 without further mention. In particular, it is clear that (1) implies (2) and (2) implies (3).

Let us prove that (3) and (4) are equivalent. Note that if Z is an affine scheme, then the morphism $Z \to \mathrm{Spec}(\mathbf{Z})$ is a separated morphism as a morphism of algebraic spaces over $\mathrm{Spec}(\mathbf{Z})$. If $Z \times_Y X \to Z$ is $\mathcal{P}$, then $Z \times_Y X \to \mathrm{Spec}(\mathbf{Z})$ is $\mathcal{P}$ as a composition (see Lemma 4.8). Hence the algebraic space $Z \times_Y X$ is $\mathcal{P}$. Conversely, if the algebraic space $Z \times_Y X$ is $\mathcal{P}$, then $Z \times_Y X \to \mathrm{Spec}(\mathbf{Z})$ is $\mathcal{P}$, and hence by Lemma 4.10 we see that $Z \times_Y X \to Z$ is $\mathcal{P}$.

Let us prove that (3) implies (5). Assume (3). Let V be a scheme and let $V \to Y$ be étale surjective. We have to show that $V \times_Y X \to V$ has property $\mathcal{P}$. In other words, we have to show that the morphism

$$V \times_Y X \longrightarrow (V \times_Y X) \times_V (V \times_Y X) = V \times_Y X \times_Y X$$

has the corresponding property (i.e., is a closed immersion, immersion, or quasi-compact). Let $V = \bigcup V_j$ be an affine open covering of V. By assumption we know that each of the morphisms

$$V_j \times_Y X \longrightarrow V_j \times_Y X \times_Y X$$

does have the corresponding property. Since being a closed immersion, immersion, quasi-compact immersion, or quasi-compact is Zariski local on the target, and since the V_j cover V we get the desired conclusion.

Let us prove that (5) implies (1). Let $V \to Y$ be as in (5). Then we have the fibre product diagram

$$\begin{array}{ccc} V \times_Y X & \longrightarrow & X \\ \downarrow & & \downarrow \\ V \times_Y X \times_Y X & \longrightarrow & X \times_Y X \end{array}$$

By assumption the left vertical arrow is a closed immersion, immersion, quasi-compact immersion, or quasi-compact. It follows from Spaces, Lemma 5.6 that also the right vertical arrow is a closed immersion, immersion, quasi-compact immersion, or quasi-compact.

It is clear that (1) implies (6) by taking the covering $Y = Y$. Assume $Y = \bigcup Y_i$ is as in (6). Choose schemes V_i and surjective étale morphisms $V_i \to Y_i$. Note that the morphisms $V_i \times_Y X \to V_i$ have $\mathcal{P}$ as they are base changes of the morphisms $f^{-1}(Y_i) \to Y_i$. Set $V = \coprod V_i$. Then $V \to Y$ is a morphism as in (5) (details omitted). Hence (6) implies (5) and we are done. □

03KY **Lemma 4.13.** *Let S be a scheme. Let $f : X \to Y$ be a representable morphism of algebraic spaces over S.*

(1) *The morphism f is locally separated.*
(2) *The morphism f is (quasi-)separated in the sense of Definition 4.2 above if and only if f is (quasi-)separated in the sense of Section 3.*

In particular, if $f : X \to Y$ is a morphism of schemes over S, then f is (quasi-)separated in the sense of Definition 4.2 if and only if f is (quasi-)separated as a morphism of schemes.

Proof. This is the equivalence of (1) and (2) of Lemma 4.12 combined with the fact that any morphism of schemes is locally separated, see Schemes, Lemma 21.2. □

5. Surjective morphisms

03MC We have already defined in Section 3 what it means for a representable morphism of algebraic spaces to be surjective.

03MD **Lemma 5.1.** *Let S be a scheme. Let $f : X \to Y$ be a representable morphism of algebraic spaces over S. Then f is surjective (in the sense of Section 3) if and only if $|f| : |X| \to |Y|$ is surjective.*

Proof. Namely, if $f : X \to Y$ is representable, then it is surjective if and only if for every scheme T and every morphism $T \to Y$ the base change $f_T : T \times_Y X \to T$ of f is a surjective morphism of schemes, in other words, if and only if $|f_T|$ is surjective. By Properties of Spaces, Lemma 4.3 the map $|T \times_Y X| \to |T| \times_{|Y|} |X|$ is always surjective. Hence $|f_T| : |T \times_Y X| \to |T|$ is surjective if $|f| : |X| \to |Y|$ is surjective. Conversely, if $|f_T|$ is surjective for every $T \to Y$ as above, then by taking T to be the spectrum of a field we conclude that $|X| \to |Y|$ is surjective. □

This clears the way for the following definition.

03ME **Definition 5.2.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. We say f is *surjective* if the map $|f| : |X| \to |Y|$ of associated topological spaces is surjective.

03MF **Lemma 5.3.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is surjective,*
(2) *for every scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is surjective,*
(3) *for every affine scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is surjective,*
(4) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is a surjective morphism,*
(5) *there exists a scheme U and a surjective étale morphism $\varphi : U \to X$ such that the composition $f \circ \varphi$ is surjective,*
(6) *there exists a commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where U, V are schemes and the vertical arrows are surjective étale such that the top horizontal arrow is surjective, and

(7) *there exists a Zariski covering* $Y = \bigcup Y_i$ *such that each of the morphisms* $f^{-1}(Y_i) \to Y_i$ *is surjective.*

Proof. Omitted. □

03MG **Lemma 5.4.** *The composition of surjective morphisms is surjective.*

Proof. This is immediate from the definition. □

03MH **Lemma 5.5.** *The base change of a surjective morphism is surjective.*

Proof. Follows immediately from Properties of Spaces, Lemma 4.3. □

6. Open morphisms

03Z0 For a representable morphism of algebraic spaces we have already defined (in Section 3) what it means to be universally open. Hence before we give the natural definition we check that it agrees with this in the representable case.

03Z1 **Lemma 6.1.** *Let* S *be a scheme. Let* $f : X \to Y$ *be a representable morphism of algebraic spaces over* S*. The following are equivalent*

(1) f *is universally open (in the sense of Section 3), and*
(2) *for every morphism of algebraic spaces* $Z \to Y$ *the morphism of topological spaces* $|Z \times_Y X| \to |Z|$ *is open.*

Proof. Assume (1), and let $Z \to Y$ be as in (2). Choose a scheme V and a surjective étale morphism $V \to Y$. By assumption the morphism of schemes $V \times_Y X \to V$ is universally open. By Properties of Spaces, Section 4 in the commutative diagram

$$\begin{array}{ccc} |V \times_Y X| & \longrightarrow & |Z \times_Y X| \\ \downarrow & & \downarrow \\ |V| & \longrightarrow & |Z| \end{array}$$

the horizontal arrows are open and surjective, and moreover

$$|V \times_Y X| \longrightarrow |V| \times_{|Z|} |Z \times_Y X|$$

is surjective. Hence as the left vertical arrow is open it follows that the right vertical arrow is open. This proves (2). The implication (2) $\Rightarrow$ (1) is immediate from the definitions. □

Thus we may use the following natural definition.

03Z2 **Definition 6.2.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say f is *open* if the map of topological spaces $|f| : |X| \to |Y|$ is open.
(2) We say f is *universally open* if for every morphism of algebraic spaces $Z \to Y$ the morphism of topological spaces
$$|Z \times_Y X| \to |Z|$$
is open, i.e., the base change $Z \times_Y X \to Z$ is open.

Note that an étale morphism of algebraic spaces is universally open, see Properties of Spaces, Definition 16.2 and Lemmas 16.7 and 16.5.

03Z3 **Lemma 6.3.** *The base change of a universally open morphism of algebraic spaces by any morphism of algebraic spaces is universally open.*

Proof. This is immediate from the definition. □

03Z4 **Lemma 6.4.** *The composition of a pair of (universally) open morphisms of algebraic spaces is (universally) open.*

Proof. Omitted. □

03Z5 **Lemma 6.5.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent*

(1) *f is universally open,*
(2) *for every scheme Z and every morphism $Z \to Y$ the projection $|Z \times_Y X| \to |Z|$ is open,*
(3) *for every affine scheme Z and every morphism $Z \to Y$ the projection $|Z \times_Y X| \to |Z|$ is open, and*
(4) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is a universally open morphism of algebraic spaces, and*
(5) *there exists a Zariski covering $Y = \bigcup Y_i$ such that each of the morphisms $f^{-1}(Y_i) \to Y_i$ is universally open.*

Proof. We omit the proof that (1) implies (2), and that (2) implies (3).

Assume (3). Choose a surjective étale morphism $V \to Y$. We are going to show that $V \times_Y X \to V$ is a universally open morphism of algebraic spaces. Let $Z \to V$ be a morphism from an algebraic space to V. Let $W \to Z$ be a surjective étale morphism where $W = \coprod W_i$ is a disjoint union of affine schemes, see Properties of Spaces, Lemma 6.1. Then we have the following commutative diagram

$$\begin{array}{ccccccc}
\coprod_i |W_i \times_Y X| & = & |W \times_Y X| & \longrightarrow & |Z \times_Y X| & = & |Z \times_V (V \times_Y X)| \\
\downarrow & & \downarrow & & \downarrow & & \swarrow \\
\coprod |W_i| & = & |W| & \longrightarrow & |Z| & &
\end{array}$$

We have to show the south-east arrow is open. The middle horizontal arrows are surjective and open (Properties of Spaces, Lemma 16.7). By assumption (3), and the fact that W_i is affine we see that the left vertical arrows are open. Hence it follows that the right vertical arrow is open.

Assume $V \to Y$ is as in (4). We will show that f is universally open. Let $Z \to Y$ be a morphism of algebraic spaces. Consider the diagram

$$\begin{array}{ccccc}
|(V \times_Y Z) \times_V (V \times_Y X)| & = & |V \times_Y X| & \longrightarrow & |Z \times_Y X| \\
& \searrow & \downarrow & & \downarrow \\
& & |V \times_Y Z| & \longrightarrow & |Z|
\end{array}$$

The south-west arrow is open by assumption. The horizontal arrows are surjective and open because the corresponding morphisms of algebraic spaces are étale (see Properties of Spaces, Lemma 16.7). It follows that the right vertical arrow is open.

Of course (1) implies (5) by taking the covering $Y = Y$. Assume $Y = \bigcup Y_i$ is as in (5). Then for any $Z \to Y$ we get a corresponding Zariski covering $Z = \bigcup Z_i$ such

that the base change of f to Z_i is open. By a simple topological argument this implies that $Z \times_Y X \to Z$ is open. Hence (1) holds. □

06DN **Lemma 6.6.** *Let S be a scheme. Let $p : X \to \mathrm{Spec}(k)$ be a morphism of algebraic spaces over S where k is a field. Then $p : X \to \mathrm{Spec}(k)$ is universally open.*

Proof. Choose a scheme U and a surjective étale morphism $U \to X$. The composition $U \to \mathrm{Spec}(k)$ is universally open (as a morphism of schemes) by Morphisms, Lemma 23.4. Let $Z \to \mathrm{Spec}(k)$ be a morphism of schemes. Then $U \times_{\mathrm{Spec}(k)} Z \to X \times_{\mathrm{Spec}(k)} Z$ is surjective, see Lemma 5.5. Hence the first of the maps

$$|U \times_{\mathrm{Spec}(k)} Z| \to |X \times_{\mathrm{Spec}(k)} Z| \to |Z|$$

is surjective. Since the composition is open by the above we conclude that the second map is open as well. Whence p is universally open by Lemma 6.5. □

7. Submersive morphisms

0411 For a representable morphism of algebraic spaces we have already defined (in Section 3) what it means to be universally submersive. Hence before we give the natural definition we check that it agrees with this in the representable case.

0CFQ **Lemma 7.1.** *Let S be a scheme. Let $f : X \to Y$ be a representable morphism of algebraic spaces over S. The following are equivalent*

1. *f is universally submersive (in the sense of Section 3), and*
2. *for every morphism of algebraic spaces $Z \to Y$ the morphism of topological spaces $|Z \times_Y X| \to |Z|$ is submersive.*

Proof. Assume (1), and let $Z \to Y$ be as in (2). Choose a scheme V and a surjective étale morphism $V \to Y$. By assumption the morphism of schemes $V \times_Y X \to V$ is universally submersive. By Properties of Spaces, Section 4 in the commutative diagram

$$\begin{array}{ccc} |V \times_Y X| & \longrightarrow & |Z \times_Y X| \\ \downarrow & & \downarrow \\ |V| & \longrightarrow & |Z| \end{array}$$

the horizontal arrows are open and surjective, and moreover

$$|V \times_Y X| \longrightarrow |V| \times_{|Z|} |Z \times_Y X|$$

is surjective. Hence as the left vertical arrow is submersive it follows that the right vertical arrow is submersive. This proves (2). The implication (2) $\Rightarrow$ (1) is immediate from the definitions. □

Thus we may use the following natural definition.

0412 **Definition 7.2.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

1. We say f is *submersive*[2] if the continuous map $|X| \to |Y|$ is submersive, see Topology, Definition 6.3.
2. We say f is *universally submersive* if for every morphism of algebraic spaces $Y' \to Y$ the base change $Y' \times_Y X \to Y'$ is submersive.

[2]This is very different from the notion of a submersion of differential manifolds.

We note that a submersive morphism is in particular surjective.

0CFR **Lemma 7.3.** *The base change of a universally submersive morphism of algebraic spaces by any morphism of algebraic spaces is universally submersive.*

Proof. This is immediate from the definition. □

0CFS **Lemma 7.4.** *The composition of a pair of (universally) submersive morphisms of algebraic spaces is (universally) submersive.*

Proof. Omitted. □

8. Quasi-compact morphisms

03HC By Section 3 we know what it means for a representable morphism of algebraic spaces to be quasi-compact. In order to formulate the definition for a general morphism of algebraic spaces we make the following observation.

03HD **Lemma 8.1.** *Let S be a scheme. Let $f : X \to Y$ be a representable morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is quasi-compact (in the sense of Section 3), and*
(2) *for every quasi-compact algebraic space Z and any morphism $Z \to Y$ the algebraic space $Z \times_Y X$ is quasi-compact.*

Proof. Assume (1), and let $Z \to Y$ be a morphism of algebraic spaces with Z quasi-compact. By Properties of Spaces, Definition 5.1 there exists a quasi-compact scheme U and a surjective étale morphism $U \to Z$. Since f is representable and quasi-compact we see by definition that $U \times_Y X$ is a scheme, and that $U \times_Y X \to U$ is quasi-compact. Hence $U \times_Y X$ is a quasi-compact scheme. The morphism $U \times_Y X \to Z \times_Y X$ is étale and surjective (as the base change of the representable étale and surjective morphism $U \to Z$, see Section 3). Hence by definition $Z \times_Y X$ is quasi-compact.

Assume (2). Let $Z \to Y$ be a morphism, where Z is a scheme. We have to show that $p : Z \times_Y X \to Z$ is quasi-compact. Let $U \subset Z$ be affine open. Then $p^{-1}(U) = U \times_Y Z$ and the scheme $U \times_Y Z$ is quasi-compact by assumption (2). Hence p is quasi-compact, see Schemes, Section 19. □

This motivates the following definition.

03HE **Definition 8.2.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. We say f is *quasi-compact* if for every quasi-compact algebraic space Z and morphism $Z \to Y$ the fibre product $Z \times_Y X$ is quasi-compact.

By Lemma 8.1 above this agrees with the already existing notion for representable morphisms of algebraic spaces.

0EMK **Lemma 8.3.** *Let S be a scheme. If $f : X \to Y$ is a quasi-compact morphism of algebraic spaces over S, then the underlying map $|f| : |X| \to |Y|$ of topological space is quasi-compact.*

Proof. Let $V \subset |Y|$ be quasi-compact open. By Properties of Spaces, Lemma 4.8 there is an open subspace $Y' \subset Y$ with $V = |Y'|$. Then Y' is a quasi-compact algebraic space by Properties of Spaces, Lemma 5.2 and hence $X' = Y' \times_Y X$ is a quasi-compact algebraic space by Definition 8.2. On the other hand, $X' \subset X$

is an open subspace (Spaces, Lemma 12.3) and $|X'| = |f|^{-1}(|X'|) = |f|^{-1}(V)$ by Properties of Spaces, Lemma 4.3. We conclude using Properties of Spaces, Lemma 5.2 again that $|X'|$ is a quasi-compact open of $|X|$ as desired. □

03HF **Lemma 8.4.** *The base change of a quasi-compact morphism of algebraic spaces by any morphism of algebraic spaces is quasi-compact.*

Proof. Omitted. Hint: Transitivity of fibre products. □

03HG **Lemma 8.5.** *The composition of a pair of quasi-compact morphisms of algebraic spaces is quasi-compact.*

Proof. Omitted. Hint: Transitivity of fibre products. □

040W **Lemma 8.6.** *Let S be a scheme.*

(1) *If $X \to Y$ is a surjective morphism of algebraic spaces over S, and X is quasi-compact then Y is quasi-compact.*
(2) *If*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & Z & \end{array}$$

is a commutative diagram of morphisms of algebraic spaces over S and f is surjective and p is quasi-compact, then q is quasi-compact.

Proof. Assume X is quasi-compact and $X \to Y$ is surjective. By Definition 5.2 the map $|X| \to |Y|$ is surjective, hence we see Y is quasi-compact by Properties of Spaces, Lemma 5.2 and the topological fact that the image of a quasi-compact space under a continuous map is quasi-compact, see Topology, Lemma 12.7. Let f, p, q be as in (2). Let $T \to Z$ be a morphism whose source is a quasi-compact algebraic space. By assumption $T \times_Z X$ is quasi-compact. By Lemma 5.5 the morphism $T \times_Z X \to T \times_Z Y$ is surjective. Hence by part (1) we see $T \times_Z Y$ is quasi-compact too. Thus q is quasi-compact. □

04ZJ **Lemma 8.7.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $g : Y' \to Y$ be a universally open and surjective morphism of algebraic spaces such that the base change $f' : X' \to Y'$ is quasi-compact. Then f is quasi-compact.*

Proof. Let $Z \to Y$ be a morphism of algebraic spaces with Z quasi-compact. As g is universally open and surjective, we see that $Y' \times_Y Z \to Z$ is open and surjective. As every point of $|Y' \times_Y Z|$ has a fundamental system of quasi-compact open neighbourhoods (see Properties of Spaces, Lemma 5.5) we can find a quasi-compact open $W \subset |Y' \times_Y Z|$ which surjects onto Z. Denote $f'' : W \times_Y X \to W$ the base change of f' by $W \to Y'$. By assumption $W \times_Y X$ is quasi-compact. As $W \to Z$ is surjective we see that $W \times_Y X \to Z \times_Y X$ is surjective. Hence $Z \times_Y X$ is quasi-compact by Lemma 8.6. Thus f is quasi-compact. □

03KG **Lemma 8.8.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is quasi-compact,*
(2) *for every scheme Z and any morphism $Z \to Y$ the morphism of algebraic spaces $Z \times_Y X \to Z$ is quasi-compact,*

(3) *for every affine scheme Z and any morphism $Z \to Y$ the algebraic space $Z \times_Y X$ is quasi-compact,*
(4) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is a quasi-compact morphism of algebraic spaces, and*
(5) *there exists a surjective étale morphism $Y' \to Y$ of algebraic spaces such that $Y' \times_Y X \to Y'$ is a quasi-compact morphism of algebraic spaces, and*
(6) *there exists a Zariski covering $Y = \bigcup Y_i$ such that each of the morphisms $f^{-1}(Y_i) \to Y_i$ is quasi-compact.*

Proof. We will use Lemma 8.4 without further mention. It is clear that (1) implies (2) and that (2) implies (3). Assume (3). Let Z be a quasi-compact algebraic space over S, and let $Z \to Y$ be a morphism. By Properties of Spaces, Lemma 6.3 there exists an affine scheme U and a surjective étale morphism $U \to Z$. Then $U \times_Y X \to Z \times_Y X$ is a surjective morphism of algebraic spaces, see Lemma 5.5. By assumption $|U \times_Y X|$ is quasi-compact. It surjects onto $|Z \times_Y X|$, hence we conclude that $|Z \times_Y X|$ is quasi-compact, see Topology, Lemma 12.7. This proves that (3) implies (1).

The implications (1) $\Rightarrow$ (4), (4) $\Rightarrow$ (5) are clear. The implication (5) $\Rightarrow$ (1) follows from Lemma 8.7 and the fact that an étale morphism of algebraic spaces is universally open (see discussion following Definition 6.2).

Of course (1) implies (6) by taking the covering $Y = Y$. Assume $Y = \bigcup Y_i$ is as in (6). Let Z be affine and let $Z \to Y$ be a morphism. Then there exists a finite standard affine covering $Z = Z_1 \cup \ldots \cup Z_n$ such that each $Z_j \to Y$ factors through Y_{i_j} for some i_j. Hence the algebraic space

$$Z_j \times_Y X = Z_j \times_{Y_{i_j}} f^{-1}(Y_{i_j})$$

is quasi-compact. Since $Z \times_Y X = \bigcup_{j=1,\ldots,n} Z_j \times_Y X$ is a Zariski covering we see that $|Z \times_Y X| = \bigcup_{j=1,\ldots,n} |Z_j \times_Y X|$ (see Properties of Spaces, Lemma 4.8) is a finite union of quasi-compact spaces, hence quasi-compact. Thus we see that (6) implies (3). □

The following (and the next) lemma guarantees in particular that a morphism $X \to \mathrm{Spec}(A)$ is quasi-compact as soon as X is a quasi-compact algebraic space

03KS **Lemma 8.9.** *Let S be a scheme. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of algebraic spaces over S. If $g \circ f$ is quasi-compact and g is quasi-separated then f is quasi-compact.*

Proof. This is true because f equals the composition $(1, f) : X \to X \times_Z Y \to Y$. The first map is quasi-compact by Lemma 4.7 because it is a section of the quasi-separated morphism $X \times_Z Y \to X$ (a base change of g, see Lemma 4.4). The second map is quasi-compact as it is the base change of f, see Lemma 8.4. And compositions of quasi-compact morphisms are quasi-compact, see Lemma 8.5. □

073B **Lemma 8.10.** *Let $f : X \to Y$ be a morphism of algebraic spaces over a scheme S.*

(1) *If X is quasi-compact and Y is quasi-separated, then f is quasi-compact.*
(2) *If X is quasi-compact and quasi-separated and Y is quasi-separated, then f is quasi-compact and quasi-separated.*

(3) *A fibre product of quasi-compact and quasi-separated algebraic spaces is quasi-compact and quasi-separated.*

Proof. Part (1) follows from Lemma 8.9 with $Z = S = \mathrm{Spec}(\mathbf{Z})$. Part (2) follows from (1) and Lemma 4.10. For (3) let $X \to Y$ and $Z \to Y$ be morphisms of quasi-compact and quasi-separated algebraic spaces. Then $X \times_Y Z \to Z$ is quasi-compact and quasi-separated as a base change of $X \to Y$ using (2) and Lemmas 8.4 and 4.4. Hence $X \times_Y Z$ is quasi-compact and quasi-separated as an algebraic space quasi-compact and quasi-separated over Z, see Lemmas 4.9 and 8.5. □

9. Universally closed morphisms

03HH For a representable morphism of algebraic spaces we have already defined (in Section 3) what it means to be universally closed. Hence before we give the natural definition we check that it agrees with this in the representable case.

03XD **Lemma 9.1.** *Let S be a scheme. Let $f : X \to Y$ be a representable morphism of algebraic spaces over S. The following are equivalent*

(1) *f is universally closed (in the sense of Section 3), and*
(2) *for every morphism of algebraic spaces $Z \to Y$ the morphism of topological spaces $|Z \times_Y X| \to |Z|$ is closed.*

Proof. Assume (1), and let $Z \to Y$ be as in (2). Choose a scheme V and a surjective étale morphism $V \to Y$. By assumption the morphism of schemes $V \times_Y X \to V$ is universally closed. By Properties of Spaces, Section 4 in the commutative diagram

$$\begin{array}{ccc} |V \times_Y X| & \longrightarrow & |Z \times_Y X| \\ \downarrow & & \downarrow \\ |V| & \longrightarrow & |Z| \end{array}$$

the horizontal arrows are open and surjective, and moreover

$$|V \times_Y X| \longrightarrow |V| \times_{|Z|} |Z \times_Y X|$$

is surjective. Hence as the left vertical arrow is closed it follows that the right vertical arrow is closed. This proves (2). The implication (2) $\Rightarrow$ (1) is immediate from the definitions. □

Thus we may use the following natural definition.

03HI **Definition 9.2.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say f is *closed* if the map of topological spaces $|X| \to |Y|$ is closed.
(2) We say f is *universally closed* if for every morphism of algebraic spaces $Z \to Y$ the morphism of topological spaces

$$|Z \times_Y X| \to |Z|$$

is closed, i.e., the base change $Z \times_Y X \to Z$ is closed.

03IS **Lemma 9.3.** *The base change of a universally closed morphism of algebraic spaces by any morphism of algebraic spaces is universally closed.*

Proof. This is immediate from the definition. □

03IU **Lemma 9.4.** *The composition of a pair of (universally) closed morphisms of algebraic spaces is (universally) closed.*

Proof. Omitted. □

03IT **Lemma 9.5.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent*

(1) *f is universally closed,*
(2) *for every scheme Z and every morphism $Z \to Y$ the projection $|Z \times_Y X| \to |Z|$ is closed,*
(3) *for every affine scheme Z and every morphism $Z \to Y$ the projection $|Z \times_Y X| \to |Z|$ is closed,*
(4) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is a universally closed morphism of algebraic spaces, and*
(5) *there exists a Zariski covering $Y = \bigcup Y_i$ such that each of the morphisms $f^{-1}(Y_i) \to Y_i$ is universally closed.*

Proof. We omit the proof that (1) implies (2), and that (2) implies (3).

Assume (3). Choose a surjective étale morphism $V \to Y$. We are going to show that $V \times_Y X \to V$ is a universally closed morphism of algebraic spaces. Let $Z \to V$ be a morphism from an algebraic space to V. Let $W \to Z$ be a surjective étale morphism where $W = \coprod W_i$ is a disjoint union of affine schemes, see Properties of Spaces, Lemma 6.1. Then we have the following commutative diagram

$$\begin{array}{ccccccc}
\coprod_i |W_i \times_Y X| & = & |W \times_Y X| & \longrightarrow & |Z \times_Y X| & = & |Z \times_V (V \times_Y X)| \\
\downarrow & & \downarrow & & \downarrow & \swarrow & \\
\coprod |W_i| & = & |W| & \longrightarrow & |Z| & &
\end{array}$$

We have to show the south-east arrow is closed. The middle horizontal arrows are surjective and open (Properties of Spaces, Lemma 16.7). By assumption (3), and the fact that W_i is affine we see that the left vertical arrows are closed. Hence it follows that the right vertical arrow is closed.

Assume (4). We will show that f is universally closed. Let $Z \to Y$ be a morphism of algebraic spaces. Consider the diagram

$$\begin{array}{ccccc}
|(V \times_Y Z) \times_V (V \times_Y X)| & = & |V \times_Y X| & \longrightarrow & |Z \times_Y X| \\
& \searrow & \downarrow & & \downarrow \\
& & |V \times_Y Z| & \longrightarrow & |Z|
\end{array}$$

The south-west arrow is closed by assumption. The horizontal arrows are surjective and open because the corresponding morphisms of algebraic spaces are étale (see Properties of Spaces, Lemma 16.7). It follows that the right vertical arrow is closed.

Of course (1) implies (5) by taking the covering $Y = Y$. Assume $Y = \bigcup Y_i$ is as in (5). Then for any $Z \to Y$ we get a corresponding Zariski covering $Z = \bigcup Z_i$ such that the base change of f to Z_i is closed. By a simple topological argument this implies that $Z \times_Y X \to Z$ is closed. Hence (1) holds. □

03IV **Example 9.6.** Strange example of a universally closed morphism. Let $\mathbf{Q} \subset k$ be a field of characteristic zero. Let $X = \mathbf{A}^1_k/\mathbf{Z}$ as in Spaces, Example 14.8. We claim the structure morphism $p : X \to \mathrm{Spec}(k)$ is universally closed. Namely, if Z/k is a scheme, and $T \subset |X \times_k Z|$ is closed, then T corresponds to a $\mathbf{Z}$-invariant closed subset of $T' \subset |\mathbf{A}^1 \times Z|$. It is easy to see that this implies that T' is the inverse image of a subset T'' of Z. By Morphisms, Lemma 25.12 we have that $T'' \subset Z$ is closed. Of course T'' is the image of T. Hence p is universally closed by Lemma 9.5.

04XW **Lemma 9.7.** *Let S be a scheme. A universally closed morphism of algebraic spaces over S is quasi-compact.*

Proof. This proof is a repeat of the proof in the case of schemes, see Morphisms, Lemma 41.8. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume that f is not quasi-compact. Our goal is to show that f is not universally closed. By Lemma 8.8 there exists an affine scheme Z and a morphism $Z \to Y$ such that $Z \times_Y X \to Z$ is not quasi-compact. To achieve our goal it suffices to show that $Z \times_Y X \to Z$ is not universally closed, hence we may assume that $Y = \mathrm{Spec}(B)$ for some ring B.

Write $X = \bigcup_{i \in I} X_i$ where the X_i are quasi-compact open subspaces of X. For example, choose a surjective étale morphism $U \to X$ where U is a scheme, choose an affine open covering $U = \bigcup U_i$ and let $X_i \subset X$ be the image of U_i. We will use later that the morphisms $X_i \to Y$ are quasi-compact, see Lemma 8.9. Let $T = \mathrm{Spec}(B[a_i; i \in I])$. Let $T_i = D(a_i) \subset T$. Let $Z \subset T \times_Y X$ be the reduced closed subspace whose underlying closed set of points is $|T \times_Y Z| \setminus \bigcup_{i \in I} |T_i \times_Y X_i|$, see Properties of Spaces, Lemma 12.3. (Note that $T_i \times_Y X_i$ is an open subspace of $T \times_Y X$ as $T_i \to T$ and $X_i \to X$ are open immersions, see Spaces, Lemmas 12.3 and 12.2.) Here is a diagram

$$\begin{array}{ccccc} Z & \longrightarrow & T \times_Y X & \xrightarrow[q]{} & X \\ & \searrow & \downarrow{\scriptstyle f_T} & & \downarrow{\scriptstyle f} \\ & & T & \xrightarrow{p} & Y \end{array}$$

It suffices to prove that the image $f_T(|Z|)$ is not closed in $|T|$.

We claim there exists a point $y \in Y$ such that there is no affine open neighborhood V of y in Y such that X_V is quasi-compact. If not then we can cover Y with finitely many such V and for each V the morphism $Y_V \to V$ is quasi-compact by Lemma 8.9 and then Lemma 8.8 implies f quasi-compact, a contradiction. Fix a $y \in Y$ as in the claim.

Let $t \in T$ be the point lying over y with $\kappa(t) = \kappa(y)$ such that $a_i = 1$ in $\kappa(t)$ for all i. Suppose $z \in |Z|$ with $f_T(z) = t$. Then $q(t) \in X_i$ for some i. Hence $f_T(z) \notin T_i$ by construction of Z, which contradicts the fact that $t \in T_i$ by construction. Hence we see that $t \in |T| \setminus f_T(|Z|)$.

Assume $f_T(|Z|)$ is closed in $|T|$. Then there exists an element $g \in B[a_i; i \in I]$ with $f_T(|Z|) \subset V(g)$ but $t \notin V(g)$. Hence the image of g in $\kappa(t)$ is nonzero. In particular some coefficient of g has nonzero image in $\kappa(y)$. Hence this coefficient is invertible on some affine open neighborhood V of y. Let J be the finite set of $j \in I$ such that the variable a_j appears in g. Since X_V is not quasi-compact and each $X_{i,V}$

is quasi-compact, we may choose a point $x \in |X_V| \setminus \bigcup_{j\in J} |X_{j,V}|$. In other words, $x \in |X| \setminus \bigcup_{j\in J} |X_j|$ and x lies above some $v \in V$. Since g has a coefficient that is invertible on V, we can find a point $t' \in T$ lying above v such that $t' \notin V(g)$ and $t' \in V(a_i)$ for all $i \notin J$. This is true because $V(a_i; i \in I \setminus J) = \mathrm{Spec}(B[a_j; j \in J])$ and the set of points of this scheme lying over v is bijective with $\mathrm{Spec}(\kappa(v)[a_j; j \in J])$ and g restricts to a nonzero element of this polynomial ring by construction. In other words $t' \notin T_i$ for each $i \notin J$. By Properties of Spaces, Lemma 4.3 we can find a point z of $X \times_Y T$ mapping to $x \in X$ and to $t' \in T$. Since $x \notin |X_j|$ for $j \in J$ and $t' \notin T_i$ for $i \in I \setminus J$ we see that $z \in |Z|$. On the other hand $f_T(z) = t' \notin V(g)$ which contradicts $f_T(Z) \subset V(g)$. Thus the assumption "$f_T(|Z|)$ closed" is wrong and we conclude indeed that f_T is not closed as desired. □

The target of a separated algebraic space under a surjective universally closed morphism is separated.

05Z2 **Lemma 9.8.** *Let S be a scheme. Let B be an algebraic space over S. Let $f : X \to Y$ be a surjective universally closed morphism of algebraic spaces over B.*

(1) *If X is quasi-separated, then Y is quasi-separated.*
(2) *If X is separated, then Y is separated.*
(3) *If X is quasi-separated over B, then Y is quasi-separated over B.*
(4) *If X is separated over B, then Y is separated over B.*

Proof. Parts (1) and (2) are a consequence of (3) and (4) for $S = B = \mathrm{Spec}(\mathbf{Z})$ (see Properties of Spaces, Definition 3.1). Consider the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\Delta_{X/B}} & X \times_B X \\ \downarrow & & \downarrow \\ Y & \xrightarrow{\Delta_{Y/B}} & Y \times_B Y \end{array}$$

The left vertical arrow is surjective (i.e., universally surjective). The right vertical arrow is universally closed as a composition of the universally closed morphisms $X \times_B X \to X \times_B Y \to Y \times_B Y$. Hence it is also quasi-compact, see Lemma 9.7.

Assume X is quasi-separated over B, i.e., $\Delta_{X/B}$ is quasi-compact. Then if Z is quasi-compact and $Z \to Y \times_B Y$ is a morphism, then $Z \times_{Y\times_B Y} X \to Z \times_{Y\times_B Y} Y$ is surjective and $Z \times_{Y\times_B Y} X$ is quasi-compact by our remarks above. We conclude that $\Delta_{Y/B}$ is quasi-compact, i.e., Y is quasi-separated over B.

Assume X is separated over B, i.e., $\Delta_{X/B}$ is a closed immersion. Then if Z is affine, and $Z \to Y \times_B Y$ is a morphism, then $Z \times_{Y\times_B Y} X \to Z \times_{Y\times_B Y} Y$ is surjective and $Z \times_{Y\times_B Y} X \to Z$ is universally closed by our remarks above. We conclude that $\Delta_{Y/B}$ is universally closed. It follows that $\Delta_{Y/B}$ is representable, locally of finite type, a monomorphism (see Lemma 4.1) and universally closed, hence a closed immersion, see Étale Morphisms, Lemma 7.2 (and also the abstract principle Spaces, Lemma 5.8). Thus Y is separated over B. □

10. Monomorphisms

042K A representable morphism $X \to Y$ of algebraic spaces is a monomorphism according to Section 3 if for every scheme Z and morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is representable by a monomorphism of schemes. This means exactly that $Z \times_Y X \to Z$ is an injective map of sheaves on $(Sch/S)_{fppf}$. Since this is supposed to hold for

all Z and all maps $Z \to Y$ this is in turn equivalent to the map $X \to Y$ being an injective map of sheaves on $(Sch/S)_{fppf}$. Thus we may define a monomorphism of a (possibly nonrepresentable[3]) morphism of algebraic spaces as follows.

042L **Definition 10.1.** Let S be a scheme. A morphism of algebraic spaces over S is called a *monomorphism* if it is an injective map of sheaves, i.e., a monomorphism in the category of sheaves on $(Sch/S)_{fppf}$.

The following lemma shows that this also means that it is a monomorphism in the category of algebraic spaces over S.

042M **Lemma 10.2.** *Let S be a scheme. Let $j : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *j is a monomorphism (as in Definition 10.1),*
(2) *j is a monomorphism in the category of algebraic spaces over S, and*
(3) *the diagonal morphism $\Delta_{X/Y} : X \to X \times_Y X$ is an isomorphism.*

Proof. Note that $X \times_Y X$ is both the fibre product in the category of sheaves on $(Sch/S)_{fppf}$ and the fibre product in the category of algebraic spaces over S, see Spaces, Lemma 7.3. The equivalence of (1) and (3) is a general characterization of injective maps of sheaves on any site. The equivalence of (2) and (3) is a characterization of monomorphisms in any category with fibre products. □

042N **Lemma 10.3.** *A monomorphism of algebraic spaces is separated.*

Proof. This is true because an isomorphism is a closed immersion, and Lemma 10.2 above. □

042O **Lemma 10.4.** *A composition of monomorphisms is a monomorphism.*

Proof. True because a composition of injective sheaf maps is injective. □

042P **Lemma 10.5.** *The base change of a monomorphism is a monomorphism.*

Proof. This is a general fact about fibre products in a category of sheaves. □

042Q **Lemma 10.6.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent*

(1) *f is a monomorphism,*
(2) *for every scheme Z and morphism $Z \to Y$ the base change $Z \times_Y X \to Z$ of f is a monomorphism,*
(3) *for every affine scheme Z and every morphism $Z \to Y$ the base change $Z \times_Y X \to Z$ of f is a monomorphism,*
(4) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that the base change $V \times_Y X \to V$ is a monomorphism, and*
(5) *there exists a Zariski covering $Y = \bigcup Y_i$ such that each of the morphisms $f^{-1}(Y_i) \to Y_i$ is a monomorphism.*

Proof. We will use without further mention that a base change of a monomorphism is a monomorphism, see Lemma 10.5. In particular it is clear that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) (by taking V to be a disjoint union of affine schemes étale over Y, see Properties of Spaces, Lemma 6.1). Let V be a scheme, and let $V \to Y$ be a

[3]We do not know whether any monomorphism of algebraic spaces is representable. For a discussion see More on Morphisms of Spaces, Section 4.

surjective étale morphism. If $V \times_Y X \to V$ is a monomorphism, then it follows that $X \to Y$ is a monomorphism. Namely, given any cartesian diagram of sheaves

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{a} & \mathcal{G} \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle c} \\ \mathcal{H} & \xrightarrow{d} & \mathcal{I} \end{array} \qquad \mathcal{F} = \mathcal{H} \times_{\mathcal{I}} \mathcal{G}$$

if c is a surjection of sheaves, and a is injective, then also d is injective. Thus (4) implies (1). Proof of the equivalence of (5) and (1) is omitted. □

042R **Lemma 10.7.** *An immersion of algebraic spaces is a monomorphism. In particular, any immersion is separated.*

Proof. Let $f : X \to Y$ be an immersion of algebraic spaces. For any morphism $Z \to Y$ with Z representable the base change $Z \times_Y X \to Z$ is an immersion of schemes, hence a monomorphism, see Schemes, Lemma 23.8. Hence f is representable, and a monomorphism. □

We will improve on the following lemma in Decent Spaces, Lemma 19.1.

06MG **Lemma 10.8.** *Let S be a scheme. Let k be a field and let $Z \to \mathrm{Spec}(k)$ be a monomorphism of algebraic spaces over S. Then either $Z = \emptyset$ or $Z = \mathrm{Spec}(k)$.*

Proof. By Lemmas 10.3 and 4.9 we see that Z is a separated algebraic space. Hence there exists an open dense subspace $Z' \subset Z$ which is a scheme, see Properties of Spaces, Proposition 13.3. By Schemes, Lemma 23.11 we see that either $Z' = \emptyset$ or $Z' \cong \mathrm{Spec}(k)$. In the first case we conclude that $Z = \emptyset$ and in the second case we conclude that $Z' = Z = \mathrm{Spec}(k)$ as $Z \to \mathrm{Spec}(k)$ is a monomorphism which is an isomorphism over Z'. □

06RV **Lemma 10.9.** *Let S be a scheme. If $X \to Y$ is a monomorphism of algebraic spaces over S, then $|X| \to |Y|$ is injective.*

Proof. Immediate from the definitions. □

11. Pushforward of quasi-coherent sheaves

03M7 We first prove a simple lemma that relates pushforward of sheaves of modules for a morphism of algebraic spaces to pushforward of sheaves of modules for a morphism of schemes.

03M8 **Lemma 11.1.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $U \to X$ be a surjective étale morphism from a scheme to X. Set $R = U \times_X U$ and denote $t, s : R \to U$ the projection morphisms as usual. Denote $a : U \to Y$ and $b : R \to Y$ the induced morphisms. For any object $\mathcal{F}$ of $Mod(\mathcal{O}_X)$ there exists an exact sequence*

$$0 \to f_*\mathcal{F} \to a_*(\mathcal{F}|_U) \to b_*(\mathcal{F}|_R)$$

where the second arrow is the difference $t^* - s^*$.

Proof. We denote $\mathcal{F}$ also its extension to a sheaf of modules on $X_{spaces,étale}$, see Properties of Spaces, Remark 18.4. Let $V \to Y$ be an object of $Y_{étale}$. Then $V \times_Y X$ is an object of $X_{spaces,étale}$, and by definition $f_*\mathcal{F}(V) = \mathcal{F}(V \times_Y X)$. Since $U \to X$ is surjective étale, we see that $\{V \times_Y U \to V \times_Y X\}$ is a covering.

Also, we have $(V \times_Y U) \times_X (V \times_Y U) = V \times_Y R$. Hence, by the sheaf condition of $\mathcal{F}$ on $X_{spaces,étale}$ we have a short exact sequence

$$0 \to \mathcal{F}(V \times_Y X) \to \mathcal{F}(V \times_Y U) \to \mathcal{F}(V \times_Y R)$$

where the second arrow is the difference of restricting via t or s. This exact sequence is functorial in V and hence we obtain the lemma. □

Let S be a scheme. Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of representable algebraic spaces X and Y over S. By Descent, Proposition 9.4 the functor $f_* : QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_Y)$ agrees with the usual functor if we think of X and Y as schemes.

More generally, suppose $f : X \to Y$ is a representable, quasi-compact, and quasi-separated morphism of algebraic spaces over S. Let V be a scheme and let $V \to Y$ be an étale surjective morphism. Let $U = V \times_Y X$ and let $f' : U \to V$ be the base change of f. Then for any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ we have

04CF (11.1.1) $$f'_*(\mathcal{F}|_U) = (f_*\mathcal{F})|_V,$$

see Properties of Spaces, Lemma 26.2. And because $f' : U \to V$ is a quasi-compact and quasi-separated morphism of schemes, by the remark of the preceding paragraph we may compute $f'_*(\mathcal{F}|_U)$ by thinking of $\mathcal{F}|_U$ as a quasi-coherent sheaf on the scheme U, and f' as a morphism of schemes. We will frequently use this without further mention.

The next level of generality is to consider an arbitrary quasi-compact and quasi-separated morphism of algebraic spaces.

03M9 **Lemma 11.2.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is quasi-compact and quasi-separated, then f_* transforms quasi-coherent $\mathcal{O}_X$-modules into quasi-coherent $\mathcal{O}_Y$-modules.*

Proof. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. We have to show that $f_*\mathcal{F}$ is a quasi-coherent sheaf on Y. For this it suffices to show that for any affine scheme V and étale morphism $V \to Y$ the restriction of $f_*\mathcal{F}$ to V is quasi-coherent, see Properties of Spaces, Lemma 29.6. Let $f' : V \times_Y X \to V$ be the base change of f by $V \to Y$. Note that f' is also quasi-compact and quasi-separated, see Lemmas 8.4 and 4.4. By (11.1.1) we know that the restriction of $f_*\mathcal{F}$ to V is f'_* of the restriction of $\mathcal{F}$ to $V \times_Y X$. Hence we may replace f by f', and assume that Y is an affine scheme.

Assume Y is an affine scheme. Since f is quasi-compact we see that X is quasi-compact. Thus we may choose an affine scheme U and a surjective étale morphism $U \to X$, see Properties of Spaces, Lemma 6.3. By Lemma 11.1 we get an exact sequence

$$0 \to f_*\mathcal{F} \to a_*(\mathcal{F}|_U) \to b_*(\mathcal{F}|_R).$$

where $R = U \times_X U$. As $X \to Y$ is quasi-separated we see that $R \to U \times_Y U$ is a quasi-compact monomorphism. This implies that R is a quasi-compact separated scheme (as U and Y are affine at this point). Hence $a : U \to Y$ and $b : R \to Y$ are quasi-compact and quasi-separated morphisms of schemes. Thus by Descent, Proposition 9.4 the sheaves $a_*(\mathcal{F}|_U)$ and $b_*(\mathcal{F}|_R)$ are quasi-coherent (see also the discussion preceding this lemma). This implies that $f_*\mathcal{F}$ is a kernel of quasi-coherent modules, and hence itself quasi-coherent, see Properties of Spaces, Lemma 29.7. □

Higher direct images are discussed in Cohomology of Spaces, Section 3.

12. Immersions

03HB Open, closed and locally closed immersions of algebraic spaces were defined in Spaces, Section 12. Namely, a morphism of algebraic spaces is a *closed immersion* (resp. *open immersion*, resp. *immersion*) if it is representable and a closed immersion (resp. open immersion, resp. immersion) in the sense of Section 3.

In particular these types of morphisms are stable under base change and compositions of morphisms in the category of algebraic spaces over S, see Spaces, Lemmas 12.2 and 12.3.

03M4 **Lemma 12.1.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

1. *f is a closed immersion (resp. open immersion, resp. immersion),*
2. *for every scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is a closed immersion (resp. open immersion, resp. immersion),*
3. *for every affine scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is a closed immersion (resp. open immersion, resp. immersion),*
4. *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is a closed immersion (resp. open immersion, resp. immersion), and*
5. *there exists a Zariski covering $Y = \bigcup Y_i$ such that each of the morphisms $f^{-1}(Y_i) \to Y_i$ is a closed immersion (resp. open immersion, resp. immersion).*

Proof. Using that a base change of a closed immersion (resp. open immersion, resp. immersion) is another one it is clear that (1) implies (2) and (2) implies (3). Also (3) implies (4) since we can take V to be a disjoint union of affines, see Properties of Spaces, Lemma 6.1.

Assume $V \to Y$ is as in (4). Let $\mathcal{P}$ be the property closed immersion (resp. open immersion, resp. immersion) of morphisms of schemes. Note that property $\mathcal{P}$ is preserved under any base change and fppf local on the base (see Section 3). Moreover, morphisms of type $\mathcal{P}$ are separated and locally quasi-finite (in each of the three cases, see Schemes, Lemma 23.8, and Morphisms, Lemma 20.16). Hence by More on Morphisms, Lemma 57.1 the morphisms of type $\mathcal{P}$ satisfy descent for fppf covering. Thus Spaces, Lemma 11.5 applies and we see that $X \to Y$ is representable and has property $\mathcal{P}$, in other words (1) holds.

The equivalence of (1) and (5) follows from the fact that $\mathcal{P}$ is Zariski local on the target (since we saw above that $\mathcal{P}$ is in fact fppf local on the target). □

0AGC **Lemma 12.2.** *Let S be a scheme. Let $Z \to Y \to X$ be morphisms of algebraic spaces over S.*

1. *If $Z \to X$ is representable, locally of finite type, locally quasi-finite, separated, and a monomorphism, then $Z \to Y$ is representable, locally of finite type, locally quasi-finite, separated, and a monomorphism.*
2. *If $Z \to X$ is an immersion and $Y \to X$ is locally separated, then $Z \to Y$ is an immersion.*

(3) *If $Z \to X$ is a closed immersion and $Y \to X$ is separated, then $Z \to Y$ is a closed immersion.*

Proof. In each case the proof is to contemplate the commutative diagram

$$\begin{array}{ccccc} Z & \longrightarrow & Y \times_X Z & \longrightarrow & Z \\ & \searrow & \downarrow & & \downarrow \\ & & Y & \longrightarrow & X \end{array}$$

where the composition of the top horizontal arrows is the identity. Let us prove (1). The first horizontal arrow is a section of $Y \times_X Z \to Z$, whence representable, locally of finite type, locally quasi-finite, separated, and a monomorphism by Lemma 4.7. The arrow $Y \times_X Z \to Y$ is a base change of $Z \to X$ hence is representable, locally of finite type, locally quasi-finite, separated, and a monomorphism (as each of these properties of morphisms of schemes is stable under base change, see Spaces, Remark 4.1). Hence the same is true for the composition (as each of these properties of morphisms of schemes is stable under composition, see Spaces, Remark 4.2). This proves (1). The other results are proved in exactly the same manner. □

04CD **Lemma 12.3.** *Let S be a scheme. Let $i : Z \to X$ be an immersion of algebraic spaces over S. Then $|i| : |Z| \to |X|$ is a homeomorphism onto a locally closed subset, and i is a closed immersion if and only if the image $|i|(|Z|) \subset |X|$ is a closed subset.*

Proof. The first statement is Properties of Spaces, Lemma 12.1. Let U be a scheme and let $U \to X$ be a surjective étale morphism. By assumption $T = U \times_X Z$ is a scheme and the morphism $j : T \to U$ is an immersion of schemes. By Lemma 12.1 the morphism i is a closed immersion if and only if j is a closed immersion. By Schemes, Lemma 10.4 this is true if and only if $j(T)$ is closed in U. However, the subset $j(T) \subset U$ is the inverse image of $|i|(|Z|) \subset |X|$, see Properties of Spaces, Lemma 4.3. This finishes the proof. □

04CE **Remark 12.4.** Let S be a scheme. Let $i : Z \to X$ be an immersion of algebraic spaces over S. Since i is a monomorphism we may think of $|Z|$ as a subset of $|X|$; in the rest of this remark we do so. Let $\partial|Z|$ be the boundary of $|Z|$ in the topological space $|X|$. In a formula

$$\partial|Z| = \overline{|Z|} \setminus |Z|.$$

Let ∂Z be the reduced closed subspace of X with $|\partial Z| = \partial|Z|$ obtained by taking the reduced induced closed subspace structure, see Properties of Spaces, Definition 12.5. By construction we see that $|Z|$ is closed in $|X| \setminus |\partial Z| = |X \setminus \partial Z|$. Hence it is true that any immersion of algebraic spaces can be factored as a closed immersion followed by an open immersion (but not the other way in general, see Morphisms, Example 3.4).

06EC **Remark 12.5.** Let S be a scheme. Let X be an algebraic space over S. Let $T \subset |X|$ be a locally closed subset. Let ∂T be the boundary of T in the topological space $|X|$. In a formula

$$\partial T = \overline{T} \setminus T.$$

Let $U \subset X$ be the open subspace of X with $|U| = |X| \setminus \partial T$, see Properties of Spaces, Lemma 4.8. Let Z be the reduced closed subspace of U with $|Z| = T$ obtained by taking the reduced induced closed subspace structure, see Properties of Spaces,

Definition 12.5. By construction $Z \to U$ is a closed immersion of algebraic spaces and $U \to X$ is an open immersion, hence $Z \to X$ is an immersion of algebraic spaces over S (see Spaces, Lemma 12.2). Note that Z is a reduced algebraic space and that $|Z| = T$ as subsets of $|X|$. We sometimes say Z is the *reduced induced subspace structure* on T.

081U **Lemma 12.6.** *Let S be a scheme. Let $Z \to X$ be an immersion of algebraic spaces over S. Assume $Z \to X$ is quasi-compact. There exists a factorization $Z \to \overline{Z} \to X$ where $Z \to \overline{Z}$ is an open immersion and $\overline{Z} \to X$ is a closed immersion.*

Proof. Let U be a scheme and let $U \to X$ be surjective étale. As usual denote $R = U \times_X U$ with projections $s, t : R \to U$. Set $T = Z \times_U X$. Let $\overline{T} \subset U$ be the scheme theoretic image of $T \to U$. Note that $s^{-1}\overline{T} = t^{-1}\overline{T}$ as taking scheme theoretic images of quasi-compact morphisms commute with flat base change, see Morphisms, Lemma 25.16. Hence we obtain a closed subspace $\overline{Z} \subset X$ whose pullback to U is $\overline{T}$, see Properties of Spaces, Lemma 12.2. By Morphisms, Lemma 7.7 the morphism $T \to \overline{T}$ is an open immersion. It follows that $Z \to \overline{Z}$ is an open immersion and we win. □

13. Closed immersions

03MA In this section we elucidate some of the results obtained previously on immersions of algebraic spaces. See Spaces, Section 12 and Section 12 in this chapter. This section is the analogue of Morphisms, Section 2 for algebraic spaces.

03MB **Lemma 13.1.** *Let S be a scheme. Let X be an algebraic space over S. For every closed immersion $i : Z \to X$ the sheaf $i_*\mathcal{O}_Z$ is a quasi-coherent $\mathcal{O}_X$-module, the map $i^\sharp : \mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective and its kernel is a quasi-coherent sheaf of ideals. The rule $Z \mapsto \mathrm{Ker}(\mathcal{O}_X \to i_*\mathcal{O}_Z)$ defines an inclusion reversing bijection*

$$\begin{array}{c} \text{closed subspaces} \\ Z \subset X \end{array} \longrightarrow \begin{array}{c} \text{quasi-coherent sheaves} \\ \text{of ideals } \mathcal{I} \subset \mathcal{O}_X \end{array}$$

Moreover, given a closed subscheme Z corresponding to the quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ a morphism of algebraic spaces $h : Y \to X$ factors through Z if and only if the map $h^*\mathcal{I} \to h^*\mathcal{O}_X = \mathcal{O}_Y$ is zero.

Proof. Let $U \to X$ be a surjective étale morphism whose source is a scheme. Consider the diagram

$$\begin{array}{ccc} U \times_X Z & \longrightarrow & Z \\ {\scriptstyle i'}\downarrow & & \downarrow{\scriptstyle i} \\ U & \longrightarrow & X \end{array}$$

By Lemma 12.1 we see that i is a closed immersion if and only if i' is a closed immersion. By Properties of Spaces, Lemma 26.2 we see that $i'_*\mathcal{O}_{U \times_X Z}$ is the restriction of $i_*\mathcal{O}_Z$ to U. Hence the assertions on $\mathcal{O}_X \to i_*\mathcal{O}_Z$ are equivalent to the corresponding assertions on $\mathcal{O}_U \to i'_*\mathcal{O}_{U \times_X Z}$. And since i' is a closed immersion of schemes, these results follow from Morphisms, Lemma 2.1.

Let us prove that given a quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ the formula

$$Z(T) = \{h : T \to X \mid h^*\mathcal{I} \to \mathcal{O}_T \text{ is zero}\}$$

defines a closed subspace of X. It is clearly a subfunctor of X. To show that $Z \to X$ is representable by closed immersions, let $\varphi : U \to X$ be a morphism from a scheme towards X. Then $Z \times_X U$ is represented by the analogous subfunctor of U corresponding to the sheaf of ideals $\mathrm{Im}(\varphi^*\mathcal{I} \to \mathcal{O}_U)$. By Properties of Spaces, Lemma 29.2 the $\mathcal{O}_U$-module $\varphi^*\mathcal{I}$ is quasi-coherent on U, and hence $\mathrm{Im}(\varphi^*\mathcal{I} \to \mathcal{O}_U)$ is a quasi-coherent sheaf of ideals on U. By Schemes, Lemma 4.6 we conclude that $Z \times_X U$ is represented by the closed subscheme of U associated to $\mathrm{Im}(\varphi^*\mathcal{I} \to \mathcal{O}_U)$. Thus Z is a closed subspace of X.

In the formula for Z above the inputs T are schemes since algebraic spaces are sheaves on $(Sch/S)_{fppf}$. We omit the verification that the same formula remains true if T is an algebraic space. □

083Q **Definition 13.2.** Let S be a scheme. Let $f : Y \to X$ be a morphism of algebraic spaces over S. Let $Z \subset X$ be a closed subspace. The *inverse image* $f^{-1}(Z)$ *of the closed subspace* Z is the closed subspace $Z \times_X Y$ of Y.

This definition makes sense by Lemma 12.1. If $\mathcal{I} \subset \mathcal{O}_X$ is the quasi-coherent sheaf of ideals corresponding to Z via Lemma 13.1 then $f^{-1}\mathcal{I}\mathcal{O}_Y = \mathrm{Im}(f^*\mathcal{I} \to \mathcal{O}_Y)$ is the sheaf of ideals corresponding to $f^{-1}(Z)$.

04CG **Lemma 13.3.** *A closed immersion of algebraic spaces is quasi-compact.*

Proof. This follows from Schemes, Lemma 19.5 by general principles, see Spaces, Lemma 5.8. □

04CH **Lemma 13.4.** *A closed immersion of algebraic spaces is separated.*

Proof. This follows from Schemes, Lemma 23.8 by general principles, see Spaces, Lemma 5.8. □

04E5 **Lemma 13.5.** *Let S be a scheme. Let $i : Z \to X$ be a closed immersion of algebraic spaces over S.*

(1) *The functor*
$$i_{small,*} : Sh(Z_{étale}) \longrightarrow Sh(X_{étale})$$
is fully faithful and its essential image is those sheaves of sets $\mathcal{F}$ on $X_{étale}$ whose restriction to $X \setminus Z$ is isomorphic to $*$, and

(2) *the functor*
$$i_{small,*} : Ab(Z_{étale}) \longrightarrow Ab(X_{étale})$$
is fully faithful and its essential image is those abelian sheaves on $X_{étale}$ whose support is contained in $|Z|$.

In both cases i_{small}^{-1} is a left inverse to the functor $i_{small,*}$.

Proof. Let U be a scheme and let $U \to X$ be surjective étale. Set $V = Z \times_X U$. Then V is a scheme and $i' : V \to U$ is a closed immersion of schemes. By Properties of Spaces, Lemma 18.12 for any sheaf $\mathcal{G}$ on Z we have
$$(i_{small}^{-1} i_{small,*}\mathcal{G})|_V = (i')_{small}^{-1} i'_{small,*}(\mathcal{G}|_V)$$
By Étale Cohomology, Proposition 46.4 the map $(i')_{small}^{-1} i'_{small,*}(\mathcal{G}|_V) \to \mathcal{G}|_V$ is an isomorphism. Since $V \to Z$ is surjective and étale this implies that $i_{small}^{-1} i_{small,*}\mathcal{G} \to \mathcal{G}$ is an isomorphism. This clearly implies that $i_{small,*}$ is fully faithful, see Sites, Lemma 41.1. To prove the statement on the essential image, consider a sheaf of

sets $\mathcal{F}$ on $X_{étale}$ whose restriction to $X \setminus Z$ is isomorphic to $*$. As in the proof of Étale Cohomology, Proposition 46.4 we consider the adjunction mapping

$$\mathcal{F} \longrightarrow i_{small,*}i_{small}^{-1}\mathcal{F}.$$

As in the first part we see that the restriction of this map to U is an isomorphism by the corresponding result for the case of schemes. Since U is an étale covering of X we conclude it is an isomorphism. □

0DK1 **Lemma 13.6.** *Let S be a scheme. Let $i : Z \to X$ be a closed immersion of algebraic spaces over S. Let $\overline{z}$ be a geometric point of Z with image $\overline{x}$ in X. Then $(i_{small,*}\mathcal{F})_{\overline{z}} = \mathcal{F}_{\overline{x}}$ for any sheaf $\mathcal{F}$ on $Z_{étale}$.*

Proof. Choose an étale neighbourhood $(U, \overline{u})$ of $\overline{x}$. Then the stalk $(i_{small,*}\mathcal{F})_{\overline{z}}$ is the stalk of $i_{small,*}\mathcal{F}|_U$ at $\overline{u}$. By Properties of Spaces, Lemma 18.12 we may replace X by U and Z by $Z \times_X U$. Then $Z \to X$ is a closed immersion of schemes and the result is Étale Cohomology, Lemma 46.3. □

The following lemma holds more generally in the setting of a closed immersion of topoi (insert future reference here).

04G0 **Lemma 13.7.** *Let S be a scheme. Let $i : Z \to X$ be a closed immersion of algebraic spaces over S. Let $\mathcal{A}$ be a sheaf of rings on $X_{étale}$. Let $\mathcal{B}$ be a sheaf of rings on $Z_{étale}$. Let $\varphi : \mathcal{A} \to i_{small,*}\mathcal{B}$ be a homomorphism of sheaves of rings so that we obtain a morphism of ringed topoi*

$$f : (Sh(Z_{étale}), \mathcal{B}) \longrightarrow (Sh(X_{étale}), \mathcal{A}).$$

For a sheaf of $\mathcal{A}$-modules $\mathcal{F}$ and a sheaf of $\mathcal{B}$-modules $\mathcal{G}$ the canonical map

$$\mathcal{F} \otimes_{\mathcal{A}} f_*\mathcal{G} \longrightarrow f_*(f^*\mathcal{F} \otimes_{\mathcal{B}} \mathcal{G}).$$

is an isomorphism.

Proof. The map is the map adjoint to the map

$$f^*\mathcal{F} \otimes_{\mathcal{B}} f^*f_*\mathcal{G} = f^*(\mathcal{F} \otimes_{\mathcal{A}} f_*\mathcal{G}) \longrightarrow f^*\mathcal{F} \otimes_{\mathcal{B}} \mathcal{G}$$

coming from $\mathrm{id} : f^*\mathcal{F} \to f^*\mathcal{F}$ and the adjunction map $f^*f_*\mathcal{G} \to \mathcal{G}$. To see this map is an isomorphism, we may check on stalks (Properties of Spaces, Theorem 19.12). Let $\overline{z} : \mathrm{Spec}(k) \to Z$ be a geometric point with image $\overline{x} = i \circ \overline{z} : \mathrm{Spec}(k) \to X$. Working out what our maps does on stalks, we see that we have to show

$$\mathcal{F}_{\overline{x}} \otimes_{\mathcal{A}_{\overline{x}}} \mathcal{G}_{\overline{z}} = (\mathcal{F}_{\overline{x}} \otimes_{\mathcal{A}_{\overline{x}}} \mathcal{B}_{\overline{z}}) \otimes_{\mathcal{B}_{\overline{z}}} \mathcal{G}_{\overline{z}}$$

which holds true. Here we have used that taking tensor products commutes with taking stalks, the behaviour of stalks under pullback Properties of Spaces, Lemma 19.9, and the behaviour of stalks under pushforward along a closed immersion Lemma 13.6. □

14. Closed immersions and quasi-coherent sheaves

04CI This section is the analogue of Morphisms, Section 4.

04CJ **Lemma 14.1.** *Let S be a scheme. Let $i : Z \to X$ be a closed immersion of algebraic spaces over S. Let $\mathcal{I} \subset \mathcal{O}_X$ be the quasi-coherent sheaf of ideals cutting out Z.*

(1) *For any $\mathcal{O}_X$-module $\mathcal{F}$ the adjunction map $\mathcal{F} \to i_*i^*\mathcal{F}$ induces an isomorphism $\mathcal{F}/\mathcal{I}\mathcal{F} \cong i_*i^*\mathcal{F}$.*

(2) *The functor i^* is a left inverse to i_*, i.e., for any $\mathcal{O}_Z$-module $\mathcal{G}$ the adjunction map $i^*i_*\mathcal{G} \to \mathcal{G}$ is an isomorphism.*
(3) *The functor*
$$i_* : QCoh(\mathcal{O}_Z) \longrightarrow QCoh(\mathcal{O}_X)$$
is exact, fully faithful, with essential image those quasi-coherent $\mathcal{O}_X$-modules $\mathcal{F}$ such that $\mathcal{I}\mathcal{F} = 0$.

Proof. During this proof we work exclusively with sheaves on the small étale sites, and we use $i_*, i^{-1}, \ldots$ to denote pushforward and pullback of sheaves of abelian groups instead of $i_{small,*}, i_{small}^{-1}$.

Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. By Lemma 13.7 applied with $\mathcal{A} = \mathcal{O}_X$ and $\mathcal{G} = \mathcal{B} = \mathcal{O}_Z$ we see that $i_*i^*\mathcal{F} = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_Z$. By Lemma 13.1 we see that we have a short exact sequence
$$0 \to \mathcal{I} \to \mathcal{O}_X \to i_*\mathcal{O}_Z \to 0$$
It follows from properties of the tensor product that $\mathcal{F} \otimes_{\mathcal{O}_X} i_*\mathcal{O}_Z = \mathcal{F}/\mathcal{I}\mathcal{F}$. This proves (1) (except that we omit the verification that the map is induced by the adjunction mapping).

Let $\mathcal{G}$ be any $\mathcal{O}_Z$-module. By Lemma 13.5 we see that $i^{-1}i_*\mathcal{G} = \mathcal{G}$. Hence to prove (2) we have to show that the canonical map $\mathcal{G} \otimes_{i^{-1}\mathcal{O}_X} \mathcal{O}_Z \to \mathcal{G}$ is an isomorphism. This follows from general properties of tensor products if we can show that $i^{-1}\mathcal{O}_X \to \mathcal{O}_Z$ is surjective. By Lemma 13.5 it suffices to prove that $i_*i^{-1}\mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective. Since the surjective map $\mathcal{O}_X \to i_*\mathcal{O}_Z$ factors through this map we see that (2) holds.

Finally we prove the most interesting part of the lemma, namely part (3). A closed immersion is quasi-compact and separated, see Lemmas 13.3 and 13.4. Hence Lemma 11.2 applies and the pushforward of a quasi-coherent sheaf on Z is indeed a quasi-coherent sheaf on X. Thus we obtain our functor $i_*^{QCoh} : QCoh(\mathcal{O}_Z) \to QCoh(\mathcal{O}_X)$. It is clear from part (2) that i_*^{QCoh} is fully faithful since it has a left inverse, namely i^*.

Now we turn to the description of the essential image of the functor i_*. It is clear that $\mathcal{I}(i_*\mathcal{G}) = 0$ for any $\mathcal{O}_Z$-module, since $\mathcal{I}$ is the kernel of the map $\mathcal{O}_X \to i_*\mathcal{O}_Z$ which is the map we use to put an $\mathcal{O}_X$-module structure on $i_*\mathcal{G}$. Next, suppose that $\mathcal{F}$ is any quasi-coherent $\mathcal{O}_X$-module such that $\mathcal{I}\mathcal{F} = 0$. Then we see that $\mathcal{F}$ is an $i_*\mathcal{O}_Z$-module because $i_*\mathcal{O}_Z = \mathcal{O}_X/\mathcal{I}$. Hence in particular its support is contained in $|Z|$. We apply Lemma 13.5 to see that $\mathcal{F} \cong i_*\mathcal{G}$ for some $\mathcal{O}_Z$-module $\mathcal{G}$. The only small detail left over is to see why $\mathcal{G}$ is quasi-coherent. This is true because $\mathcal{G} \cong i^*\mathcal{F}$ by part (2) and Properties of Spaces, Lemma 29.2. □

Let $i : Z \to X$ be a closed immersion of algebraic spaces. Because of the lemma above we often, by abuse of notation, denote $\mathcal{F}$ the sheaf $i_*\mathcal{F}$ on X.

04CK **Lemma 14.2.** *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\mathcal{G} \subset \mathcal{F}$ be a $\mathcal{O}_X$-submodule. There exists a unique quasi-coherent $\mathcal{O}_X$-submodule $\mathcal{G}' \subset \mathcal{G}$ with the following property: For every quasi-coherent $\mathcal{O}_X$-module $\mathcal{H}$ the map*
$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{H}, \mathcal{G}') \longrightarrow \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{H}, \mathcal{G})$$
is bijective. In particular $\mathcal{G}'$ is the largest quasi-coherent $\mathcal{O}_X$-submodule of $\mathcal{F}$ contained in $\mathcal{G}$.

Proof. Let $\mathcal{G}_a$, $a \in A$ be the set of quasi-coherent $\mathcal{O}_X$-submodules contained in $\mathcal{G}$. Then the image $\mathcal{G}'$ of

$$\bigoplus\nolimits_{a \in A} \mathcal{G}_a \longrightarrow \mathcal{F}$$

is quasi-coherent as the image of a map of quasi-coherent sheaves on X is quasi-coherent and since a direct sum of quasi-coherent sheaves is quasi-coherent, see Properties of Spaces, Lemma 29.7. The module $\mathcal{G}'$ is contained in $\mathcal{G}$. Hence this is the largest quasi-coherent $\mathcal{O}_X$-module contained in $\mathcal{G}$.

To prove the formula, let $\mathcal{H}$ be a quasi-coherent $\mathcal{O}_X$-module and let $\alpha : \mathcal{H} \to \mathcal{G}$ be an $\mathcal{O}_X$-module map. The image of the composition $\mathcal{H} \to \mathcal{G} \to \mathcal{F}$ is quasi-coherent as the image of a map of quasi-coherent sheaves. Hence it is contained in $\mathcal{G}'$. Hence α factors through $\mathcal{G}'$ as desired. □

04CL **Lemma 14.3.** *Let S be a scheme. Let $i : Z \to X$ be a closed immersion of algebraic spaces over S. There is a functor[4] $i^! : QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_Z)$ which is a right adjoint to i_*. (Compare Modules, Lemma 6.3.)*

Proof. Given quasi-coherent $\mathcal{O}_X$-module $\mathcal{G}$ we consider the subsheaf $\mathcal{H}_Z(\mathcal{G})$ of $\mathcal{G}$ of local sections annihilated by $\mathcal{I}$. By Lemma 14.2 there is a canonical largest quasi-coherent $\mathcal{O}_X$-submodule $\mathcal{H}_Z(\mathcal{G})'$. By construction we have

$$\mathrm{Hom}_{\mathcal{O}_X}(i_*\mathcal{F}, \mathcal{H}_Z(\mathcal{G})') = \mathrm{Hom}_{\mathcal{O}_X}(i_*\mathcal{F}, \mathcal{G})$$

for any quasi-coherent $\mathcal{O}_Z$-module $\mathcal{F}$. Hence we can set $i^!\mathcal{G} = i^*(\mathcal{H}_Z(\mathcal{G})')$. Details omitted. □

Using the 1-to-1 corresponding between quasi-coherent sheaves of ideals and closed subspaces (see Lemma 13.1) we can define scheme theoretic intersections and unions of closed subschemes.

0CYZ **Definition 14.4.** Let S be a scheme. Let X be an algebraic space over S. Let $Z, Y \subset X$ be closed subspaces corresponding to quasi-coherent ideal sheaves $\mathcal{I}, \mathcal{J} \subset \mathcal{O}_X$. The *scheme theoretic intersection* of Z and Y is the closed subspace of X cut out by $\mathcal{I} + \mathcal{J}$. Then *scheme theoretic union* of Z and Y is the closed subspace of X cut out by $\mathcal{I} \cap \mathcal{J}$.

It is clear that formation of scheme theoretic intersection commutes with étale localization and the same is true for scheme theoretic union.

0CZ0 **Lemma 14.5.** *Let S be a scheme. Let X be an algebraic space over S. Let $Z, Y \subset X$ be closed subspaces. Let $Z \cap Y$ be the scheme theoretic intersection of Z and Y. Then $Z \cap Y \to Z$ and $Z \cap Y \to Y$ are closed immersions and*

$$\begin{array}{ccc} Z \cap Y & \longrightarrow & Z \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

is a cartesian diagram of algebraic spaces over S, i.e., $Z \cap Y = Z \times_X Y$.

Proof. The morphisms $Z \cap Y \to Z$ and $Z \cap Y \to Y$ are closed immersions by Lemma 13.1. Since formation of the scheme theoretic intersection commutes with étale localization we conclude the diagram is cartesian by the case of schemes. See Morphisms, Lemma 4.5. □

[4] This is likely nonstandard notation.

0CZ1 **Lemma 14.6.** *Let S be a scheme. Let X be an algebraic space over S. Let $Y, Z \subset X$ be closed subspaces. Let $Y \cup Z$ be the scheme theoretic union of Y and Z. Let $Y \cap Z$ be the scheme theoretic intersection of Y and Z. Then $Y \to Y \cup Z$ and $Z \to Y \cup Z$ are closed immersions, there is a short exact sequence*

$$0 \to \mathcal{O}_{Y \cup Z} \to \mathcal{O}_Y \times \mathcal{O}_Z \to \mathcal{O}_{Y \cap Z} \to 0$$

of $\mathcal{O}_Z$-modules, and the diagram

$$\begin{array}{ccc} Y \cap Z & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Z & \longrightarrow & Y \cup Z \end{array}$$

is cocartesian in the category of algebraic spaces over S, i.e., $Y \cup Z = Y \amalg_{Y \cap Z} Z$.

Proof. The morphisms $Y \to Y \cup Z$ and $Z \to Y \cup Z$ are closed immersions by Lemma 13.1. In the short exact sequence we use the equivalence of Lemma 14.1 to think of quasi-coherent modules on closed subspaces of X as quasi-coherent modules on X. For the first map in the sequence we use the canonical maps $\mathcal{O}_{Y \cup Z} \to \mathcal{O}_Y$ and $\mathcal{O}_{Y \cup Z} \to \mathcal{O}_Z$ and for the second map we use the canonical map $\mathcal{O}_Y \to \mathcal{O}_{Y \cap Z}$ and the negative of the canonical map $\mathcal{O}_Z \to \mathcal{O}_{Y \cap Z}$. Then to check exactness we may work étale locally and deduce exactness from the case of schemes (Morphisms, Lemma 4.6).

To show the diagram is cocartesian, suppose we are given an algebraic space T over S and morphisms $f : Y \to T$, $g : Z \to T$ agreeing as morphisms $Y \cap Z \to T$. Goal: Show there exists a unique morphism $h : Y \cup Z \to T$ agreeing with f and g. To construct h we may work étale locally on $Y \cup Z$ (as $Y \cup Z$ is an étale sheaf being an algebraic space). Hence we may assume that X is a scheme. In this case we know that $Y \cup Z$ is the pushout of Y and Z along $Y \cap Z$ in the category of schemes by Morphisms, Lemma 4.6. Choose a scheme T' and a surjective étale morphism $T' \to T$. Set $Y' = T' \times_{T,f} Y$ and $Z' = T' \times_{T,g} Z$. Then Y' and Z' are schemes and we have a canonical isomorphism $\varphi : Y' \times_Y (Y \cap Z) \to Z' \times_Z (Y \cap Z)$ of schemes. By More on Morphisms, Lemma 67.8 the pushout $W' = Y' \amalg_{Y' \times_Y (Y \cap Z), \varphi} Z'$ exists in the category of schemes. The morphism $W' \to Y \cup Z$ is étale by More on Morphisms, Lemma 67.9. It is surjective as $Y' \to Y$ and $Z' \to Z$ are surjective. The morphisms $f' : Y' \to T'$ and $g' : Z' \to T'$ glue to a unique morphism of schemes $h' : W' \to T'$. By uniqueness the composition $W' \to T' \to T$ descends to the desired morphism $h : Y \cup Z \to T$. Some details omitted. □

15. Supports of modules

07TX In this section we collect some elementary results on supports of quasi-coherent modules on algebraic spaces. Let X be an algebraic space. The support of an abelian sheaf on $X_{étale}$ has been defined in Properties of Spaces, Section 20. We use the same definition for supports of modules. The following lemma tells us this agrees with the notion as defined for quasi-coherent modules on schemes.

07TY **Lemma 15.1.** *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let U be a scheme and let $\varphi : U \to X$ be an étale morphism. Then*

$$Supp(\varphi^* \mathcal{F}) = |\varphi|^{-1}(Supp(\mathcal{F}))$$

where the left hand side is the support of $\varphi^*\mathcal{F}$ as a quasi-coherent module on the scheme U.

Proof. Let $u \in U$ be a (usual) point and let $\overline{x}$ be a geometric point lying over u. By Properties of Spaces, Lemma 29.4 we have $(\varphi^*\mathcal{F})_u \otimes_{\mathcal{O}_{U,u}} \mathcal{O}_{X,\overline{x}} = \mathcal{F}_{\overline{x}}$. Since $\mathcal{O}_{U,u} \to \mathcal{O}_{X,\overline{x}}$ is the strict henselization by Properties of Spaces, Lemma 22.1 we see that it is faithfully flat (see More on Algebra, Lemma 45.1). Thus we see that $(\varphi^*\mathcal{F})_u = 0$ if and only if $\mathcal{F}_{\overline{x}} = 0$. This proves the lemma. □

For finite type quasi-coherent modules the support is closed, can be checked on fibres, and commutes with base change.

07TZ **Lemma 15.2.** *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Then*

1. *The support of $\mathcal{F}$ is closed.*
2. *For a geometric point $\overline{x}$ lying over $x \in |X|$ we have*
$$x \in \mathit{Supp}(\mathcal{F}) \Leftrightarrow \mathcal{F}_{\overline{x}} \neq 0 \Leftrightarrow \mathcal{F}_{\overline{x}} \otimes_{\mathcal{O}_{X,\overline{x}}} \kappa(\overline{x}) \neq 0.$$
3. *For any morphism of algebraic spaces $f : Y \to X$ the pullback $f^*\mathcal{F}$ is of finite type as well and we have $\mathit{Supp}(f^*\mathcal{F}) = f^{-1}(\mathit{Supp}(\mathcal{F}))$.*

Proof. Choose a scheme U and a surjective étale morphism $\varphi : U \to X$. By Lemma 15.1 the inverse image of the support of $\mathcal{F}$ is the support of $\varphi^*\mathcal{F}$ which is closed by Morphisms, Lemma 5.3. Thus (1) follows from the definition of the topology on $|X|$.

The first equivalence in (2) is the definition of support. The second equivalence follows from Nakayama's lemma, see Algebra, Lemma 20.1.

Let $f : Y \to X$ be as in (3). Note that $f^*\mathcal{F}$ is of finite type by Properties of Spaces, Section 30. For the final assertion, let $\overline{y}$ be a geometric point of Y mapping to the geometric point $\overline{x}$ on X. Recall that
$$(f^*\mathcal{F})_{\overline{y}} = \mathcal{F}_{\overline{x}} \otimes_{\mathcal{O}_{X,\overline{x}}} \mathcal{O}_{Y,\overline{y}},$$
see Properties of Spaces, Lemma 29.5. Hence $(f^*\mathcal{F})_{\overline{y}} \otimes \kappa(\overline{y})$ is nonzero if and only if $\mathcal{F}_{\overline{x}} \otimes \kappa(\overline{x})$ is nonzero. By (2) this implies $x \in \mathrm{Supp}(\mathcal{F})$ if and only if $y \in \mathrm{Supp}(f^*\mathcal{F})$, which is the content of assertion (3). □

Our next task is to show that the scheme theoretic support of a finite type quasi-coherent module (see Morphisms, Definition 5.5) also makes sense for finite type quasi-coherent modules on algebraic spaces.

07U0 **Lemma 15.3.** *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. There exists a smallest closed subspace $i : Z \to X$ such that there exists a quasi-coherent $\mathcal{O}_Z$-module $\mathcal{G}$ with $i_*\mathcal{G} \cong \mathcal{F}$. Moreover:*

1. *If U is a scheme and $\varphi : U \to X$ is an étale morphism then $Z \times_X U$ is the scheme theoretic support of $\varphi^*\mathcal{F}$.*
2. *The quasi-coherent sheaf $\mathcal{G}$ is unique up to unique isomorphism.*
3. *The quasi-coherent sheaf $\mathcal{G}$ is of finite type.*
4. *The support of $\mathcal{G}$ and of $\mathcal{F}$ is $|Z|$.*

Proof. Choose a scheme U and a surjective étale morphism $\varphi : U \to X$. Let $R = U \times_X U$ with projections $s, t : R \to U$. Let $i' : Z' \to U$ be the scheme theoretic support of $\varphi^*\mathcal{F}$ and let $\mathcal{G}'$ be the (unique up to unique isomorphism) finite type quasi-coherent $\mathcal{O}_{Z'}$-module with $i'_*\mathcal{G}' = \varphi^*\mathcal{F}$, see Morphisms, Lemma 5.4. As $s^*\varphi^*\mathcal{F} = t^*\varphi^*\mathcal{F}$ we see that $R' = s^{-1}Z' = t^{-1}Z'$ as closed subschemes of R by Morphisms, Lemma 25.14. Thus we may apply Properties of Spaces, Lemma 12.2 to find a closed subspace $i : Z \to X$ whose pullback to U is Z'. Writing $s', t' : R' \to Z'$ the projections and $j' : R' \to R$ the given closed immersion, we see that

$$j'_*(s')^*\mathcal{G}' = s^*i'_*\mathcal{G}' = s^*\varphi^*\mathcal{F} = t^*\varphi^*\mathcal{F} = t^*i'_*\mathcal{G}' = j'_*(t')^*\mathcal{G}'$$

(the first and the last equality by Cohomology of Schemes, Lemma 5.2). Hence the uniqueness of Morphisms, Lemma 25.14 applied to $R' \to R$ gives an isomorphism $\alpha : (t')^*\mathcal{G}' \to (s')^*\mathcal{G}'$ compatible with the canonical isomorphism $t^*\varphi^*\mathcal{F} = s^*\varphi^*\mathcal{F}$ via j'_*. Clearly α satisfies the cocycle condition, hence we may apply Properties of Spaces, Proposition 32.1 to obtain a quasi-coherent module $\mathcal{G}$ on Z whose restriction to Z' is $\mathcal{G}'$ compatible with α. Again using the equivalence of the proposition mentioned above (this time for X) we conclude that $i_*\mathcal{G} \cong \mathcal{F}$.

This proves existence. The other properties of the lemma follow by comparing with the result for schemes using Lemma 15.1. Detailed proofs omitted. □

07U1 **Definition 15.4.** Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. The *scheme theoretic support of* $\mathcal{F}$ is the closed subspace $Z \subset X$ constructed in Lemma 15.3.

In this situation we often think of $\mathcal{F}$ as a quasi-coherent sheaf of finite type on Z (via the equivalence of categories of Lemma 14.1).

16. Scheme theoretic image

082W Caution: Some of the material in this section is ultra-general and behaves differently from what you might expect.

082X **Lemma 16.1.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. There exists a closed subspace $Z \subset Y$ such that f factors through Z and such that for any other closed subspace $Z' \subset Y$ such that f factors through Z' we have $Z \subset Z'$.*

Proof. Let $\mathcal{I} = \mathrm{Ker}(\mathcal{O}_Y \to f_*\mathcal{O}_X)$. If $\mathcal{I}$ is quasi-coherent then we just take Z to be the closed subscheme determined by $\mathcal{I}$, see Lemma 13.1. In general the lemma requires us to show that there exists a largest quasi-coherent sheaf of ideals $\mathcal{I}'$ contained in $\mathcal{I}$. This follows from Lemma 14.2. □

Suppose that in the situation of Lemma 16.1 above X and Y are representable. Then the closed subspace $Z \subset Y$ found in the lemma agrees with the closed subscheme $Z \subset Y$ found in Morphisms, Lemma 6.1. The reason is that closed subspaces (or subschemes) are in a inclusion reversing correspondence with quasi-coherent ideal sheaves on $X_{étale}$ and X. As the category of quasi-coherent modules on $X_{étale}$ and X are the same (Properties of Spaces, Section 29) we conclude. Thus the following definition agrees with the earlier definition for morphisms of schemes.

082Y **Definition 16.2.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The *scheme theoretic image* of f is the smallest closed subspace $Z \subset Y$ through which f factors, see Lemma 16.1 above.

We often just denote $f : X \to Z$ the factorization of f. If the morphism f is not quasi-compact, then (in general) the construction of the scheme theoretic image does not commute with restriction to open subspaces of Y.

082Z **Lemma 16.3.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $Z \subset Y$ be the scheme theoretic image of f. If f is quasi-compact then*

1. *the sheaf of ideals $\mathcal{I} = \mathrm{Ker}(\mathcal{O}_Y \to f_*\mathcal{O}_X)$ is quasi-coherent,*
2. *the scheme theoretic image Z is the closed subspace corresponding to $\mathcal{I}$,*
3. *for any étale morphism $V \to Y$ the scheme theoretic image of $X \times_Y V \to V$ is equal to $Z \times_Y V$, and*
4. *the image $|f|(|X|) \subset |Z|$ is a dense subset of $|Z|$.*

Proof. To prove (3) it suffices to prove (1) and (2) since the formation of $\mathcal{I}$ commutes with étale localization. If (1) holds then in the proof of Lemma 16.1 we showed (2). Let us prove that $\mathcal{I}$ is quasi-coherent. Since the property of being quasi-coherent is étale local we may assume Y is an affine scheme. As f is quasi-compact, we can find an affine scheme U and a surjective étale morphism $U \to X$. Denote f' the composition $U \to X \to Y$. Then $f_*\mathcal{O}_X$ is a subsheaf of $f'_*\mathcal{O}_U$, and hence $\mathcal{I} = \mathrm{Ker}(\mathcal{O}_Y \to \mathcal{O}_{X'})$. By Lemma 11.2 the sheaf $f'_*\mathcal{O}_U$ is quasi-coherent on Y. Hence $\mathcal{I}$ is quasi-coherent as a kernel of a map between coherent modules. Finally, part (4) follows from parts (1), (2), and (3) as the ideal $\mathcal{I}$ will be the unit ideal in any point of $|Y|$ which is not contained in the closure of $|f|(|X|)$. □

0830 **Lemma 16.4.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume X is reduced. Then*

1. *the scheme theoretic image Z of f is the reduced induced algebraic space structure on $\overline{|f|(|X|)}$, and*
2. *for any étale morphism $V \to Y$ the scheme theoretic image of $X \times_Y V \to V$ is equal to $Z \times_Y V$.*

Proof. Part (1) is true because the reduced induced algebraic space structure on $\overline{|f|(|X|)}$ is the smallest closed subspace of Y through which f factors, see Properties of Spaces, Lemma 12.4. Part (2) follows from (1), the fact that $|V| \to |Y|$ is open, and the fact that being reduced is preserved under étale localization. □

089B **Lemma 16.5.** *Let S be a scheme. Let $f : X \to Y$ be a quasi-compact morphism of algebraic spaces over S. Let Z be the scheme theoretic image of f. Let $z \in |Z|$. There exists a valuation ring A with fraction field K and a commutative diagram*

$$\begin{array}{ccccc} \mathrm{Spec}(K) & & \longrightarrow & & X \\ \downarrow & & & \swarrow & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & Z & \longrightarrow & Y \end{array}$$

such that the closed point of $\mathrm{Spec}(A)$ maps to z.

Proof. Choose an affine scheme V with a point $z' \in V$ and an étale morphism $V \to Y$ mapping z' to z. Let $Z' \subset V$ be the scheme theoretic image of $X \times_Y V \to V$. By Lemma 16.3 we have $Z' = Z \times_Y V$. Thus $z' \in Z'$. Since f is quasi-compact and V is affine we see that $X \times_Y V$ is quasi-compact. Hence there exists an affine scheme W and a surjective étale morphism $W \to X \times_Y V$. Then $Z' \subset V$ is also the scheme theoretic image of $W \to V$. By Morphisms, Lemma 6.5 we can choose a diagram

$$\begin{array}{ccccccc} \mathrm{Spec}(K) & \longrightarrow & W & \longrightarrow & X \times_Y V & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & Z' & \longrightarrow & V & \longrightarrow & Y \end{array}$$

such that the closed point of $\mathrm{Spec}(A)$ maps to z'. Composing with $Z' \to Z$ and $W \to X \times_Y V \to X$ we obtain a solution. □

0CP2 **Lemma 16.6.** *Let S be a scheme. Let*

$$\begin{array}{ccc} X_1 & \xrightarrow[f_1]{} & Y_1 \\ \downarrow & & \downarrow \\ X_2 & \xrightarrow{f_2} & Y_2 \end{array}$$

be a commutative diagram of algebraic spaces over S. Let $Z_i \subset Y_i$, $i = 1, 2$ be the scheme theoretic image of f_i. Then the morphism $Y_1 \to Y_2$ induces a morphism $Z_1 \to Z_2$ and a commutative diagram

$$\begin{array}{ccccc} X_1 & \longrightarrow & Z_1 & \longrightarrow & Y_1 \\ \downarrow & & \downarrow & & \downarrow \\ X_2 & \longrightarrow & Z_2 & \longrightarrow & Y_2 \end{array}$$

Proof. The scheme theoretic inverse image of Z_2 in Y_1 is a closed subspace of Y_1 through which f_1 factors. Hence Z_1 is contained in this. This proves the lemma. □

0CP3 **Lemma 16.7.** *Let S be a scheme. Let $f : X \to Y$ be a separated morphism of algebraic spaces over S. Let $V \subset Y$ be an open subspace such that $V \to Y$ is quasi-compact. Let $s : V \to X$ be a morphism such that $f \circ s = id_V$. Let Y' be the scheme theoretic image of s. Then $Y' \to Y$ is an isomorphism over V.*

Proof. By Lemma 8.9 the morphism $s : V \to X$ is quasi-compact. Hence the construction of the scheme theoretic image Y' of s commutes with restriction to opens by Lemma 16.3. In particular, we see that $Y' \cap f^{-1}(V)$ is the scheme theoretic image of a section of the separated morphism $f^{-1}(V) \to V$. Since a section of a separated morphism is a closed immersion (Lemma 4.7), we conclude that $Y' \cap f^{-1}(V) \to V$ is an isomorphism as desired. □

17. Scheme theoretic closure and density

0831 This section is the analogue of Morphisms, Section 7.

0832 **Lemma 17.1.** *Let S be a scheme. Let $W \subset S$ be a scheme theoretically dense open subscheme (Morphisms, Definition 7.1). Let $f : X \to S$ be a morphism of schemes which is flat, locally of finite presentation, and locally quasi-finite. Then $f^{-1}(W)$ is scheme theoretically dense in X.*

Proof. We will use the characterization of Morphisms, Lemma 7.5. Assume $V \subset X$ is an open and $g \in \Gamma(V, \mathcal{O}_V)$ is a function which restricts to zero on $f^{-1}(W) \cap V$. We have to show that $g = 0$. Assume $g \neq 0$ to get a contradiction. By More on Morphisms, Lemma 45.6 we may shrink V, find an open $U \subset S$ fitting into a commutative diagram

$$\begin{array}{ccc} V & \longrightarrow & X \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle f} \\ U & \longrightarrow & S, \end{array}$$

a quasi-coherent subsheaf $\mathcal{F} \subset \mathcal{O}_U$, an integer $r > 0$, and an injective $\mathcal{O}_U$-module map $\mathcal{F}^{\oplus r} \to \pi_*\mathcal{O}_V$ whose image contains $g|_V$. Say $(g_1, \ldots, g_r) \in \Gamma(U, \mathcal{F}^{\oplus r})$ maps to g. Then we see that $g_i|_{W \cap U} = 0$ because $g|_{f^{-1}W \cap V} = 0$. Hence $g_i = 0$ because $\mathcal{F} \subset \mathcal{O}_U$ and W is scheme theoretically dense in S. This implies $g = 0$ which is the desired contradiction. □

0833 **Lemma 17.2.** *Let S be a scheme. Let X be an algebraic space over S. Let $U \subset X$ be an open subspace. The following are equivalent*

1. *for every étale morphism $\varphi : V \to X$ (of algebraic spaces) the scheme theoretic closure of $\varphi^{-1}(U)$ in V is equal to V,*
2. *there exists a scheme V and a surjective étale morphism $\varphi : V \to X$ such that the scheme theoretic closure of $\varphi^{-1}(U)$ in V is equal to V,*

Proof. Observe that if $V \to V'$ is a morphism of algebraic spaces étale over X, and $Z \subset V$, resp. $Z' \subset V'$ is the scheme theoretic closure of $U \times_X V$, resp. $U \times_X V'$ in V, resp. V', then Z maps into Z'. Thus if $V \to V'$ is surjective and étale then $Z = V$ implies $Z' = V'$. Next, note that an étale morphism is flat, locally of finite presentation, and locally quasi-finite (see Morphisms, Section 36). Thus Lemma 17.1 implies that if V and V' are schemes, then $Z' = V'$ implies $Z = V$. A formal argument using that every algebraic space has an étale covering by a scheme shows that (1) and (2) are equivalent. □

It follows from Lemma 17.2 that the following definition is compatible with the definition in the case of schemes.

0834 **Definition 17.3.** Let S be a scheme. Let X be an algebraic space over S. Let $U \subset X$ be an open subspace.

1. The scheme theoretic image of the morphism $U \to X$ is called the *scheme theoretic closure of U in X*.
2. We say U is *scheme theoretically dense in X* if the equivalent conditions of Lemma 17.2 are satisfied.

With this definition it is **not** the case that U is scheme theoretically dense in X if and only if the scheme theoretic closure of U is X. This is somewhat inelegant. But with suitable finiteness conditions we will see that it does hold.

0835 **Lemma 17.4.** *Let S be a scheme. Let X be an algebraic space over S. Let $U \subset X$ be an open subspace. If $U \to X$ is quasi-compact, then U is scheme theoretically dense in X if and only if the scheme theoretic closure of U in X is X.*

Proof. Follows from Lemma 16.3 part (3). □

0836 **Lemma 17.5.** *Let S be a scheme. Let $j : U \to X$ be an open immersion of algebraic spaces over S. Then U is scheme theoretically dense in X if and only if $\mathcal{O}_X \to j_*\mathcal{O}_U$ is injective.*

Proof. If $\mathcal{O}_X \to j_*\mathcal{O}_U$ is injective, then the same is true when restricted to any algebraic space V étale over X. Hence the scheme theoretic closure of $U \times_X V$ in V is equal to V, see proof of Lemma 16.1. Conversely, assume the scheme theoretic closure of $U \times_X V$ is equal to V for all V étale over X. Suppose that $\mathcal{O}_X \to j_*\mathcal{O}_U$ is not injective. Then we can find an affine, say $V = \mathrm{Spec}(A)$, étale over X and a nonzero element $f \in A$ such that f maps to zero in $\Gamma(V \times_X U, \mathcal{O})$. In this case the scheme theoretic closure of $V \times_X U$ in V is clearly contained in $\mathrm{Spec}(A/(f))$ a contradiction. □

0837 **Lemma 17.6.** *Let S be a scheme. Let X be an algebraic space over S. If U, V are scheme theoretically dense open subspaces of X, then so is $U \cap V$.*

Proof. Let $W \to X$ be any étale morphism. Consider the map $\mathcal{O}(W) \to \mathcal{O}(W \times_X V) \to \mathcal{O}(W \times_X (V \cap U))$. By Lemma 17.5 both maps are injective. Hence the composite is injective. Hence by Lemma 17.5 $U \cap V$ is scheme theoretically dense in X. □

088G **Lemma 17.7.** *Let S be a scheme. Let $h : Z \to X$ be an immersion of algebraic spaces over S. Assume either $Z \to X$ is quasi-compact or Z is reduced. Let $\overline{Z} \subset X$ be the scheme theoretic image of h. Then the morphism $Z \to \overline{Z}$ is an open immersion which identifies Z with a scheme theoretically dense open subspace of $\overline{Z}$. Moreover, Z is topologically dense in $\overline{Z}$.*

Proof. In both cases the formation of $\overline{Z}$ commutes with étale localization, see Lemmas 16.3 and 16.4. Hence this lemma follows from the case of schemes, see Morphisms, Lemma 7.7. □

084N **Lemma 17.8.** *Let S be a scheme. Let B be an algebraic space over S. Let $f, g : X \to Y$ be morphisms of algebraic spaces over B. Let $U \subset X$ be an open subspace such that $f|_U = g|_U$. If the scheme theoretic closure of U in X is X and $Y \to B$ is separated, then $f = g$.*

Proof. As $Y \to B$ is separated the fibre product $Y \times_{\Delta, Y \times_B Y, (f,g)} X$ is a closed subspace $Z \subset X$. As $f|_U = g|_U$ we see that $U \subset Z$. Hence $Z = X$ as U is assumed scheme theoretically dense in X. □

18. Dominant morphisms

0ABK We copy the definition of a dominant morphism of schemes to get the notion of a dominant morphism of algebraic spaces. We caution the reader that this definition is not well behaved unless the morphism is quasi-compact and the algebraic spaces satisfy some separation axioms.

0ABL **Definition 18.1.** Let S be a scheme. A morphism $f : X \to Y$ of algebraic spaces over S is called *dominant* if the image of $|f| : |X| \to |Y|$ is dense in $|Y|$.

19. Universally injective morphisms

03MT We have already defined in Section 3 what it means for a representable morphism of algebraic spaces to be universally injective. For a field K over S (recall this means that we are given a structure morphism $\mathrm{Spec}(K) \to S$) and an algebraic space X over S we write $X(K) = \mathrm{Mor}_S(\mathrm{Spec}(K), X)$. We first translate the condition for representable morphisms into a condition on the functor of points.

03MU **Lemma 19.1.** *Let S be a scheme. Let $f : X \to Y$ be a representable morphism of algebraic spaces over S. Then f is universally injective (in the sense of Section 3) if and only if for all fields K the map $X(K) \to Y(K)$ is injective.*

Proof. We are going to use Morphisms, Lemma 10.2 without further mention. Suppose that f is universally injective. Then for any field K and any morphism $\mathrm{Spec}(K) \to Y$ the morphism of schemes $\mathrm{Spec}(K) \times_Y X \to \mathrm{Spec}(K)$ is universally injective. Hence there exists at most one section of the morphism $\mathrm{Spec}(K) \times_Y X \to \mathrm{Spec}(K)$. Hence the map $X(K) \to Y(K)$ is injective. Conversely, suppose that for every field K the map $X(K) \to Y(K)$ is injective. Let $T \to Y$ be a morphism from a scheme into Y, and consider the base change $f_T : T \times_Y X \to T$. For any field K we have

$$(T \times_Y X)(K) = T(K) \times_{Y(K)} X(K)$$

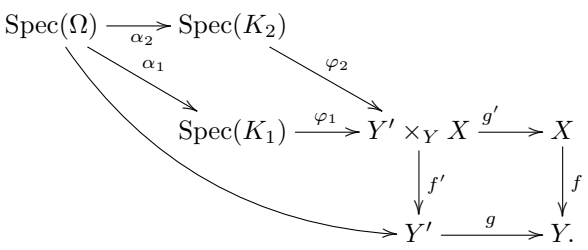
by definition of the fibre product, and hence the injectivity of $X(K) \to Y(K)$ guarantees the injectivity of $(T \times_Y X)(K) \to T(K)$ which means that f_T is universally injective as desired. □

Next, we translate the property that the transformation between field valued points is injective into something more geometric.

040X **Lemma 19.2.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

1. *the map $X(K) \to Y(K)$ is injective for every field K over S*
2. *for every morphism $Y' \to Y$ of algebraic spaces over S the induced map $|Y' \times_Y X| \to |Y'|$ is injective, and*
3. *the diagonal morphism $X \to X \times_Y X$ is surjective.*

Proof. Assume (1). Let $g : Y' \to Y$ be a morphism of algebraic spaces, and denote $f' : Y' \times_Y X \to Y'$ the base change of f. Let K_i, $i = 1, 2$ be fields and let $\varphi_i : \mathrm{Spec}(K_i) \to Y' \times_Y X$ be morphisms such that $f' \circ \varphi_1$ and $f' \circ \varphi_2$ define the same element of $|Y'|$. By definition this means there exists a field Ω and embeddings $\alpha_i : K_i \subset \Omega$ such that the two morphisms $f' \circ \varphi_i \circ \alpha_i : \mathrm{Spec}(\Omega) \to Y'$ are equal. Here is the corresponding commutative diagram



In particular the compositions $g \circ f' \circ \varphi_i \circ \alpha_i$ are equal. By assumption (1) this implies that the morphism $g' \circ \varphi_i \circ \alpha_i$ are equal, where $g' : Y' \times_Y X \to X$ is the

projection. By the universal property of the fibre product we conclude that the morphisms $\varphi_i \circ \alpha_i : \mathrm{Spec}(\Omega) \to Y' \times_Y X$ are equal. In other words φ_1 and φ_2 define the same point of $Y' \times_Y X$. We conclude that (2) holds.

Assume (2). Let K be a field over S, and let $a, b : \mathrm{Spec}(K) \to X$ be two morphisms such that $f \circ a = f \circ b$. Denote $c : \mathrm{Spec}(K) \to Y$ the common value. By assumption $|\mathrm{Spec}(K) \times_{c,Y} X| \to |\mathrm{Spec}(K)|$ is injective. This means there exists a field Ω and embeddings $\alpha_i : K \to \Omega$ such that

$$\begin{array}{ccc} \mathrm{Spec}(\Omega) & \xrightarrow[\alpha_1]{} & \mathrm{Spec}(K) \\ {\scriptstyle \alpha_2}\downarrow & & \downarrow{\scriptstyle a} \\ \mathrm{Spec}(K) & \xrightarrow{b} & \mathrm{Spec}(K) \times_{c,Y} X \end{array}$$

is commutative. Composing with the projection to $\mathrm{Spec}(K)$ we see that $\alpha_1 = \alpha_2$. Denote the common value α. Then we see that $\{\alpha : \mathrm{Spec}(\Omega) \to \mathrm{Spec}(K)\}$ is a fpqc covering of $\mathrm{Spec}(K)$ such that the two morphisms a, b become equal on the members of the covering. By Properties of Spaces, Proposition 17.1 we conclude that $a = b$. We conclude that (1) holds.

Assume (3). Let $x, x' \in |X|$ be a pair of points such that $f(x) = f(x')$ in $|Y|$. By Properties of Spaces, Lemma 4.3 we see there exists a $x'' \in |X \times_Y X|$ whose projections are x and x'. By assumption and Properties of Spaces, Lemma 4.4 there exists a $x''' \in |X|$ with $\Delta_{X/Y}(x''') = x''$. Thus $x = x'$. In other words f is injective. Since condition (3) is stable under base change we see that f satisfies (2).

Assume (2). Then in particular $|X \times_Y X| \to |X|$ is injective which implies immediately that $|\Delta_{X/Y}| : |X| \to |X \times_Y X|$ is surjective, which implies that $\Delta_{X/Y}$ is surjective by Properties of Spaces, Lemma 4.4. □

By the two lemmas above the following definition does not conflict with the already defined notion of a universally injective representable morphism of algebraic spaces.

03MV **Definition 19.3.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. We say f is *universally injective* if for every morphism $Y' \to Y$ the induced map $|Y' \times_Y X| \to |Y'|$ is injective.

To be sure this means that any or all of the equivalent conditions of Lemma 19.2 hold.

05VS **Remark 19.4.** A universally injective morphism of schemes is separated, see Morphisms, Lemma 10.3. This is not the case for morphisms of algebraic spaces. Namely, the algebraic space $X = \mathbf{A}^1_k/\{x \sim -x \mid x \neq 0\}$ constructed in Spaces, Example 14.1 comes equipped with a morphism $X \to \mathbf{A}^1_k$ which maps the point with coordinate x to the point with coordinate x^2. This is an isomorphism away from 0, and there is a unique point of X lying above 0. As X isn't separated this is a universally injective morphism of algebraic spaces which is not separated.

03MW **Lemma 19.5.** *The base change of a universally injective morphism is universally injective.*

Proof. Omitted. Hint: This is formal. □

03MX **Lemma 19.6.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is universally injective,*
(2) *for every scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is universally injective,*
(3) *for every affine scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is universally injective,*
(4) *there exists a scheme Z and a surjective morphism $Z \to Y$ such that $Z \times_Y X \to Z$ is universally injective, and*
(5) *there exists a Zariski covering $Y = \bigcup Y_i$ such that each of the morphisms $f^{-1}(Y_i) \to Y_i$ is universally injective.*

Proof. We will use that being universally injective is preserved under base change (Lemma 19.5) without further mention in this proof. It is clear that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4).

Assume $g : Z \to Y$ as in (4). Let $y : \mathrm{Spec}(K) \to Y$ be a morphism from the spectrum of a field into Y. By assumption we can find an extension field $\alpha : K \subset K'$ and a morphism $z : \mathrm{Spec}(K') \to Z$ such that $y \circ \alpha = g \circ z$ (with obvious abuse of notation). By assumption the morphism $Z \times_Y X \to Z$ is universally injective, hence there is at most one lift of $g \circ z : \mathrm{Spec}(K') \to Y$ to a morphism into X. Since $\{\alpha : \mathrm{Spec}(K') \to \mathrm{Spec}(K)\}$ is a fpqc covering this implies there is at most one lift of $y : \mathrm{Spec}(K) \to Y$ to a morphism into X, see Properties of Spaces, Proposition 17.1. Thus we see that (1) holds.

We omit the verification that (5) is equivalent to (1). □

03MY **Lemma 19.7.** *A composition of universally injective morphisms is universally injective.*

Proof. Omitted. □

20. Affine morphisms

03WD We have already defined in Section 3 what it means for a representable morphism of algebraic spaces to be affine.

03WE **Lemma 20.1.** *Let S be a scheme. Let $f : X \to Y$ be a representable morphism of algebraic spaces over S. Then f is affine (in the sense of Section 3) if and only if for all affine schemes Z and morphisms $Z \to Y$ the scheme $X \times_Y Z$ is affine.*

Proof. This follows directly from the definition of an affine morphism of schemes (Morphisms, Definition 11.1). □

This clears the way for the following definition.

03WF **Definition 20.2.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. We say f is *affine* if for every affine scheme Z and morphism $Z \to Y$ the algebraic space $X \times_Y Z$ is representable by an affine scheme.

03WG **Lemma 20.3.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is representable and affine,*
(2) *f is affine,*
(3) *for every affine scheme V and étale morphism $V \to Y$ the scheme $X \times_Y V$ is affine,*

(4) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is affine, and*
(5) *there exists a Zariski covering $Y = \bigcup Y_i$ such that each of the morphisms $f^{-1}(Y_i) \to Y_i$ is affine.*

Proof. It is clear that (1) implies (2), that (2) implies (3), and that (3) implies (4) by taking V to be a disjoint union of affines étale over Y, see Properties of Spaces, Lemma 6.1. Assume $V \to Y$ is as in (4). Then for every affine open W of V we see that $W \times_Y X$ is an affine open of $V \times_Y X$. Hence by Properties of Spaces, Lemma 13.1 we conclude that $V \times_Y X$ is a scheme. Moreover the morphism $V \times_Y X \to V$ is affine. This means we can apply Spaces, Lemma 11.5 because the class of affine morphisms satisfies all the required properties (see Morphisms, Lemmas 11.8 and Descent, Lemmas 23.18 and 37.1). The conclusion of applying this lemma is that f is representable and affine, i.e., (1) holds.

The equivalence of (1) and (5) follows from the fact that being affine is Zariski local on the target (the reference above shows that being affine is in fact fpqc local on the target). □

03WH **Lemma 20.4.** *The composition of affine morphisms is affine.*

Proof. Omitted. Hint: Transitivity of fibre products. □

03WI **Lemma 20.5.** *The base change of an affine morphism is affine.*

Proof. Omitted. Hint: Transitivity of fibre products. □

07U2 **Lemma 20.6.** *A closed immersion is affine.*

Proof. Follows immediately from the corresponding statement for morphisms of schemes, see Morphisms, Lemma 11.9. □

081V **Lemma 20.7.** *Let S be a scheme. Let X be an algebraic space over S. There is an anti-equivalence of categories*

$$\begin{matrix} \text{algebraic spaces} \\ \text{affine over } X \end{matrix} \longleftrightarrow \begin{matrix} \text{quasi-coherent sheaves} \\ \text{of } \mathcal{O}_X\text{-algebras} \end{matrix}$$

which associates to $f : Y \to X$ the sheaf $f_*\mathcal{O}_Y$. Moreover, this equivalence is compatible with arbitrary base change.

Proof. This lemma is the analogue of Morphisms, Lemma 11.5. Let $\mathcal{A}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-algebras. We will construct an affine morphism of algebraic spaces $\pi : Y = \underline{\mathrm{Spec}}_X(\mathcal{A}) \to X$ with $\pi_*\mathcal{O}_Y \cong \mathcal{A}$. To do this, choose a scheme U and a surjective étale morphism $\varphi : U \to X$. As usual denote $R = U \times_X U$ with projections $s, t : R \to U$. Denote $\psi : R \to X$ the composition $\psi = \varphi \circ s = \varphi \circ t$. By the aforementioned lemma there exists an affine morphisms of schemes $\pi_0 : V \to U$ and $\pi_1 : W \to R$ with $\pi_{0,*}\mathcal{O}_V \cong \varphi^*\mathcal{A}$ and $\pi_{1,*}\mathcal{O}_W \cong \psi^*\mathcal{A}$. Since the construction is compatible with base change there exist morphisms $s', t' : W \to V$ such that the diagrams

$$\begin{array}{ccc} W & \xrightarrow[s']{} & V \\ \downarrow & & \downarrow \\ R & \xrightarrow{s} & U \end{array} \quad \text{and} \quad \begin{array}{ccc} W & \xrightarrow[t']{} & V \\ \downarrow & & \downarrow \\ R & \xrightarrow{t} & U \end{array}$$

are cartesian. It follows that s', t' are étale. It is a formal consequence of the above that $(t', s') : W \to V \times_S V$ is a monomorphism. We omit the verification that $W \to V \times_S V$ is an equivalence relation (hint: think about the pullback of $\mathcal{A}$ to $U \times_X U \times_X U = R \times_{s,U,t} R$). The quotient sheaf $Y = V/W$ is an algebraic space, see Spaces, Theorem 10.5. By Groupoids, Lemma 20.7 we see that $Y \times_X U \cong V$. Hence $Y \to X$ is affine by Lemma 20.3. Finally, the isomorphism of

$$(Y \times_X U \to U)_* \mathcal{O}_{Y \times_X U} = \pi_{0,*} \mathcal{O}_V \cong \varphi^* \mathcal{A}$$

is compatible with glueing isomorphisms, whence $(Y \to X)_* \mathcal{O}_Y \cong \mathcal{A}$ by Properties of Spaces, Proposition 32.1. We omit the verification that this construction is compatible with base change. □

081W **Definition 20.8.** Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{A}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-algebras. The *relative spectrum of $\mathcal{A}$ over X*, or simply the *spectrum of $\mathcal{A}$ over X* is the affine morphism $\underline{\mathrm{Spec}}(\mathcal{A}) \to X$ corresponding to $\mathcal{A}$ under the equivalence of categories of Lemma 20.7.

Forming the relative spectrum commutes with arbitrary base change.

081X **Remark 20.9.** Let S be a scheme. Let $f : Y \to X$ be a quasi-compact and quasi-separated morphism of algebraic spaces over S. Then f has a canonical factorization

$$Y \longrightarrow \underline{\mathrm{Spec}}_X(f_* \mathcal{O}_Y) \longrightarrow X$$

This makes sense because $f_* \mathcal{O}_Y$ is quasi-coherent by Lemma 11.2. The morphism $Y \to \underline{\mathrm{Spec}}_X(f_* \mathcal{O}_Y)$ comes from the canonical $\mathcal{O}_Y$-algebra map $f^* f_* \mathcal{O}_Y \to \mathcal{O}_Y$ which corresponds to a canonical morphism $Y \to Y \times_X \underline{\mathrm{Spec}}_X(f_* \mathcal{O}_Y)$ over Y (see Lemma 20.7) whence a factorization of f as above.

08AI **Lemma 20.10.** *Let S be a scheme. Let $f : Y \to X$ be an affine morphism of algebraic spaces over S. Let $\mathcal{A} = f_* \mathcal{O}_Y$. The functor $\mathcal{F} \mapsto f_* \mathcal{F}$ induces an equivalence of categories*

$$\left\{ \begin{array}{c} \text{category of quasi-coherent} \\ \mathcal{O}_Y\text{-modules} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{category of quasi-coherent} \\ \mathcal{A}\text{-modules} \end{array} \right\}$$

Moreover, an $\mathcal{A}$-module is quasi-coherent as an $\mathcal{O}_X$-module if and only if it is quasi-coherent as an $\mathcal{A}$-module.

Proof. Omitted. □

08GB **Lemma 20.11.** *Let S be a scheme. Let B be an algebraic space over S. Suppose $g : X \to Y$ is a morphism of algebraic spaces over B.*

1. *If X is affine over B and $\Delta : Y \to Y \times_B Y$ is affine, then g is affine.*
2. *If X is affine over B and Y is separated over B, then g is affine.*
3. *A morphism from an affine scheme to an algebraic space with affine diagonal over $\mathbf{Z}$ (as in Properties of Spaces, Definition 3.1) is affine.*
4. *A morphism from an affine scheme to a separated algebraic space is affine.*

Proof. Proof of (1). The base change $X \times_B Y \to Y$ is affine by Lemma 20.5. The morphism $(1, g) : X \to X \times_B Y$ is the base change of $Y \to Y \times_B Y$ by the morphism $X \times_B Y \to Y \times_B Y$. Hence it is affine by Lemma 20.5. The composition of affine morphisms is affine (see Lemma 20.4) and (1) follows. Part (2) follows from (1) as a closed immersion is affine (see Lemma 20.6) and Y/B separated means Δ is a closed immersion. Parts (3) and (4) are special cases of (1) and (2). □

09TF **Lemma 20.12.** *Let S be a scheme. Let X be a quasi-separated algebraic space over S. Let A be an Artinian ring. Any morphism $\mathrm{Spec}(A) \to X$ is affine.*

Proof. Let $U \to X$ be an étale morphism with U affine. To prove the lemma we have to show that $\mathrm{Spec}(A) \times_X U$ is affine, see Lemma 20.3. Since X is quasi-separated the scheme $\mathrm{Spec}(A) \times_X U$ is quasi-compact. Moreover, the projection morphism $\mathrm{Spec}(A) \times_X U \to \mathrm{Spec}(A)$ is étale. Hence this morphism has finite discrete fibers and moreover the topology on $\mathrm{Spec}(A)$ is discrete. Thus $\mathrm{Spec}(A) \times_X U$ is a scheme whose underlying topological space is a finite discrete set. We are done by Schemes, Lemma 11.8. □

21. Quasi-affine morphisms

03WJ We have already defined in Section 3 what it means for a representable morphism of algebraic spaces to be quasi-affine.

03WK **Lemma 21.1.** *Let S be a scheme. Let $f : X \to Y$ be a representable morphism of algebraic spaces over S. Then f is quasi-affine (in the sense of Section 3) if and only if for all affine schemes Z and morphisms $Z \to Y$ the scheme $X \times_Y Z$ is quasi-affine.*

Proof. This follows directly from the definition of a quasi-affine morphism of schemes (Morphisms, Definition 13.1). □

This clears the way for the following definition.

03WL **Definition 21.2.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. We say f is *quasi-affine* if for every affine scheme Z and morphism $Z \to Y$ the algebraic space $X \times_Y Z$ is representable by a quasi-affine scheme.

03WM **Lemma 21.3.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

1. *f is representable and quasi-affine,*
2. *f is quasi-affine,*
3. *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is quasi-affine, and*
4. *there exists a Zariski covering $Y = \bigcup Y_i$ such that each of the morphisms $f^{-1}(Y_i) \to Y_i$ is quasi-affine.*

Proof. It is clear that (1) implies (2) and that (2) implies (3) by taking V to be a disjoint union of affines étale over Y, see Properties of Spaces, Lemma 6.1. Assume $V \to Y$ is as in (3). Then for every affine open W of V we see that $W \times_Y X$ is a quasi-affine open of $V \times_Y X$. Hence by Properties of Spaces, Lemma 13.1 we conclude that $V \times_Y X$ is a scheme. Moreover the morphism $V \times_Y X \to V$ is quasi-affine. This means we can apply Spaces, Lemma 11.5 because the class of quasi-affine morphisms satisfies all the required properties (see Morphisms, Lemmas 13.5 and Descent, Lemmas 23.20 and 38.1). The conclusion of applying this lemma is that f is representable and quasi-affine, i.e., (1) holds.

The equivalence of (1) and (4) follows from the fact that being quasi-affine is Zariski local on the target (the reference above shows that being quasi-affine is in fact fpqc local on the target). □

03WN **Lemma 21.4.** *The composition of quasi-affine morphisms is quasi-affine.*

Proof. Omitted. □

03WO **Lemma 21.5.** *The base change of a quasi-affine morphism is quasi-affine.*

Proof. Omitted. □

086S **Lemma 21.6.** *Let S be a scheme. A quasi-compact and quasi-separated morphism of algebraic spaces $f : Y \to X$ is quasi-affine if and only if the canonical factorization $Y \to \underline{\mathrm{Spec}}_X(f_*\mathcal{O}_Y)$ (Remark 20.9) is an open immersion.*

Proof. Let $U \to X$ be a surjective morphism where U is a scheme. Since we may check whether f is quasi-affine after base change to U (Lemma 21.3), since $f_*\mathcal{O}_Y|_V$ is equal to $(Y \times_X U \to U)_*\mathcal{O}_{Y \times_X U}$ (Properties of Spaces, Lemma 26.2), and since formation of relative spectrum commutes with base change (Lemma 20.7), we see that the assertion reduces to the case that X is a scheme. If X is a scheme and either f is quasi-affine or $Y \to \underline{\mathrm{Spec}}_X(f_*\mathcal{O}_Y)$ is an open immersion, then Y is a scheme as well. Thus we reduce to Morphisms, Lemma 13.3. □

22. Types of morphisms étale local on source-and-target

03MI Given a property of morphisms of schemes which is *étale local on the source-and-target*, see Descent, Definition 32.3 we may use it to define a corresponding property of morphisms of algebraic spaces, namely by imposing either of the equivalent conditions of the lemma below.

03MJ **Lemma 22.1.** *Let $\mathcal{P}$ be a property of morphisms of schemes which is étale local on the source-and-target. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Consider commutative diagrams*

$$\begin{array}{ccc} U & \xrightarrow{h} & V \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ X & \xrightarrow{f} & Y \end{array}$$

where U and V are schemes and the vertical arrows are étale. The following are equivalent

1. *for any diagram as above the morphism h has property $\mathcal{P}$, and*
2. *for some diagram as above with $a : U \to X$ surjective the morphism h has property $\mathcal{P}$.*

If X and Y are representable, then this is also equivalent to f (as a morphism of schemes) having property $\mathcal{P}$. If $\mathcal{P}$ is also preserved under any base change, and fppf local on the base, then for representable morphisms f this is also equivalent to f having property $\mathcal{P}$ in the sense of Section 3.

Proof. Let us prove the equivalence of (1) and (2). The implication (1) $\Rightarrow$ (2) is immediate (taking into account Spaces, Lemma 11.6). Assume

$$\begin{array}{ccc} U & \xrightarrow{h} & V \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array} \qquad \begin{array}{ccc} U' & \xrightarrow{h'} & V' \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

are two diagrams as in the lemma. Assume $U \to X$ is surjective and h has property $\mathcal{P}$. To show that (2) implies (1) we have to prove that h' has $\mathcal{P}$. To do this consider the diagram

$$\begin{array}{ccccc} U & \longleftarrow & U \times_X U' & \longrightarrow & U' \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle (h,h')} & & \downarrow{\scriptstyle h'} \\ V & \longleftarrow & V \times_Y V' & \longrightarrow & V' \end{array}$$

By Descent, Lemma 32.5 we see that h has $\mathcal{P}$ implies (h, h') has $\mathcal{P}$ and since $U \times_X U' \to U'$ is surjective this implies (by the same lemma) that h' has $\mathcal{P}$.

If X and Y are representable, then Descent, Lemma 32.5 applies which shows that (1) and (2) are equivalent to f having $\mathcal{P}$.

Finally, suppose f is representable, and U, V, a, b, h are as in part (2) of the lemma, and that $\mathcal{P}$ is preserved under arbitrary base change. We have to show that for any scheme Z and morphism $Z \to X$ the base change $Z \times_Y X \to Z$ has property $\mathcal{P}$. Consider the diagram

$$\begin{array}{ccc} Z \times_Y U & \longrightarrow & Z \times_Y V \\ \downarrow & & \downarrow \\ Z \times_Y X & \longrightarrow & Z \end{array}$$

Note that the top horizontal arrow is a base change of h and hence has property $\mathcal{P}$. The left vertical arrow is étale and surjective and the right vertical arrow is étale. Thus Descent, Lemma 32.5 once again kicks in and shows that $Z \times_Y X \to Z$ has property $\mathcal{P}$. □

04RD **Definition 22.2.** Let S be a scheme. Let $\mathcal{P}$ be a property of morphisms of schemes which is étale local on the source-and-target. We say a morphism $f : X \to Y$ of algebraic spaces over S *has property* $\mathcal{P}$ if the equivalent conditions of Lemma 22.1 hold.

Here are a couple of obvious remarks.

0AML **Remark 22.3.** Let S be a scheme. Let $\mathcal{P}$ be a property of morphisms of schemes which is étale local on the source-and-target. Suppose that moreover $\mathcal{P}$ is stable under compositions. Then the class of morphisms of algebraic spaces having property $\mathcal{P}$ is stable under composition.

0AMM **Remark 22.4.** Let S be a scheme. Let $\mathcal{P}$ be a property of morphisms of schemes which is étale local on the source-and-target. Suppose that moreover $\mathcal{P}$ is stable under base change. Then the class of morphisms of algebraic spaces having property $\mathcal{P}$ is stable under base change.

Given a property of morphisms of germs of schemes which is *étale local on the source-and-target*, see Descent, Definition 33.1 we may use it to define a corresponding property of morphisms of algebraic spaces at a point, namely by imposing either of the equivalent conditions of the lemma below.

04NC **Lemma 22.5.** *Let $\mathcal{Q}$ be a property of morphisms of germs which is étale local on the source-and-target. Let S be a scheme. Let $f : X \to Y$ be a morphism of*

algebraic spaces over S. Let $x \in |X|$ be a point of X. Consider the diagrams

$$\begin{array}{ccc} U & \xrightarrow{h} & V \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ X & \xrightarrow{f} & Y \end{array} \qquad \begin{array}{ccc} u & \longrightarrow & v \\ \downarrow & & \downarrow \\ x & \longrightarrow & y \end{array}$$

where U and V are schemes, a, b are étale, and u, v, x, y are points of the corresponding spaces. The following are equivalent

(1) *for any diagram as above we have $\mathcal{Q}((U,u) \to (V,v))$, and*
(2) *for some diagram as above we have $\mathcal{Q}((U,u) \to (V,v))$.*

If X and Y are representable, then this is also equivalent to $\mathcal{Q}((X,x) \to (Y,y))$.

Proof. Omitted. Hint: Very similar to the proof of Lemma 22.1. □

04RE **Definition 22.6.** Let $\mathcal{Q}$ be a property of morphisms of germs of schemes which is étale local on the source-and-target. Let S be a scheme. Given a morphism $f : X \to Y$ of algebraic spaces over S and a point $x \in |X|$ we say that f *has property $\mathcal{Q}$ at x* if the equivalent conditions of Lemma 22.5 hold.

The following lemma should not be used blindly to go from a property of morphisms to a property of morphisms at a point. For example if $\mathcal{P}$ is the property of being flat, then the property Q in the following lemma means "f is flat in an open neighbourhood of x" which is not the same as "f is flat at x".

04RF **Lemma 22.7.** *Let $\mathcal{P}$ be a property of morphisms of schemes which is étale local on the source-and-target. Consider the property $\mathcal{Q}$ of morphisms of germs associated to $\mathcal{P}$ in Descent, Lemma 33.2. Then*

(1) *$\mathcal{Q}$ is étale local on the source-and-target.*
(2) *given a morphism of algebraic spaces $f : X \to Y$ and $x \in |X|$ the following are equivalent*
   (a) *f has $\mathcal{Q}$ at x, and*
   (b) *there is an open neighbourhood $X' \subset X$ of x such that $X' \to Y$ has $\mathcal{P}$.*
(3) *given a morphism of algebraic spaces $f : X \to Y$ the following are equivalent:*
   (a) *f has $\mathcal{P}$,*
   (b) *for every $x \in |X|$ the morphism f has $\mathcal{Q}$ at x.*

Proof. See Descent, Lemma 33.2 for (1). The implication (1)(a) $\Rightarrow$ (2)(b) follows on letting $X' = a(U) \subset X$ given a diagram as in Lemma 22.5. The implication (2)(b) $\Rightarrow$ (1)(a) is clear. The equivalence of (3)(a) and (3)(b) follows from the corresponding result for morphisms of schemes, see Descent, Lemma 33.3. □

04RG **Remark 22.8.** We will apply Lemma 22.7 above to all cases listed in Descent, Remark 32.7 except "flat". In each case we will do this by defining f to have property $\mathcal{P}$ at x if f has $\mathcal{P}$ in a neighbourhood of x.

23. Morphisms of finite type

03XE The property "locally of finite type" of morphisms of schemes is étale local on the source-and-target, see Descent, Remark 32.7. It is also stable under base change and fpqc local on the target, see Morphisms, Lemma 15.4, and Descent, Lemmas 23.10. Hence, by Lemma 22.1 above, we may define what it means for a morphism

of algebraic spaces to be locally of finite type as follows and it agrees with the already existing notion defined in Section 3 when the morphism is representable.

03XF **Definition 23.1.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say f *locally of finite type* if the equivalent conditions of Lemma 22.1 hold with $\mathcal{P} =$ locally of finite type.
(2) Let $x \in |X|$. We say f is of *finite type at* x if there exists an open neighbourhood $X' \subset X$ of x such that $f|_{X'} : X' \to Y$ is locally of finite type.
(3) We say f is *of finite type* if it is locally of finite type and quasi-compact.

Consider the algebraic space $\mathbf{A}^1_k/\mathbf{Z}$ of Spaces, Example 14.8. The morphism $\mathbf{A}^1_k/\mathbf{Z} \to \mathrm{Spec}(k)$ is of finite type.

03XG **Lemma 23.2.** *The composition of finite type morphisms is of finite type. The same holds for locally of finite type.*

Proof. See Remark 22.3 and Morphisms, Lemma 15.3. □

03XH **Lemma 23.3.** *A base change of a finite type morphism is finite type. The same holds for locally of finite type.*

Proof. See Remark 22.4 and Morphisms, Lemma 15.4. □

040Y **Lemma 23.4.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is locally of finite type,*
(2) *for every $x \in |X|$ the morphism f is of finite type at x,*
(3) *for every scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is locally of finite type,*
(4) *for every affine scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is locally of finite type,*
(5) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is locally of finite type,*
(6) *there exists a scheme U and a surjective étale morphism $\varphi : U \to X$ such that the composition $f \circ \varphi$ is locally of finite type,*
(7) *for every commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where U, V are schemes and the vertical arrows are étale the top horizontal arrow is locally of finite type,
(8) *there exists a commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where U, V are schemes, the vertical arrows are étale, $U \to X$ is surjective, and the top horizontal arrow is locally of finite type, and

(9) *there exist Zariski coverings* $Y = \bigcup_{i \in I} Y_i$, *and* $f^{-1}(Y_i) = \bigcup X_{ij}$ *such that each morphism* $X_{ij} \to Y_i$ *is locally of finite type.*

Proof. Each of the conditions (2), (3), (4), (5), (6), (7), and (9) imply condition (8) in a straightforward manner. For example, if (5) holds, then we can choose a scheme V which is a disjoint union of affines and a surjective morphism $V \to Y$ (see Properties of Spaces, Lemma 6.1). Then $V \times_Y X \to V$ is locally of finite type by (5). Choose a scheme U and a surjective étale morphism $U \to V \times_Y X$. Then $U \to V$ is locally of finite type by Lemma 23.2. Hence (8) is true.

The conditions (1), (7), and (8) are equivalent by definition.

To finish the proof, we show that (1) implies all of the conditions (2), (3), (4), (5), (6), and (9). For (2) this is immediate. For (3), (4), (5), and (9) this follows from the fact that being locally of finite type is preserved under base change, see Lemma 23.3. For (6) we can take $U = X$ and we're done. □

04ZK **Lemma 23.5.** *Let* S *be a scheme. Let* $f : X \to Y$ *be a morphism of algebraic spaces over* S. *If* f *is locally of finite type and* Y *is locally Noetherian, then* X *is locally Noetherian.*

Proof. Let

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

be a commutative diagram where U, V are schemes and the vertical arrows are surjective étale. If f is locally of finite type, then $U \to V$ is locally of finite type. If Y is locally Noetherian, then V is locally Noetherian. By Morphisms, Lemma 15.6 we see that U is locally Noetherian, which means that X is locally Noetherian. □

0462 **Lemma 23.6.** *Let* S *be a scheme. Let* $f : X \to Y$, $g : Y \to Z$ *be morphisms of algebraic spaces over* S. *If* $g \circ f : X \to Z$ *is locally of finite type, then* $f : X \to Y$ *is locally of finite type.*

Proof. We can find a diagram

$$\begin{array}{ccccc} U & \longrightarrow & V & \longrightarrow & W \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & Y & \longrightarrow & Z \end{array}$$

where U, V, W are schemes, the vertical arrows are étale and surjective, see Spaces, Lemma 11.6. At this point we can use Lemma 23.4 and Morphisms, Lemma 15.8 to conclude. □

06ED **Lemma 23.7.** *An immersion is locally of finite type.*

Proof. Follows from the general principle Spaces, Lemma 5.8 and Morphisms, Lemmas 15.5. □

24. Points and geometric points

0485 In this section we make some remarks on points and geometric points (see Properties of Spaces, Definition 19.1). One way to think about a geometric point of X is to consider a geometric point $\overline{s} : \mathrm{Spec}(k) \to S$ of S and a lift of $\overline{s}$ to a morphism $\overline{x}$ into X. Here is a diagram

$$\begin{array}{ccc} \mathrm{Spec}(k) & \xrightarrow{\overline{x}} & X \\ & \searrow^{\overline{s}} & \downarrow \\ & & S. \end{array}$$

We often say "let k be an algebraically closed field over S" to indicate that $\mathrm{Spec}(k)$ comes equipped with a morphism $\mathrm{Spec}(k) \to S$. In this situation we write

$$X(k) = \mathrm{Mor}_S(\mathrm{Spec}(k), X) = \{\overline{x} \in X \text{ lying over } \overline{s}\}$$

for the set of k-valued points of X. In this case the map $X(k) \to |X|$ maps into the subset $|X_s| \subset |X|$. Here $X_s = \mathrm{Spec}(\kappa(s)) \times_S X$, where $s \in S$ is the point corresponding to $\overline{s}$. As $\mathrm{Spec}(\kappa(s)) \to S$ is a monomorphism, also the base change $X_s \to X$ is a monomorphism, and $|X_s|$ is indeed a subset of $|X|$.

0487 **Lemma 24.1.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is locally of finite type. The following are equivalent:*

1. *f is surjective, and*
2. *for every algebraically closed field k over S the induced map $X(k) \to Y(k)$ is surjective.*

Proof. Choose a diagram

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

with U, V schemes over S and vertical arrows surjective and étale, see Spaces, Lemma 11.6. Since f is locally of finite type we see that $U \to V$ is locally of finite type.

Assume (1) and let $\overline{y} \in Y(k)$. Then $U \to Y$ is surjective and locally of finite type by Lemmas 5.4 and 23.2. Let $Z = U \times_{Y,\overline{y}} \mathrm{Spec}(k)$. This is a scheme. The projection $Z \to \mathrm{Spec}(k)$ is surjective and locally of finite type by Lemmas 5.5 and 23.3. It follows from Varieties, Lemma 14.1 that Z has a k valued point $\overline{z}$. The image $\overline{x} \in X(k)$ of $\overline{z}$ maps to $\overline{y}$ as desired.

Assume (2). By Properties of Spaces, Lemma 4.4 it suffices to show that $|X| \to |Y|$ is surjective. Let $y \in |Y|$. Choose a $u \in U$ mapping to y. Let $k \supset \kappa(u)$ be an algebraic closure. Denote $\overline{u} \in U(k)$ the corresponding point and $\overline{y} \in Y(k)$ its image. By assumption there exists a $\overline{x} \in X(k)$ mapping to $\overline{y}$. Then it is clear that the image $x \in |X|$ of $\overline{x}$ maps to y. □

In order to state the next lemma we introduce the following notation. Given a scheme T we denote

$$\lambda(T) = \sup\{\aleph_0, |\kappa(t)|; t \in T\}.$$

In words $\lambda(T)$ is the smallest infinite cardinal bounding all the cardinalities of residue fields ot T. Note that if R is a ring then the cardinality of any residue

field $\kappa(\mathfrak{p})$ of R is bounded by the cardinality of R (details omitted). This implies that $\lambda(T) \leq \text{size}(T)$ where $\text{size}(T)$ is the size of the scheme T as introduced in Sets, Section 9. If L/K is a finitely generated field extension then $|K| \leq |L| \leq \max\{\aleph_0, |K|\}$. It follows that if $T' \to T$ is a morphism of schemes which is locally of finite type then $\lambda(T') \leq \lambda(T)$, and if $T' \to T$ is also surjective then equality holds. Next, suppose that S is a scheme and that X is an algebraic space over S. In this case we define

$$\lambda(X) := \lambda(U)$$

where U is any scheme over S which has a surjective étale morphism towards X. The reason that this is independent of the choice of U is that given a pair of such schemes U and U' the fibre product $U \times_X U'$ is a scheme which admits a surjective étale morphism to both U and U', whence $\lambda(U) = \lambda(U \times_X U') = \lambda(U')$ by the discussion above.

0488 **Lemma 24.2.** *Let S be a scheme. Let X, Y be algebraic spaces over S.*

(1) *As k ranges over all algebraically closed fields over S the collection of geometric points $\overline{y} \in Y(k)$ cover all of $|Y|$.*
(2) *As k ranges over all algebraically closed fields over S with $|k| \geq \lambda(Y)$ and $|k| > \lambda(X)$ the geometric points $\overline{y} \in Y(k)$ cover all of $|Y|$.*
(3) *For any geometric point $\overline{s} : \text{Spec}(k) \to S$ where k has cardinality $> \lambda(X)$ the map*
$$X(k) \longrightarrow |X_s|$$
is surjective.
(4) *Let $X \to Y$ be a morphism of algebraic spaces over S. For any geometric point $\overline{s} : \text{Spec}(k) \to S$ where k has cardinality $> \lambda(X)$ the map*
$$X(k) \longrightarrow |X| \times_{|Y|} Y(k)$$
is surjective.
(5) *Let $X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*
   (a) *the map $X \to Y$ is surjective,*
   (b) *for all algebraically closed fields k over S with $|k| > \lambda(X)$, and $|k| \geq \lambda(Y)$ the map $X(k) \to Y(k)$ is surjective.*

Proof. To prove part (1) choose a surjective étale morphism $V \to Y$ where V is a scheme. For each $v \in V$ choose an algebraic closure $\kappa(v) \subset k_v$. Consider the morphisms $\overline{x} : \text{Spec}(k_v) \to V \to Y$. By construction of $|Y|$ these cover $|Y|$.

To prove part (2) we will use the following two facts whose proofs we omit: (i) If K is a field and $\overline{K}$ is algebraic closure then $|\overline{K}| \leq \max\{\aleph_0, |K|\}$. (ii) For any algebraically closed field k and any cardinal $\aleph$, $\aleph \geq |k|$ there exists an extension of algebraically closed fields k'/k with $|k'| = \aleph$. Now we set $\aleph = \max\{\lambda(X), \lambda(Y)\}^+$. Here $\lambda^+ > \lambda$ indicates the next bigger cardinal, see Sets, Section 6. Now (i) implies that the fields k_u constructed in the first paragraph of the proof all have cardinality bounded by $\lambda(X)$. Hence by (ii) we can find extensions $k_u \subset k'_u$ such that $|k'_u| = \aleph$. The morphisms $\overline{x}' : \text{Spec}(k'_u) \to X$ cover $|X|$ as desired. To really finish the proof of (2) we need to show that the schemes $\text{Spec}(k'_u)$ are (isomorphic to) objects of Sch_{fppf} because our conventions are that all schemes are objects of Sch_{fppf}; the rest of this paragraph should be skipped by anyone who is not interested in set theoretical considerations. By construction there exists an object T of Sch_{fppf}

such that $\lambda(X)$ and $\lambda(Y)$ are bounded by $\text{size}(T)$. By our construction of the category Sch_{fppf} in Topologies, Definitions 7.6 as the category Sch_α constructed in Sets, Lemma 9.2 we see that any scheme whose size is $\leq \text{size}(T)^+$ is isomorphic to an object of Sch_{fppf}. See the expression for the function *Bound* in Sets, Equation (9.1.1). Since $\aleph \leq \text{size}(T)^+$ we conclude.

The notation X_s in part (3) means the fibre product $\text{Spec}(\kappa(s)) \times_S X$, where $s \in S$ is the point corresponding to $\overline{s}$. Hence part (2) follows from (4) with $Y = \text{Spec}(\kappa(s))$.

Let us prove (4). Let $X \to Y$ be a morphism of algebraic spaces over S. Let k be an algebraically closed field over S of cardinality $> \lambda(X)$. Let $\overline{y} \in Y(k)$ and $x \in |X|$ which map to the same element y of $|Y|$. We have to find $\overline{x} \in X(k)$ mapping to x and $\overline{y}$. Choose a commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

with U, V schemes over S and vertical arrows surjective and étale, see Spaces, Lemma 11.6. Choose a $u \in |U|$ which maps to x, and denote $v \in |V|$ the image. We will think of $u = \text{Spec}(\kappa(u))$ and $v = \text{Spec}(\kappa(v))$ as schemes. Note that $V \times_Y \text{Spec}(k)$ is a scheme étale over k. Hence it is a disjoint union of spectra of finite separable extensions of k, see Morphisms, Lemma 36.7. As v maps to y we see that $v \times_Y \text{Spec}(k)$ is a nonempty scheme. As $v \to V$ is a monomorphism, we see that $v \times_Y \text{Spec}(k) \to V \times_Y \text{Spec}(k)$ is a monomorphism. Hence $v \times_Y \text{Spec}(k)$ is a disjoint union of spectra of finite separable extensions of k, by Schemes, Lemma 23.11. We conclude that the morphism $v \times_Y \text{Spec}(k) \to \text{Spec}(k)$ has a section, i.e., we can find a morphism $\overline{v} : \text{Spec}(k) \to V$ lying over v and over $\overline{y}$. Finally we consider the scheme

$$u \times_{V,\overline{v}} \text{Spec}(k) = \text{Spec}(\kappa(u) \otimes_{\kappa(v)} k)$$

where $\kappa(v) \to k$ is the field map defining the morphism $\overline{v}$. Since the cardinality of k is larger than the cardinality of $\kappa(u)$ by assumption we may apply Algebra, Lemma 35.12 to see that any maximal ideal $\mathfrak{m} \subset \kappa(u) \otimes_{\kappa(v)} k$ has a residue field which is algebraic over k and hence equal to k. Such a maximal ideal will hence produce a morphism $\overline{u} : \text{Spec}(k) \to U$ lying over u and mapping to $\overline{v}$. The composition $\text{Spec}(k) \to U \to X$ will be the desired geometric point $\overline{x} \in X(k)$. This concludes the proof of part (4).

Part (5) is a formal consequence of parts (2) and (4) and Properties of Spaces, Lemma 4.4. □

25. Points of finite type

06EE Let S be a scheme. Let X be an algebraic space over S. A finite type point $x \in |X|$ is a point which can be represented by a morphism $\text{Spec}(k) \to X$ which is locally of finite type. Finite type points are a suitable replacement of closed points for algebraic spaces and algebraic stacks. There are always "enough of them" for example.

06EF **Lemma 25.1.** *Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$. The following are equivalent:*

(1) *There exists a morphism* $\mathrm{Spec}(k) \to X$ *which is locally of finite type and represents* x*.*
(2) *There exists a scheme* U*, a closed point* $u \in U$*, and an étale morphism* $\varphi : U \to X$ *such that* $\varphi(u) = x$*.*

Proof. Let $u \in U$ and $U \to X$ be as in (2). Then $\mathrm{Spec}(\kappa(u)) \to U$ is of finite type, and $U \to X$ is representable and locally of finite type (by the general principle Spaces, Lemma 5.8 and Morphisms, Lemmas 36.11 and 21.8). Hence we see (1) holds by Lemma 23.2.

Conversely, assume $\mathrm{Spec}(k) \to X$ is locally of finite type and represents x. Let $U \to X$ be a surjective étale morphism where U is a scheme. By assumption $U \times_X \mathrm{Spec}(k) \to U$ is locally of finite type. Pick a finite type point v of $U \times_X \mathrm{Spec}(k)$ (there exists at least one, see Morphisms, Lemma 16.4). By Morphisms, Lemma 16.5 the image $u \in U$ of v is a finite type point of U. Hence by Morphisms, Lemma 16.4 after shrinking U we may assume that u is a closed point of U, i.e., (2) holds. □

06EG **Definition 25.2.** Let S be a scheme. Let X be an algebraic space over S. We say a point $x \in |X|$ is a *finite type point*[5] if the equivalent conditions of Lemma 25.1 are satisfied. We denote $X_{\text{ft-pts}}$ the set of finite type points of X.

We can describe the set of finite type points as follows.

06EH **Lemma 25.3.** *Let* S *be a scheme. Let* X *be an algebraic space over* S*. We have*

$$X_{ft\text{-}pts} = \bigcup\nolimits_{\varphi : U \to X \text{ étale}} |\varphi|(U_0)$$

where U_0 *is the set of closed points of* U*. Here we may let* U *range over all schemes étale over* X *or over all affine schemes étale over* X*.*

Proof. Immediate from Lemma 25.1. □

06EI **Lemma 25.4.** *Let* S *be a scheme. Let* $f : X \to Y$ *be a morphism of algebraic spaces over* S*. If* f *is locally of finite type, then* $f(X_{ft\text{-}pts}) \subset Y_{ft\text{-}pts}$*.*

Proof. Take $x \in X_{\text{ft-pts}}$. Represent x by a locally finite type morphism $x : \mathrm{Spec}(k) \to X$. Then $f \circ x$ is locally of finite type by Lemma 23.2. Hence $f(x) \in Y_{\text{ft-pts}}$. □

06EJ **Lemma 25.5.** *Let* S *be a scheme. Let* $f : X \to Y$ *be a morphism of algebraic spaces over* S*. If* f *is locally of finite type and surjective, then* $f(X_{ft\text{-}pts}) = Y_{ft\text{-}pts}$*.*

Proof. We have $f(X_{\text{ft-pts}}) \subset Y_{\text{ft-pts}}$ by Lemma 25.4. Let $y \in |Y|$ be a finite type point. Represent y by a morphism $\mathrm{Spec}(k) \to Y$ which is locally of finite type. As f is surjective the algebraic space $X_k = \mathrm{Spec}(k) \times_Y X$ is nonempty, therefore has a finite type point $x \in |X_k|$ by Lemma 25.3. Now $X_k \to X$ is a morphism which is locally of finite type as a base change of $\mathrm{Spec}(k) \to Y$ (Lemma 23.3). Hence the image of x in X is a finite type point by Lemma 25.4 which maps to y by construction. □

[5]This is a slight abuse of language as it would perhaps be more correct to say "locally finite type point".

06EK **Lemma 25.6.** *Let S be a scheme. Let X be an algebraic space over S. For any locally closed subset $T \subset |X|$ we have*

$$T \neq \emptyset \Rightarrow T \cap X_{ft\text{-}pts} \neq \emptyset.$$

In particular, for any closed subset $T \subset |X|$ we see that $T \cap X_{ft\text{-}pts}$ is dense in T.

Proof. Let $i : Z \to X$ be the reduced induce subspace structure on T, see Remark 12.5. Any immersion is locally of finite type, see Lemma 23.7. Hence by Lemma 25.4 we see $Z_{\text{ft-pts}} \subset X_{\text{ft-pts}} \cap T$. Finally, any nonempty affine scheme U with an étale morphism towards Z has at least one closed point. Hence Z has at least one finite type point by Lemma 25.3. The lemma follows. □

Here is another, more technical, characterization of a finite type point on an algebraic space.

06EL **Lemma 25.7.** *Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$. The following are equivalent:*

1. *x is a finite type point,*
2. *there exists an algebraic space Z whose underlying topological space $|Z|$ is a singleton, and a morphism $f : Z \to X$ which is locally of finite type such that $\{x\} = |f|(|Z|)$, and*
3. *there exists an algebraic space Z and a morphism $f : Z \to X$ with the following properties:*
   (a) *there is a surjective étale morphism $z : \text{Spec}(k) \to Z$ where k is a field,*
   (b) *f is locally of finite type,*
   (c) *f is a monomorphism, and*
   (d) *$x = f(z)$.*

Proof. Assume x is a finite type point. Choose an affine scheme U, a closed point $u \in U$, and an étale morphism $\varphi : U \to X$ with $\varphi(u) = x$, see Lemma 25.3. Set $u = \text{Spec}(\kappa(u))$ as usual. The projection morphisms $u \times_X u \to u$ are the compositions

$$u \times_X u \to u \times_X U \to u \times_X X = u$$

where the first arrow is a closed immersion (a base change of $u \to U$) and the second arrow is étale (a base change of the étale morphism $U \to X$). Hence $u \times_X U$ is a disjoint union of spectra of finite separable extensions of k (see Morphisms, Lemma 36.7) and therefore the closed subscheme $u \times_X u$ is a disjoint union of finite separable extension of k, i.e., $u \times_X u \to u$ is étale. By Spaces, Theorem 10.5 we see that $Z = u/u \times_X u$ is an algebraic space. By construction the diagram

$$\begin{array}{ccc} u & \longrightarrow & U \\ \downarrow & & \downarrow \\ Z & \longrightarrow & X \end{array}$$

is commutative with étale vertical arrows. Hence $Z \to X$ is locally of finite type (see Lemma 23.4). By construction the morphism $Z \to X$ is a monomorphism and the image of z is x. Thus (3) holds.

It is clear that (3) implies (2). If (2) holds then x is a finite type point of X by Lemma 25.4 (and Lemma 25.6 to see that $Z_{\text{ft-pts}}$ is nonempty, i.e., the unique point of Z is a finite type point of Z). □

26. Nagata spaces

0BAT See Properties of Spaces, Section 7 for the definition of a Nagata algebraic space.

0BAU **Lemma 26.1.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If Y is Nagata and f locally of finite type then X is Nagata.*

Proof. Let V be a scheme and let $V \to Y$ be a surjective étale morphism. Let U be a scheme and let $U \to X \times_Y V$ be a surjective étale morphism. If Y is Nagata, then V is a Nagata scheme. If $X \to Y$ is locally of finite type, then $U \to V$ is locally of finite type. Hence V is a Nagata scheme by Morphisms, Lemma 18.1. Then X is Nagata by definition. □

0BAV **Lemma 26.2.** *The following types of algebraic spaces are Nagata.*

(1) *Any algebraic space locally of finite type over a Nagata scheme.*
(2) *Any algebraic space locally of finite type over a field.*
(3) *Any algebraic space locally of finite type over a Noetherian complete local ring.*
(4) *Any algebraic space locally of finite type over $\mathbf{Z}$.*
(5) *Any algebraic space locally of finite type over a Dedekind ring of characteristic zero.*
(6) *And so on.*

Proof. The first property holds by Lemma 26.1. Thus the others hold as well, see Morphisms, Lemma 18.2. □

27. Quasi-finite morphisms

03XI The property "locally quasi-finite" of morphisms of schemes is étale local on the source-and-target, see Descent, Remark 32.7. It is also stable under base change and fpqc local on the target, see Morphisms, Lemma 20.13, and Descent, Lemma 23.24. Hence, by Lemma 22.1 above, we may define what it means for a morphism of algebraic spaces to be locally quasi-finite as follows and it agrees with the already existing notion defined in Section 3 when the morphism is representable.

03XJ **Definition 27.1.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say f is *locally quasi-finite* if the equivalent conditions of Lemma 22.1 hold with $\mathcal{P} =$ locally quasi-finite.
(2) Let $x \in |X|$. We say f is *quasi-finite at x* if there exists an open neighbourhood $X' \subset X$ of x such that $f|_{X'} : X' \to Y$ is locally quasi-finite.
(3) A morphism of algebraic spaces $f : X \to Y$ is *quasi-finite* if it is locally quasi-finite and quasi-compact.

The last part is compatible with the notion of quasi-finiteness for morphisms of schemes by Morphisms, Lemma 20.9.

0ABM **Lemma 27.2.** *Let S be a scheme. Let $f : X \to Y$ and $g : Y' \to Y$ be morphisms of algebraic spaces over S. Denote $f' : X' \to Y'$ the base change of f by g. Denote $g' : X' \to X$ the projection. Assume f is locally of finite type. Let $W \subset |X|$, resp. $W' \subset |X'|$ be the set of points where f, resp. f' is quasi-finite.*

(1) *$W \subset |X|$ and $W' \subset |X'|$ are open,*

(2) $W' = (g')^{-1}(W)$, *i.e., formation of the locus where f is quasi-finite commutes with base change,*
(3) *the base change of a locally quasi-finite morphism is locally quasi-finite, and*
(4) *the base change of a quasi-finite morphism is quasi-finite.*

Proof. Choose a scheme V and a surjective étale morphism $V \to Y$. Choose a scheme U and a surjective étale morphism $U \to V \times_Y X$. Choose a scheme V' and a surjective étale morphism $V' \to Y' \times_Y V$. Set $U' = V' \times_V U$ so that $U' \to X'$ is a surjective étale morphism as well. Picture

$$\begin{array}{ccc} U' & \longrightarrow & U \\ \downarrow & & \downarrow \\ V' & \longrightarrow & V \end{array} \quad \text{lying over} \quad \begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y \end{array}$$

Choose $u \in |U|$ with image $x \in |X|$. The property of being "locally quasi-finite" is étale local on the source-and-target, see Descent, Remark 32.7. Hence Lemmas 22.5 and 22.7 apply and we see that $f : X \to Y$ is quasi-finite at x if and only if $U \to V$ is quasi-finite at u. Similarly for $f' : X' \to Y'$ and the morphism $U' \to V'$. Hence parts (1), (2), and (3) reduce to Morphisms, Lemmas 20.13 and 56.2. Part (4) follows from (3) and Lemma 8.4. □

03XK **Lemma 27.3.** *The composition of quasi-finite morphisms is quasi-finite. The same holds for locally quasi-finite.*

Proof. See Remark 22.3 and Morphisms, Lemma 20.12. □

03XL **Lemma 27.4.** *A base change of a quasi-finite morphism is quasi-finite. The same holds for locally quasi-finite.*

Proof. Immediate consequence of Lemma 27.2. □

The following lemma characterizes locally quasi-finite morphisms as those morphisms which are locally of finite type and have "discrete fibres". However, this is not the same thing as asking $|X| \to |Y|$ to have discrete fibres as the discussion in Examples, Section 50 shows.

06RW **Lemma 27.5.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces. Assume f is locally of finite type. The following are equivalent*

(1) *f is locally quasi-finite,*
(2) *for every morphism $\mathrm{Spec}(k) \to Y$ where k is a field the space $|X_k|$ is discrete. Here $X_k = \mathrm{Spec}(k) \times_Y X$.*

Proof. Assume f is locally quasi-finite. Let $\mathrm{Spec}(k) \to Y$ be as in (2). Choose a surjective étale morphism $U \to X$ where U is a scheme. Then $U_k = \mathrm{Spec}(k) \times_Y U \to X_k$ is an étale morphism of algebraic spaces by Properties of Spaces, Lemma 16.5. By Lemma 27.4 we see that $X_k \to \mathrm{Spec}(k)$ is locally quasi-finite. By definition this means that $U_k \to \mathrm{Spec}(k)$ is locally quasi-finite. Hence $|U_k|$ is discrete by Morphisms, Lemma 20.8. Since $|U_k| \to |X_k|$ is surjective and open we conclude that $|X_k|$ is discrete.

Conversely, assume (2). Choose a surjective étale morphism $V \to Y$ where V is a scheme. Choose a surjective étale morphism $U \to V \times_Y X$ where U is a scheme.

Note that $U \to V$ is locally of finite type as f is locally of finite type. Picture

$$\begin{array}{ccccc} U & \longrightarrow & X \times_Y V & \longrightarrow & V \\ & \searrow & \downarrow & & \downarrow \\ & & X & \longrightarrow & Y \end{array}$$

If f is not locally quasi-finite then $U \to V$ is not locally quasi-finite. Hence there exists a specialization $u \rightsquigarrow u'$ for some $u, u' \in U$ lying over the same point $v \in V$, see Morphisms, Lemma 20.6. We claim that u, u' do not have the same image in $X_v = \mathrm{Spec}(\kappa(v)) \times_Y X$ which will contradict the assumption that $|X_v|$ is discrete as desired. Let $d = \mathrm{trdeg}_{\kappa(v)}(\kappa(u))$ and $d' = \mathrm{trdeg}_{\kappa(v)}(\kappa(u'))$. Then we see that $d > d'$ by Morphisms, Lemma 28.7. Note that U_v (the fibre of $U \to V$ over v) is the fibre product of U and X_v over $X \times_Y V$, hence $U_v \to X_v$ is étale (as a base change of the étale morphism $U \to X \times_Y V$). If $u, u' \in U_v$ map to the same element of $|X_v|$ then there exists a point $r \in R_v = U_v \times_{X_v} U_v$ with $t(r) = u$ and $s(r) = u'$, see Properties of Spaces, Lemma 4.3. Note that $s, t : R_v \to U_v$ are étale morphisms of schemes over $\kappa(v)$, hence $\kappa(u) \subset \kappa(r) \supset \kappa(u')$ are finite separable extensions of fields over $\kappa(v)$ (see Morphisms, Lemma 36.7). We conclude that the transcendence degrees are equal. This contradiction finishes the proof. □

040Z **Lemma 27.6.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is locally quasi-finite,*
(2) *for every $x \in |X|$ the morphism f is quasi-finite at x,*
(3) *for every scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is locally quasi-finite,*
(4) *for every affine scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is locally quasi-finite,*
(5) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is locally quasi-finite,*
(6) *there exists a scheme U and a surjective étale morphism $\varphi : U \to X$ such that the composition $f \circ \varphi$ is locally quasi-finite,*
(7) *for every commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where U, V are schemes and the vertical arrows are étale the top horizontal arrow is locally quasi-finite,
(8) *there exists a commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where U, V are schemes, the vertical arrows are étale, and $U \to X$ is surjective such that the top horizontal arrow is locally quasi-finite, and

(9) *there exist Zariski coverings* $Y = \bigcup_{i \in I} Y_i$, *and* $f^{-1}(Y_i) = \bigcup X_{ij}$ *such that each morphism* $X_{ij} \to Y_i$ *is locally quasi-finite.*

Proof. Omitted. □

03XM **Lemma 27.7.** *An immersion is locally quasi-finite.*

Proof. Omitted. □

03XN **Lemma 27.8.** *Let* S *be a scheme. Let* $X \to Y \to Z$ *be morphisms of algebraic spaces over* S. *If* $X \to Z$ *is locally quasi-finite, then* $X \to Y$ *is locally quasi-finite.*

Proof. Choose a commutative diagram

$$\begin{array}{ccccc} U & \longrightarrow & V & \longrightarrow & W \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & Y & \longrightarrow & Z \end{array}$$

with vertical arrows étale and surjective. (See Spaces, Lemma 11.6.) Apply Morphisms, Lemma 20.17 to the top row. □

0ABN **Lemma 27.9.** *Let* S *be a scheme. Let* $f : X \to Y$ *be a finite type morphism of algebraic spaces over* S. *Let* $y \in |Y|$. *There are at most finitely many points of* $|X|$ *lying over* y *at which* f *is quasi-finite.*

Proof. Choose a field k and a morphism $\mathrm{Spec}(k) \to Y$ in the equivalence class determined by y. The fibre $X_k = \mathrm{Spec}(k) \times_Y X$ is an algebraic space of finite type over a field, in particular quasi-compact. The map $|X_k| \to |X|$ surjects onto the fibre of $|X| \to |Y|$ over y (Properties of Spaces, Lemma 4.3). Moreover, the set of points where $X_k \to \mathrm{Spec}(k)$ is quasi-finite maps onto the set of points lying over y where f is quasi-finite by Lemma 27.2. Choose an affine scheme U and a surjective étale morphism $U \to X_k$ (Properties of Spaces, Lemma 6.3). Then $U \to \mathrm{Spec}(k)$ is a morphism of finite type and there are at most a finite number of points where this morphism is quasi-finite, see Morphisms, Lemma 20.14. Since $X_k \to \mathrm{Spec}(k)$ is quasi-finite at a point x' if and only if it is the image of a point of U where $U \to \mathrm{Spec}(k)$ is quasi-finite, we conclude. □

0463 **Lemma 27.10.** *Let* S *be a scheme. Let* $f : X \to Y$ *be a morphism of algebraic spaces over* S. *If* f *is locally of finite type and a monomorphism, then* f *is separated and locally quasi-finite.*

Proof. A monomorphism is separated, see Lemma 10.3. By Lemma 27.6 it suffices to prove the lemma after performing a base change by $Z \to Y$ with Z affine. Hence we may assume that Y is an affine scheme. Choose an affine scheme U and an étale morphism $U \to X$. Since $X \to Y$ is locally of finite type the morphism of affine schemes $U \to Y$ is of finite type. Since $X \to Y$ is a monomorphism we have $U \times_X U = U \times_Y U$. In particular the maps $U \times_Y U \to U$ are étale. Let $y \in Y$. Then either U_y is empty, or $\mathrm{Spec}(\kappa(u)) \times_{\mathrm{Spec}(\kappa(y))} U_y$ is isomorphic to the fibre of $U \times_Y U \to U$ over u for some $u \in U$ lying over y. This implies that the fibres of $U \to Y$ are finite discrete sets (as $U \times_Y U \to U$ is an étale morphism of affine schemes, see Morphisms, Lemma 36.7). Hence $U \to Y$ is quasi-finite, see Morphisms, Lemma 20.6. As $U \to X$ was an arbitrary étale morphism with U affine this implies that $X \to Y$ is locally quasi-finite. □

28. Morphisms of finite presentation

03XO The property "locally of finite presentation" of morphisms of schemes is étale local on the source-and-target, see Descent, Remark 32.7. It is also stable under base change and fpqc local on the target, see Morphisms, Lemma 21.4, and Descent, Lemma 23.11. Hence, by Lemma 22.1 above, we may define what it means for a morphism of algebraic spaces to be locally of finite presentation as follows and it agrees with the already existing notion defined in Section 3 when the morphism is representable.

03XP **Definition 28.1.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say f is *locally of finite presentation* if the equivalent conditions of Lemma 22.1 hold with $\mathcal{P}$ ="locally of finite presentation".
(2) Let $x \in |X|$. We say f is of *finite presentation at* x if there exists an open neighbourhood $X' \subset X$ of x such that $f|_{X'} : X' \to Y$ is locally of finite presentation[6].
(3) A morphism of algebraic spaces $f : X \to Y$ is *of finite presentation* if it is locally of finite presentation, quasi-compact and quasi-separated.

Note that a morphism of finite presentation is **not** just a quasi-compact morphism which is locally of finite presentation.

03XQ **Lemma 28.2.** *The composition of morphisms of finite presentation is of finite presentation. The same holds for locally of finite presentation.*

Proof. See Remark 22.3 and Morphisms, Lemma 21.3. Also use the result for quasi-compact and for quasi-separated morphisms (Lemmas 8.5 and 4.8). □

03XR **Lemma 28.3.** *A base change of a morphism of finite presentation is of finite presentation. The same holds for locally of finite presentation.*

Proof. See Remark 22.4 and Morphisms, Lemma 21.4. Also use the result for quasi-compact and for quasi-separated morphisms (Lemmas 8.4 and 4.4). □

0410 **Lemma 28.4.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is locally of finite presentation,*
(2) *for every $x \in |X|$ the morphism f is of finite presentation at x,*
(3) *for every scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is locally of finite presentation,*
(4) *for every affine scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is locally of finite presentation,*
(5) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is locally of finite presentation,*
(6) *there exists a scheme U and a surjective étale morphism $\varphi : U \to X$ such that the composition $f \circ \varphi$ is locally of finite presentation,*

[6]It seems awkward to use "locally of finite presentation at x", but the current terminology may be misleading in the sense that "of finite presentation at x" does **not** mean that there is an open neighbourhood $X' \subset X$ such that $f|_{X'}$ is of finite presentation.

(7) *for every commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where U, V are schemes and the vertical arrows are étale the top horizontal arrow is locally of finite presentation,

(8) *there exists a commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where U, V are schemes, the vertical arrows are étale, and $U \to X$ is surjective such that the top horizontal arrow is locally of finite presentation, and

(9) *there exist Zariski coverings $Y = \bigcup_{i \in I} Y_i$, and $f^{-1}(Y_i) = \bigcup X_{ij}$ such that each morphism $X_{ij} \to Y_i$ is locally of finite presentation.*

Proof. Omitted. □

0464 **Lemma 28.5.** *A morphism which is locally of finite presentation is locally of finite type. A morphism of finite presentation is of finite type.*

Proof. Let $f : X \to Y$ be a morphism of algebraic spaces which is locally of finite presentation. This means there exists a diagram as in Lemma 22.1 with h locally of finite presentation and surjective vertical arrow a. By Morphisms, Lemma 21.8 h is locally of finite type. Hence $X \to Y$ is locally of finite type by definition. If f is of finite presentation then it is quasi-compact and it follows that f is of finite type. □

04ZL **Lemma 28.6.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is of finite presentation and Y is Noetherian, then X is Noetherian.*

Proof. Assume f is of finite presentation and Y Noetherian. By Lemmas 28.5 and 23.5 we see that X is locally Noetherian. As f is quasi-compact and Y is quasi-compact we see that X is quasi-compact. As f is of finite presentation it is quasi-separated (see Definition 28.1) and as Y is Noetherian it is quasi-separated (see Properties of Spaces, Definition 24.1). Hence X is quasi-separated by Lemma 4.9. Hence we have checked all three conditions of Properties of Spaces, Definition 24.1 and we win. □

06G4 **Lemma 28.7.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.*

(1) *If Y is locally Noetherian and f locally of finite type then f is locally of finite presentation.*
(2) *If Y is locally Noetherian and f of finite type and quasi-separated then f is of finite presentation.*

Proof. Assume $f : X \to Y$ locally of finite type and Y locally Noetherian. This means there exists a diagram as in Lemma 22.1 with h locally of finite type and surjective vertical arrow a. By Morphisms, Lemma 21.9 h is locally of finite presentation. Hence $X \to Y$ is locally of finite presentation by definition. This proves (1). If f is of finite type and quasi-separated then it is also quasi-compact and quasi-separated and (2) follows immediately. □

06G5 **Lemma 28.8.** *Let S be a scheme. Let Y be an algebraic space over S which is quasi-compact and quasi-separated. If X is of finite presentation over Y, then X is quasi-compact and quasi-separated.*

Proof. Omitted. □

05WT **Lemma 28.9.** *Let S be a scheme. Let $f : X \to Y$ and $Y \to Z$ be morphisms of algebraic spaces over S. If X is locally of finite presentation over Z, and Y is locally of finite type over Z, then f is locally of finite presentation.*

Proof. Choose a scheme W and a surjective étale morphism $W \to Z$. Then choose a scheme V and a surjective étale morphism $V \to W \times_Z Y$. Finally choose a scheme U and a surjective étale morphism $U \to V \times_Y X$. By definition U is locally of finite presentation over W and V is locally of finite type over W. By Morphisms, Lemma 21.11 the morphism $U \to V$ is locally of finite presentation. Hence f is locally of finite presentation. □

084P **Lemma 28.10.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S with diagonal $\Delta : X \to X \times_Y X$. If f is locally of finite type then Δ is locally of finite presentation. If f is quasi-separated and locally of finite type, then Δ is of finite presentation.*

Proof. Note that Δ is a morphism over X (via the second projection $X \times_Y X \to X$). Assume f is locally of finite type. Note that X is of finite presentation over X and $X \times_Y X$ is of finite type over X (by Lemma 23.3). Thus the first statement holds by Lemma 28.9. The second statement follows from the first, the definitions, and the fact that a diagonal morphism is separated (Lemma 4.1). □

06CN **Lemma 28.11.** *An open immersion of algebraic spaces is locally of finite presentation.*

Proof. An open immersion is by definition representable, hence we can use the general principle Spaces, Lemma 5.8 and Morphisms, Lemma 21.5. □

084Q **Lemma 28.12.** *A closed immersion $i : Z \to X$ is of finite presentation if and only if the associated quasi-coherent sheaf of ideals $\mathcal{I} = \mathrm{Ker}(\mathcal{O}_X \to i_*\mathcal{O}_Z)$ is of finite type (as an $\mathcal{O}_X$-module).*

Proof. Let U be a scheme and let $U \to X$ be a surjective étale morphism. By Lemma 28.4 we see that $i' : Z \times_X U \to U$ is of finite presentation if and only if i is. By Properties of Spaces, Section 30 we see that $\mathcal{I}$ is of finite type if and only if $\mathcal{I}|_U = \mathrm{Ker}(\mathcal{O}_U \to i'_*\mathcal{O}_{Z\times_X U})$ is. Hence the result follows from the case of schemes, see Morphisms, Lemma 21.7. □

29. Constructible sets

0ECV This section is the continuation of Properties of Spaces, Section 8.

0ECW **Lemma 29.1.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $E \subset |Y|$ be a subset. If E is étale locally constructible in Y, then $f^{-1}(E)$ is étale locally constructible in X.*

Proof. Choose a scheme V and a surjective étale morphism $\varphi : V \to Y$. Choose a scheme U and a surjective étale morphism $U \to V \times_Y X$. Then $U \to X$ is surjective étale and the inverse image of $f^{-1}(E)$ in U is the inverse image of $\varphi^{-1}(E)$ by $U \to V$. Thus the lemma follows from the case of schemes for $U \to V$ (Morphisms, Lemma 22.1) and the definition (Properties of Spaces, Definition 8.2). □

0ECX **Theorem 29.2** (Chevalley's Theorem)**.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is quasi-compact and locally of finite presentation. Then the image of every étale locally constructible subset of $|X|$ is an étale locally constructible subset of $|Y|$.*

Proof. Let $E \subset |X|$ be étale locally constructible. Let $V \to Y$ be an étale morphism with V affine. It suffices to show that the inverse image of $f(E)$ in V is constructible, see Properties of Spaces, Definition 8.2. Since f is quasi-compact $V \times_Y X$ is a quasi-compact algebraic space. Choose an affine scheme U and a surjective étale morphism $U \to V \times_Y X$ (Properties of Spaces, Lemma 6.3). By Properties of Spaces, Lemma 4.3 the inverse image of $f(E)$ in V is the image under $U \to V$ of the inverse image of E in U. Thus the result follows from the case of schemes, see Morphisms, Lemma 22.2. □

30. Flat morphisms

03MK The property "flat" of morphisms of schemes is étale local on the source-and-target, see Descent, Remark 32.7. It is also stable under base change and fpqc local on the target, see Morphisms, Lemma 25.8 and Descent, Lemma 23.15. Hence, by Lemma 22.1 above, we may define the notion of a flat morphism of algebraic spaces as follows and it agrees with the already existing notion defined in Section 3 when the morphism is representable.

03ML **Definition 30.1.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say f is *flat* if the equivalent conditions of Lemma 22.1 with $\mathcal{P}$ ="flat".
(2) Let $x \in |X|$. We say f is *flat at x* if the equivalent conditions of Lemma 22.5 hold with $\mathcal{Q}$ ="induced map local rings is flat".

Note that the second part makes sense by Descent, Lemma 33.4.

We do a quick sanity check.

08EW **Lemma 30.2.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Then f is flat if and only if f is flat at all points of $|X|$.*

Proof. Choose a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{h} & V \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ X & \xrightarrow{f} & Y \end{array}$$

where U and V are schemes, the vertical arrows are étale, and a is surjective. By definition f is flat if and only if h is flat (Definition 22.2). By definition f is flat at $x \in |X|$ if and only if h is flat at some (equivalently any) $u \in U$ which maps to x (Definition 22.6). Thus the lemma follows from the fact that a morphism of schemes is flat if and only if it is flat at all points of the source (Morphisms, Definition 25.1). □

03MN **Lemma 30.3.** *The composition of flat morphisms is flat.*

Proof. See Remark 22.3 and Morphisms, Lemma 25.6. □

03MO **Lemma 30.4.** *The base change of a flat morphism is flat.*

Proof. See Remark 22.4 and Morphisms, Lemma 25.8. □

03MM **Lemma 30.5.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is flat,*
(2) *for every $x \in |X|$ the morphism f is flat at x,*
(3) *for every scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is flat,*
(4) *for every affine scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is flat,*
(5) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is flat,*
(6) *there exists a scheme U and a surjective étale morphism $\varphi : U \to X$ such that the composition $f \circ \varphi$ is flat,*
(7) *for every commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where U, V are schemes and the vertical arrows are étale the top horizontal arrow is flat,
(8) *there exists a commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where U, V are schemes, the vertical arrows are étale, and $U \to X$ is surjective such that the top horizontal arrow is flat, and
(9) *there exists a Zariski coverings $Y = \bigcup Y_i$ and $f^{-1}(Y_i) = \bigcup X_{ij}$ such that each morphism $X_{ij} \to Y_i$ is flat.*

Proof. Omitted. □

042S **Lemma 30.6.** *A flat morphism locally of finite presentation is universally open.*

Proof. Let $f : X \to Y$ be a flat morphism locally of finite presentation of algebraic spaces over S. Choose a diagram

$$\begin{array}{ccc} U & \xrightarrow[\alpha]{} & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where U and V are schemes and the vertical arrows are surjective and étale, see Spaces, Lemma 11.6. By Lemmas 30.5 and 28.4 the morphism α is flat and locally of finite presentation. Hence by Morphisms, Lemma 25.10 we see that α is universally open. Hence $X \to Y$ is universally open according to Lemma 6.5. □

0413 **Lemma 30.7.** *Let S be a scheme. Let $f : X \to Y$ be a flat, quasi-compact, surjective morphism of algebraic spaces over S. A subset $T \subset |Y|$ is open (resp. closed) if and only $f^{-1}(|T|)$ is open (resp. closed) in $|X|$. In other words f is submersive, and in fact universally submersive.*

Proof. Choose affine schemes V_i and étale morphisms $V_i \to Y$ such that $V = \coprod V_i \to Y$ is surjective, see Properties of Spaces, Lemma 6.1. For each i the algebraic space $V_i \times_Y X$ is quasi-compact. Hence we can find an affine scheme U_i and a surjective étale morphism $U_i \to V_i \times_Y X$, see Properties of Spaces, Lemma 6.3. Then the composition $U_i \to V_i \times_Y X \to V_i$ is a surjective, flat morphism of affines. Of course then $U = \coprod U_i \to X$ is surjective and étale and $U = V \times_Y X$. Moreover, the morphism $U \to V$ is the disjoint union of the morphisms $U_i \to V_i$. Hence $U \to V$ is surjective, quasi-compact and flat. Consider the diagram

$$\begin{array}{ccc} U & \longrightarrow & X \\ \downarrow & & \downarrow \\ V & \longrightarrow & Y \end{array}$$

By definition of the topology on $|Y|$ the set T is closed (resp. open) if and only if $g^{-1}(T) \subset |V|$ is closed (resp. open). The same holds for $f^{-1}(T)$ and its inverse image in $|U|$. Since $U \to V$ is quasi-compact, surjective, and flat we win by Morphisms, Lemma 25.12. □

04NG **Lemma 30.8.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\overline{x}$ be a geometric point of X lying over the point $x \in |X|$. Let $\overline{y} = f \circ \overline{x}$. The following are equivalent*

1. *f is flat at x, and*
2. *the map on étale local rings $\mathcal{O}_{Y,\overline{y}} \to \mathcal{O}_{X,\overline{x}}$ is flat.*

Proof. Choose a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow[h]{} & V \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ X & \xrightarrow{f} & Y \end{array}$$

where U and V are schemes, a, b are étale, and $u \in U$ mapping to x. We can find a geometric point $\overline{u} : \mathrm{Spec}(k) \to U$ lying over u with $\overline{x} = a \circ \overline{u}$, see Properties of Spaces, Lemma 19.4. Set $\overline{v} = h \circ \overline{u}$ with image $v \in V$. We know that

$$\mathcal{O}_{X,\overline{x}} = \mathcal{O}^{sh}_{U,u} \quad \text{and} \quad \mathcal{O}_{Y,\overline{y}} = \mathcal{O}^{sh}_{V,v}$$

see Properties of Spaces, Lemma 22.1. We obtain a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{U,u} & \longrightarrow & \mathcal{O}_{X,\overline{x}} \\ \uparrow & & \uparrow \\ \mathcal{O}_{V,v} & \longrightarrow & \mathcal{O}_{Y,\overline{y}} \end{array}$$

of local rings with flat horizontal arrows. We have to show that the left vertical arrow is flat if and only if the right vertical arrow is. Algebra, Lemma 39.9 tells us $\mathcal{O}_{U,u}$ is flat over $\mathcal{O}_{V,v}$ if and only if $\mathcal{O}_{X,\overline{x}}$ is flat over $\mathcal{O}_{V,v}$. Hence the result follows from More on Flatness, Lemma 2.5. □

073C **Lemma 30.9.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Then f is flat if and only if the morphism of sites $(f_{small}, f^\sharp) : (X_{étale}, \mathcal{O}_X) \to (Y_{étale}, \mathcal{O}_Y)$ associated to f is flat.*

Proof. Flatness of $(f_{small}, f^\sharp)$ is defined in terms of flatness of $\mathcal{O}_X$ as a $f_{small}^{-1}\mathcal{O}_Y$-module. This can be checked at stalks, see Modules on Sites, Lemma 39.3 and Properties of Spaces, Theorem 19.12. But we've already seen that flatness of f can be checked on stalks, see Lemma 30.8. □

089C **Lemma 30.10.** *Let S be a scheme. Let $f : Y \to X$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module with scheme theoretic support $Z \subset X$. If f is flat, then $f^{-1}(Z)$ is the scheme theoretic support of $f^*\mathcal{F}$.*

Proof. Using the characterization of the scheme theoretic support as given in Lemma 15.3 and using the characterization of flat morphisms in terms of étale coverings in Lemma 30.5 we reduce to the case of schemes which is Morphisms, Lemma 25.14. □

089D **Lemma 30.11.** *Let S be a scheme. Let $f : X \to Y$ be a flat morphism of algebraic spaces over S. Let $V \to Y$ be a quasi-compact open immersion. If V is scheme theoretically dense in Y, then $f^{-1}V$ is scheme theoretically dense in X.*

Proof. Using the characterization of scheme theoretically dense opens in Lemma 17.2 and using the characterization of flat morphisms in terms of étale coverings in Lemma 30.5 we reduce to the case of schemes which is Morphisms, Lemma 25.15. □

089E **Lemma 30.12.** *Let S be a scheme. Let $f : X \to Y$ be a flat morphism of algebraic spaces over S. Let $g : V \to Y$ be a quasi-compact morphism of algebraic spaces. Let $Z \subset Y$ be the scheme theoretic image of g and let $Z' \subset X$ be the scheme theoretic image of the base change $V \times_Y X \to X$. Then $Z' = f^{-1}Z$.*

Proof. Let $Y' \to Y$ be a surjective étale morphism such that Y' is a disjoint union of affine schemes (Properties of Spaces, Lemma 6.1). Let $X' \to X \times_Y Y'$ be a surjective étale morphism such that X' is a disjoint union of affine schemes. By Lemma 30.5 the morphism $X' \to Y'$ is flat. Set $V' = V \times_Y Y'$. By Lemma 16.3 the inverse image of Z in Y' is the scheme theoretic image of $V' \to Y'$ and the inverse image of Z' in X' is the scheme theoretic image of $V' \times_{Y'} X' \to X'$. Since $X' \to X$ is surjective étale, it suffices to prove the result in the case of the morphisms $X' \to Y'$ and $V' \to Y'$. Thus we may assume X and Y are affine schemes. In this case V is

a quasi-compact algebraic space. Choose an affine scheme W and a surjective étale morphism $W \to V$ (Properties of Spaces, Lemma 6.3). It is clear that the scheme theoretic image of $V \to Y$ agrees with the scheme theoretic image of $W \to Y$ and similarly for $V \times_Y X \to Y$ and $W \times_Y X \to X$. Thus we reduce to the case of schemes which is Morphisms, Lemma 25.16. □

31. Flat modules

05VT In this section we define what it means for a module to be flat at a point. To do this we will use the notion of the stalk of a sheaf on the small étale site $X_{étale}$ of an algebraic space, see Properties of Spaces, Definition 19.6.

05VU **Lemma 31.1.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $x \in |X|$. The following are equivalent*

(1) *for some commutative diagram*

$$\begin{array}{ccc} U & \xrightarrow{h} & V \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ X & \xrightarrow{f} & Y \end{array}$$

where U and V are schemes, a, b are étale, and $u \in U$ mapping to x the module $a^*\mathcal{F}$ is flat at u over V,

(2) *the stalk $\mathcal{F}_{\overline{x}}$ is flat over the étale local ring $\mathcal{O}_{Y,\overline{y}}$ where $\overline{x}$ is any geometric point lying over x and $\overline{y} = f \circ \overline{x}$.*

Proof. During this proof we fix a geometric proof $\overline{x} : \mathrm{Spec}(k) \to X$ over x and we denote $\overline{y} = f \circ \overline{x}$ its image in Y. Given a diagram as in (1) we can find a geometric point $\overline{u} : \mathrm{Spec}(k) \to U$ lying over u with $\overline{x} = a \circ \overline{u}$, see Properties of Spaces, Lemma 19.4. Set $\overline{v} = h \circ \overline{u}$ with image $v \in V$. We know that

$$\mathcal{O}_{X,\overline{x}} = \mathcal{O}^{sh}_{U,u} \quad \text{and} \quad \mathcal{O}_{Y,\overline{y}} = \mathcal{O}^{sh}_{V,v}$$

see Properties of Spaces, Lemma 22.1. We obtain a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{U,u} & \longrightarrow & \mathcal{O}_{X,\overline{x}} \\ \uparrow & & \uparrow \\ \mathcal{O}_{V,v} & \longrightarrow & \mathcal{O}_{Y,\overline{y}} \end{array}$$

of local rings. Finally, we have

$$\mathcal{F}_{\overline{x}} = (\varphi^*\mathcal{F})_u \otimes_{\mathcal{O}_{U,u}} \mathcal{O}_{X,\overline{x}}$$

by Properties of Spaces, Lemma 29.4. Thus Algebra, Lemma 39.9 tells us $(\varphi^*\mathcal{F})_u$ is flat over $\mathcal{O}_{V,v}$ if and only if $\mathcal{F}_{\overline{x}}$ is flat over $\mathcal{O}_{V,v}$. Hence the result follows from More on Flatness, Lemma 2.5. □

05VV **Definition 31.2.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent sheaf on X.

(1) Let $x \in |X|$. We say $\mathcal{F}$ is *flat at x over Y* if the equivalent conditions of Lemma 31.1 hold.
(2) We say $\mathcal{F}$ is *flat over Y* if $\mathcal{F}$ is flat over Y at all $x \in |X|$.

Having defined this we have the obligatory base change lemma. This lemma implies that formation of the flat locus of a quasi-coherent sheaf commutes with flat base change.

05VW **Lemma 31.3.** *Let S be a scheme. Let*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

be a cartesian diagram of algebraic spaces over S. Let $x' \in |X'|$ with image $x \in |X|$. Let $\mathcal{F}$ be a quasi-coherent sheaf on X and denote $\mathcal{F}' = (g')^*\mathcal{F}$.

(1) *If $\mathcal{F}$ is flat at x over Y then $\mathcal{F}'$ is flat at x' over Y'.*
(2) *If g is flat at $f'(x')$ and $\mathcal{F}'$ is flat at x' over Y', then $\mathcal{F}$ is flat at x over Y.*

In particular, if $\mathcal{F}$ is flat over Y, then $\mathcal{F}'$ is flat over Y'.

Proof. Choose a scheme V and a surjective étale morphism $V \to Y$. Choose a scheme U and a surjective étale morphism $U \to V \times_Y X$. Choose a scheme V' and a surjective étale morphism $V' \to V \times_Y Y'$. Then $U' = V' \times_V U$ is a scheme endowed with a surjective étale morphism $U' = V' \times_V U \to Y' \times_Y X = X'$. Pick $u' \in U'$ mapping to $x' \in |X'|$. Then we can check flatness of $\mathcal{F}'$ at x' over Y' in terms of flatness of $\mathcal{F}'|_{U'}$ at u' over V'. Hence the lemma follows from More on Morphisms, Lemma 15.2. □

The following lemma discusses "composition" of flat morphisms in terms of modules. It also shows that flatness satisfies a kind of top down descent.

05VX **Lemma 31.4.** *Let S be a scheme. Let $X \to Y \to Z$ be morphisms of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $x \in |X|$ with image $y \in |Y|$.*

(1) *If $\mathcal{F}$ is flat at x over Y and Y is flat at y over Z, then $\mathcal{F}$ is flat at x over Z.*
(2) *Let $x : \mathrm{Spec}(K) \to X$ be a representative of x. If*
   (a) *$\mathcal{F}$ is flat at x over Y,*
   (b) *$x^*\mathcal{F} \neq 0$, and*
   (c) *$\mathcal{F}$ is flat at x over Z,*

   then Y is flat at y over Z.
(3) *Let $\overline{x}$ be a geometric point of X lying over x with image $\overline{y}$ in Y. If $\mathcal{F}_{\overline{x}}$ is a faithfully flat $\mathcal{O}_{Y,\overline{y}}$-module and $\mathcal{F}$ is flat at x over Z, then Y is flat at y over Z.*

Proof. Pick $\overline{x}$ and $\overline{y}$ as in part (3) and denote $\overline{z}$ the induced geometric point of Z. Via the characterization of flatness in Lemmas 31.1 and 30.8 the lemma reduces to a purely algebraic question on the local ring map $\mathcal{O}_{Z,\overline{z}} \to \mathcal{O}_{Y,\overline{y}}$ and the module $\mathcal{F}_{\overline{x}}$. Part (1) follows from Algebra, Lemma 39.4. We remark that condition (2)(b) guarantees that $\mathcal{F}_{\overline{x}}/\mathfrak{m}_{\overline{y}}\mathcal{F}_{\overline{x}}$ is nonzero. Hence (2)(a) + (2)(b) imply that $\mathcal{F}_{\overline{x}}$ is a faithfully flat $\mathcal{O}_{Y,\overline{y}}$-module, see Algebra, Lemma 39.15. Thus (2) is a special case of (3). Finally, (3) follows from Algebra, Lemma 39.10. □

Sometimes the base change happens "up on top". Here is a precise statement.

05VY **Lemma 31.5.** *Let S be a scheme. Let $f : X \to Y$, $g : Y \to Z$ be morphisms of algebraic spaces over S. Let $\mathcal{G}$ be a quasi-coherent sheaf on Y. Let $x \in |X|$ with image $y \in |Y|$. If f is flat at x, then*

$$\mathcal{G} \text{ flat over } Z \text{ at } y \Leftrightarrow f^*\mathcal{G} \text{ flat over } Z \text{ at } x.$$

In particular: If f is surjective and flat, then $\mathcal{G}$ is flat over Z, if and only if $f^*\mathcal{G}$ is flat over Z.

Proof. Pick a geometric point $\overline{x}$ of X and denote $\overline{y}$ the image in Y and $\overline{z}$ the image in Z. Via the characterization of flatness in Lemmas 31.1 and 30.8 and the description of the stalk of $f^*\mathcal{G}$ at $\overline{x}$ of Properties of Spaces, Lemma 29.5 the lemma reduces to a purely algebraic question on the local ring maps $\mathcal{O}_{Z,\overline{z}} \to \mathcal{O}_{Y,\overline{y}} \to \mathcal{O}_{X,\overline{x}}$ and the module $\mathcal{G}_{\overline{y}}$. This algebraic statement is Algebra, Lemma 39.9. □

0CVU **Lemma 31.6.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume f locally finite presentation, $\mathcal{F}$ of finite type, $X = Supp(\mathcal{F})$, and $\mathcal{F}$ flat over Y. Then f is universally open.*

Proof. Choose a surjective étale morphism $\varphi : V \to Y$ where V is a scheme. Choose a surjective étale morphism $U \to V \times_Y X$ where U is a scheme. Then it suffices to prove the lemma for $U \to V$ and the quasi-coherent $\mathcal{O}_V$-module $\varphi^*\mathcal{F}$. Hence this lemma follows from the case of schemes, see Morphisms, Lemma 25.11. □

32. Generic flatness

06QR This section is the analogue of Morphisms, Section 27.

06QS **Proposition 32.1.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules. Assume*

1. *Y is reduced,*
2. *f is of finite type, and*
3. *$\mathcal{F}$ is a finite type $\mathcal{O}_X$-module.*

Then there exists an open dense subspace $W \subset Y$ such that the base change $X_W \to W$ of f is flat, locally of finite presentation, and quasi-compact and such that $\mathcal{F}|_{X_W}$ is flat over W and of finite presentation over $\mathcal{O}_{X_W}$.

Proof. Let V be a scheme and let $V \to Y$ be a surjective étale morphism. Let $X_V = V \times_Y X$ and let $\mathcal{F}_V$ be the restriction of $\mathcal{F}$ to X_V. Suppose that the result holds for the morphism $X_V \to V$ and the sheaf $\mathcal{F}_V$. Then there exists an open subscheme $V' \subset V$ such that $X_{V'} \to V'$ is flat and of finite presentation and $\mathcal{F}_{V'}$ is an $\mathcal{O}_{X_{V'}}$-module of finite presentation flat over V'. Let $W \subset Y$ be the image of the étale morphism $V' \to Y$, see Properties of Spaces, Lemma 4.10. Then $V' \to W$ is a surjective étale morphism, hence we see that $X_W \to W$ is flat, locally of finite presentation, and quasi-compact by Lemmas 28.4, 30.5, and 8.8. By the discussion in Properties of Spaces, Section 30 we see that $\mathcal{F}_W$ is of finite presentation as a $\mathcal{O}_{X_W}$-module and by Lemma 31.3 we see that $\mathcal{F}_W$ is flat over W. This argument reduces the proposition to the case where Y is a scheme.

Suppose we can prove the proposition when Y is an affine scheme. Let $f : X \to Y$ be a finite type morphism of algebraic spaces over S with Y a scheme, and let

$\mathcal{F}$ be a finite type, quasi-coherent $\mathcal{O}_X$-module. Choose an affine open covering $Y = \bigcup V_j$. By assumption we can find dense open $W_j \subset V_j$ such that $X_{W_j} \to W_j$ is flat, locally of finite presentation, and quasi-compact and such that $\mathcal{F}|_{X_{W_j}}$ is flat over W_j and of finite presentation as an $\mathcal{O}_{X_{W_j}}$-module. In this situation we simply take $W = \bigcup W_j$ and we win. Hence we reduce the proposition to the case where Y is an affine scheme.

Let Y be an affine scheme over S, let $f : X \to Y$ be a finite type morphism of algebraic spaces over S, and let $\mathcal{F}$ be a finite type, quasi-coherent $\mathcal{O}_X$-module. Since f is of finite type it is quasi-compact, hence X is quasi-compact. Thus we can find an affine scheme U and a surjective étale morphism $U \to X$, see Properties of Spaces, Lemma 6.3. Note that $U \to Y$ is of finite type (this is what it means for f to be of finite type in this case). Hence we can apply Morphisms, Proposition 27.2 to see that there exists a dense open $W \subset Y$ such that $U_W \to W$ is flat and of finite presentation and such that $\mathcal{F}|_{U_W}$ is flat over W and of finite presentation as an $\mathcal{O}_{U_W}$-module. According to our definitions this means that the base change $X_W \to W$ of f is flat, locally of finite presentation, and quasi-compact and $\mathcal{F}|_{X_W}$ is flat over W and of finite presentation over $\mathcal{O}_{X_W}$. □

We cannot improve the result of the lemma above to requiring $X_W \to W$ to be of finite presentation as $\mathbf{A}^1_{\mathbf{Q}}/\mathbf{Z} \to \mathrm{Spec}(\mathbf{Q})$ gives a counter example. The problem is that the diagonal morphism $\Delta_{X/Y}$ may not be quasi-compact, i.e., f may not be quasi-separated. Clearly, this is also the only problem.

06QT **Proposition 32.2.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules. Assume*

(1) *Y is reduced,*
(2) *f is quasi-separated,*
(3) *f is of finite type, and*
(4) *$\mathcal{F}$ is a finite type $\mathcal{O}_X$-module.*

Then there exists an open dense subspace $W \subset Y$ such that the base change $X_W \to W$ of f is flat and of finite presentation and such that $\mathcal{F}|_{X_W}$ is flat over W and of finite presentation over $\mathcal{O}_{X_W}$.

Proof. This follows immediately from Proposition 32.1 and the fact that "of finite presentation" = "locally of finite presentation" + "quasi-compact" + "quasi-separated". □

33. Relative dimension

04NH In this section we define the relative dimension of a morphism of algebraic spaces at a point, and some closely related properties.

04NM **Definition 33.1.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $x \in |X|$. Let $d, r \in \{0, 1, 2, \ldots, \infty\}$.

(1) We say the *dimension of the local ring of the fibre of f at x* is d if the equivalent conditions of Lemma 22.5 hold for the property $\mathcal{P}_d$ described in Descent, Lemma 33.6.
(2) We say the *transcendence degree of $x/f(x)$* is r if the equivalent conditions of Lemma 22.5 hold for the property $\mathcal{P}_r$ described in Descent, Lemma 33.7.

(3) We say f *has relative dimension* d *at* x if the equivalent conditions of Lemma 22.5 hold for the property $\mathcal{P}_d$ described in Descent, Lemma 33.8.

Let us spell out what this means. Namely, choose some diagrams

$$\begin{array}{ccc} U & \xrightarrow{h} & V \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ X & \xrightarrow{f} & Y \end{array} \qquad \begin{array}{ccc} u & \longrightarrow & v \\ \downarrow & & \downarrow \\ x & \longrightarrow & y \end{array}$$

as in Lemma 22.5. Then we have

$$\begin{array}{rcl} \text{relative dimension of } f \text{ at } x & = & \dim_u(U_v) \\ \text{dimension of local ring of the fibre of } f \text{ at } x & = & \dim(\mathcal{O}_{U_v,u}) \\ \text{transcendence degree of } x/f(x) & = & \mathrm{trdeg}_{\kappa(v)}(\kappa(u)) \end{array}$$

Note that if $Y = \mathrm{Spec}(k)$ is the spectrum of a field, then the relative dimension of X/Y at x is the same as $\dim_x(X)$, the transcendence degree of $x/f(x)$ is the transcendence degree over k, and the dimension of the local ring of the fibre of f at x is just the dimension of the local ring at x, i.e., the relative notions become absolute notions in that case.

06LR **Definition 33.2.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $d \in \{0, 1, 2, \ldots\}$.

(1) We say f has *relative dimension* $\leq d$ if f has relative dimension $\leq d$ at all $x \in |X|$.
(2) We say f has *relative dimension* d if f has relative dimension d at all $x \in |X|$.

Having relative dimension *equal* to d means roughly speaking that all nonempty fibres are equidimensional of dimension d.

06RX **Lemma 33.3.** *Let S be a scheme. Let $X \to Y \to Z$ be morphisms of algebraic spaces over S. Let $x \in |X|$ and let $y \in |Y|$, $z \in |Z|$ be the images. Assume $X \to Y$ is locally quasi-finite and $Y \to Z$ locally of finite type. Then the transcendence degree of x/z is equal to the transcendence degree of y/z.*

Proof. We can choose commutative diagrams

$$\begin{array}{ccccc} U & \longrightarrow & V & \longrightarrow & W \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & Y & \longrightarrow & Z \end{array} \qquad \begin{array}{ccccc} u & \longrightarrow & v & \longrightarrow & w \\ \downarrow & & \downarrow & & \downarrow \\ x & \longrightarrow & y & \longrightarrow & z \end{array}$$

where U, V, W are schemes and the vertical arrows are étale. By definition the morphism $U \to V$ is locally quasi-finite which implies that $\kappa(v) \subset \kappa(u)$ is finite, see Morphisms, Lemma 20.5. Hence the result is clear. □

0ECY **Lemma 33.4.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is locally of finite type, Y is Jacobson (Properties of Spaces, Remark 7.3), and $x \in |X|$ is a finite type point of X, then the transcendence degree of $x/f(x)$ is 0.*

Proof. Choose a scheme V and a surjective étale morphism $V \to Y$. Choose a scheme U and a surjective étale morphism $U \to X \times_Y V$. By Lemma 25.5 we can find a finite type point $u \in U$ mapping to x. After shrinking U we may

assume $u \in U$ is closed (Morphisms, Lemma 16.4). Let $v \in V$ be the image of u. By Morphisms, Lemma 16.8 the extension $\kappa(u)/\kappa(v)$ is finite. This finishes the proof. □

0AFH **Lemma 33.5.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of locally Noetherian algebraic spaces over S which is flat, locally of finite type and of relative dimension d. For every point x in $|X|$ with image y in $|Y|$ we have $\dim_x(X) = \dim_y(Y) + d$.*

Proof. By definition of the dimension of an algebraic space at a point (Properties of Spaces, Definition 9.1) and by definition of having relative dimension d, this reduces to the corresponding statement for schemes (Morphisms, Lemma 29.6). □

34. Morphisms and dimensions of fibres

04NP This section is the analogue of Morphisms, Section 28. The formulations in this section are a bit awkward since we do not have local rings of algebraic spaces at points.

04NQ **Lemma 34.1.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $x \in |X|$. Assume f is locally of finite type. Then we have*

$$\begin{array}{c} \textit{relative dimension of } f \textit{ at } x \\ = \\ \textit{dimension of local ring of the fibre of } f \textit{ at } x \\ + \\ \textit{transcendence degree of } x/f(x) \end{array}$$

where the notation is as in Definition 33.1.

Proof. This follows immediately from Morphisms, Lemma 28.1 applied to $h : U \to V$ and $u \in U$ as in Lemma 22.5. □

04NR **Lemma 34.2.** *Let S be a scheme. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of algebraic spaces over S. Let $x \in |X|$ and set $y = f(x)$. Assume f and g locally of finite type. Then*

(1)
$$\begin{array}{c} \textit{relative dimension of } g \circ f \textit{ at } x \\ \leq \\ \textit{relative dimension of } f \textit{ at } x \\ + \\ \textit{relative dimension of } g \textit{ at } y \end{array}$$

(2) *equality holds in (1) if for some morphism $\mathrm{Spec}(k) \to Z$ from the spectrum of a field in the class of $g(f(x)) = g(y)$ the morphism $X_k \to Y_k$ is flat at x, for example if f is flat at x,*

(3)
$$\begin{array}{c} \textit{transcendence degree of } x/g(f(x)) \\ = \\ \textit{transcendence degree of } x/f(x) \\ + \\ \textit{transcendence degree of } f(x)/g(f(x)) \end{array}$$

Proof. Choose a diagram

$$\begin{array}{ccccc} U & \longrightarrow & V & \longrightarrow & W \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & Y & \longrightarrow & Z \end{array}$$

with U, V, W schemes and vertical arrows étale and surjective. (See Spaces, Lemma 11.6.) Choose $u \in U$ mapping to x. Set v, w equal to the images of u in V, W. Apply Morphisms, Lemma 28.2 to the top row and the points u, v, w. Details omitted. □

04NS **Lemma 34.3.** *Let S be a scheme. Let*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

be a fibre product diagram of algebraic spaces over S. Let $x' \in |X'|$. Set $x = g'(x')$. Assume f locally of finite type. Then

(1)
$$\begin{array}{c} \textit{relative dimension of } f \textit{ at } x \\ = \\ \textit{relative dimension of } f' \textit{ at } x' \end{array}$$

(2) *we have*
$$\begin{array}{c} \textit{dimension of local ring of the fibre of } f' \textit{ at } x' \\ - \\ \textit{dimension of local ring of the fibre of } f \textit{ at } x \\ = \\ \textit{transcendence degree of } x/f(x) \\ - \\ \textit{transcendence degree of } x'/f'(x') \end{array}$$
and the common value is ≥ 0,

(3) *given x and $y' \in |Y'|$ mapping to the same $y \in |Y|$ there exists a choice of x' such that the integer in (2) is 0.*

Proof. Choose a surjective étale morphism $V \to Y$ with V a scheme. Choose a surjective étale morphism $U \to V \times_Y X$ with U a scheme. Choose a surjective étale morphism $V' \to V \times_Y Y'$ with V' a scheme. Set $U' = V' \times_V U$. Then the induced morphism $U' \to X'$ is also surjective and étale (argument omitted). Choose $u' \in U'$ mapping to x'. At this point parts (1) and (2) follow by applying Morphisms, Lemma 28.3 to the diagram of schemes involving U', U, V', V and the point u'. To prove (3) first choose $v \in V$ mapping to y. Then using Properties of Spaces, Lemma 4.3 we can choose $v' \in V'$ mapping to y' and v and $u \in U$ mapping to x and v. Finally, according to Morphisms, Lemma 28.3 we can choose $u' \in U'$ mapping to v' and u such that the integer is zero. Then taking $x' \in |X'|$ the image of u' works. □

04NT **Lemma 34.4.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $n \geq 0$. Assume f is locally of finite type. The set*

$$W_n = \{x \in |X| \text{ such that the relative dimension of } f \text{ at } x \leq n\}$$

is open in $|X|$.

Proof. Choose a diagram

$$\begin{array}{ccc} U & \xrightarrow[h]{} & V \\ {\scriptstyle a}\downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where U and V are schemes and the vertical arrows are surjective and étale, see Spaces, Lemma 11.6. By Morphisms, Lemma 28.4 the set U_n of points where h has relative dimension $\leq n$ is open in U. By our definition of relative dimension for morphisms of algebraic spaces at points we see that $U_n = a^{-1}(W_n)$. The lemma follows by definition of the topology on $|X|$. □

04NU **Lemma 34.5.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S Let $n \geq 0$. Assume f is locally of finite presentation. The open*

$$W_n = \{x \in |X| \text{ such that the relative dimension of } f \text{ at } x \leq n\}$$

of Lemma 34.4 is retrocompact in $|X|$. (See Topology, Definition 12.1.)

Proof. Choose a diagram

$$\begin{array}{ccc} U & \xrightarrow[h]{} & V \\ {\scriptstyle a}\downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where U and V are schemes and the vertical arrows are surjective and étale, see Spaces, Lemma 11.6. In the proof of Lemma 34.4 we have seen that $a^{-1}(W_n) = U_n$ is the corresponding set for the morphism h. By Morphisms, Lemma 28.6 we see that U_n is retrocompact in U. The lemma follows by definition of the topology on $|X|$, compare with Properties of Spaces, Lemma 5.5 and its proof. □

04NV **Lemma 34.6.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is locally of finite type. Then f is locally quasi-finite if and only if f has relative dimension 0 at each $x \in |X|$.*

Proof. Choose a diagram

$$\begin{array}{ccc} U & \xrightarrow[h]{} & V \\ {\scriptstyle a}\downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where U and V are schemes and the vertical arrows are surjective and étale, see Spaces, Lemma 11.6. The definitions imply that h is locally quasi-finite if and only if f is locally quasi-finite, and that f has relative dimension 0 at all $x \in |X|$ if and only if h has relative dimension 0 at all $u \in U$. Hence the result follows from the result for h which is Morphisms, Lemma 29.5. □

04NW **Lemma 34.7.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is locally of finite type. Then there exists a canonical open subspace $X' \subset X$ such that $f|_{X'} : X' \to Y$ is locally quasi-finite, and such that the relative dimension of f at any $x \in |X|$, $x \notin |X'|$ is ≥ 1. Formation of X' commutes with arbitrary base change.*

Proof. Combine Lemmas 34.4, 34.6, and 34.3. □

06LS **Lemma 34.8.** *Let S be a scheme. Consider a cartesian diagram*

$$\begin{array}{ccc} X & \xleftarrow{p} & F \\ \downarrow & & \downarrow \\ Y & \longleftarrow & \mathrm{Spec}(k) \end{array}$$

where $X \to Y$ is a morphism of algebraic spaces over S which is locally of finite type and where k is a field over S. Let $z \in |F|$ be such that $\dim_z(F) = 0$. Then, after replacing X by an open subspace containing $p(z)$, the morphism

$$X \longrightarrow Y$$

is locally quasi-finite.

Proof. Let $X' \subset X$ be the open subspace over which f is locally quasi-finite found in Lemma 34.7. Since the formation of X' commutes with arbitrary base change we see that $z \in X' \times_Y \mathrm{Spec}(k)$. Hence the lemma is clear. □

35. The dimension formula

0BAW The analog of the dimension formula (Morphisms, Lemma 52.1) is a bit tricky to formulate, because we would have to define integral algebraic spaces (we do this later) as well as universally catenary algebraic spaces. However, the following version is straightforward.

0BAX **Lemma 35.1.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume Y is locally Noetherian and f locally of finite type. Let $x \in |X|$ with image $y \in |Y|$. Then we have*

$$\begin{aligned} &\textit{the dimension of the local ring of } X \textit{ at } x \leq \\ &\textit{the dimension of the local ring of } Y \textit{ at } y + E - \\ &\textit{the transcendence degree of } x/y \end{aligned}$$

Here E is the maximum of the transcendence degrees of $\xi/f(\xi)$ where $\xi \in |X|$ runs over the points specializing to x at which the local ring of X has dimension 0.

Proof. Choose an affine scheme V, an étale morphism $V \to Y$, and a point $v \in V$ mapping to y. Choose an affine scheme U , an étale morphism $U \to X \times_Y V$ and a point $u \in U$ mapping to v in V and x in X. Unwinding Definition 33.1 and Properties of Spaces, Definition 10.2 we have to show that

$$\dim(\mathcal{O}_{U,u}) \leq \dim(\mathcal{O}_{V,v}) + E - \mathrm{trdeg}_{\kappa(v)}(\kappa(u))$$

Let $\xi_U \in U$ be a generic point of an irreducible component of U which contains u. Then ξ_U maps to a point $\xi \in |X|$ which is in the list used to define the quantity E and in fact every ξ used in the definition of E occurs in this manner (small detail omitted). In particular, there are only a finite number of these ξ and we can take the maximum (i.e., it really is a maximum and not a supremum). The transcendence degree of ξ over $f(\xi)$ is $\mathrm{trdeg}_{\kappa(\xi_V)}(\kappa(\xi_U))$ where $\xi_V \in V$ is the image of ξ_U. Thus the lemma follows from Morphisms, Lemma 52.2. □

0BAY **Lemma 35.2.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume Y is locally Noetherian and f is locally of finite type. Then*

$$\dim(X) \leq \dim(Y) + E$$

where E is the supremum of the transcendence degrees of $\xi/f(\xi)$ where ξ runs through the points at which the local ring of X has dimension 0.

Proof. Immediate consequence of Lemma 35.1 and Properties of Spaces, Lemma 10.3. □

36. Syntomic morphisms

03Z6 The property "syntomic" of morphisms of schemes is étale local on the source-and-target, see Descent, Remark 32.7. It is also stable under base change and fpqc local on the target, see Morphisms, Lemma 30.4 and Descent, Lemma 23.26. Hence, by Lemma 22.1 above, we may define the notion of a syntomic morphism of algebraic spaces as follows and it agrees with the already existing notion defined in Section 3 when the morphism is representable.

03Z7 **Definition 36.1.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say f is *syntomic* if the equivalent conditions of Lemma 22.1 hold with $\mathcal{P}$ ="syntomic".
(2) Let $x \in |X|$. We say f is *syntomic at* x if there exists an open neighbourhood $X' \subset X$ of x such that $f|_{X'} : X' \to Y$ is syntomic.

03Z8 **Lemma 36.2.** *The composition of syntomic morphisms is syntomic.*

Proof. See Remark 22.3 and Morphisms, Lemma 30.3. □

03Z9 **Lemma 36.3.** *The base change of a syntomic morphism is syntomic.*

Proof. See Remark 22.4 and Morphisms, Lemma 30.4. □

03ZA **Lemma 36.4.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is syntomic,*
(2) *for every $x \in |X|$ the morphism f is syntomic at x,*
(3) *for every scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is syntomic,*
(4) *for every affine scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is syntomic,*
(5) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is a syntomic morphism,*
(6) *there exists a scheme U and a surjective étale morphism $\varphi : U \to X$ such that the composition $f \circ \varphi$ is syntomic,*
(7) *for every commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where U, V are schemes and the vertical arrows are étale the top horizontal arrow is syntomic,

(8) *there exists a commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where U, V are schemes, the vertical arrows are étale, and $U \to X$ is surjective such that the top horizontal arrow is syntomic, and

(9) *there exist Zariski coverings $Y = \bigcup_{i \in I} Y_i$, and $f^{-1}(Y_i) = \bigcup X_{ij}$ such that each morphism $X_{ij} \to Y_i$ is syntomic.*

Proof. Omitted. □

0DEY **Lemma 36.5.** *A syntomic morphism is locally of finite presentation.*

Proof. Follows immediately from the case of schemes (Morphisms, Lemma 30.6). □

0DEZ **Lemma 36.6.** *A syntomic morphism is flat.*

Proof. Follows immediately from the case of schemes (Morphisms, Lemma 30.7). □

0DF0 **Lemma 36.7.** *A syntomic morphism is universally open.*

Proof. Combine Lemmas 36.5, 36.6, and 30.6. □

37. Smooth morphisms

03ZB The property "smooth" of morphisms of schemes is étale local on the source-and-target, see Descent, Remark 32.7. It is also stable under base change and fpqc local on the target, see Morphisms, Lemma 34.5 and Descent, Lemma 23.27. Hence, by Lemma 22.1 above, we may define the notion of a smooth morphism of algebraic spaces as follows and it agrees with the already existing notion defined in Section 3 when the morphism is representable.

03ZC **Definition 37.1.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say f is *smooth* if the equivalent conditions of Lemma 22.1 hold with $\mathcal{P}$ ="smooth".
(2) Let $x \in |X|$. We say f is *smooth at x* if there exists an open neighbourhood $X' \subset X$ of x such that $f|_{X'} : X' \to Y$ is smooth.

03ZD **Lemma 37.2.** *The composition of smooth morphisms is smooth.*

Proof. See Remark 22.3 and Morphisms, Lemma 34.4. □

03ZE **Lemma 37.3.** *The base change of a smooth morphism is smooth.*

Proof. See Remark 22.4 and Morphisms, Lemma 34.5. □

03ZF **Lemma 37.4.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is smooth,*
(2) *for every $x \in |X|$ the morphism f is smooth at x,*

(3) *for every scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is smooth,*
(4) *for every affine scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is smooth,*
(5) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is a smooth morphism,*
(6) *there exists a scheme U and a surjective étale morphism $\varphi : U \to X$ such that the composition $f \circ \varphi$ is smooth,*
(7) *for every commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where U, V are schemes and the vertical arrows are étale the top horizontal arrow is smooth,
(8) *there exists a commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where U, V are schemes, the vertical arrows are étale, and $U \to X$ is surjective such that the top horizontal arrow is smooth, and
(9) *there exist Zariski coverings $Y = \bigcup_{i \in I} Y_i$, and $f^{-1}(Y_i) = \bigcup X_{ij}$ such that each morphism $X_{ij} \to Y_i$ is smooth.*

Proof. Omitted. □

04AJ **Lemma 37.5.** *A smooth morphism of algebraic spaces is locally of finite presentation.*

Proof. Let $X \to Y$ be a smooth morphism of algebraic spaces. By definition this means there exists a diagram as in Lemma 22.1 with h smooth and surjective vertical arrow a. By Morphisms, Lemma 34.8 h is locally of finite presentation. Hence $X \to Y$ is locally of finite presentation by definition. □

06MH **Lemma 37.6.** *A smooth morphism of algebraic spaces is locally of finite type.*

Proof. Combine Lemmas 37.5 and 28.5. □

04TA **Lemma 37.7.** *A smooth morphism of algebraic spaces is flat.*

Proof. Let $X \to Y$ be a smooth morphism of algebraic spaces. By definition this means there exists a diagram as in Lemma 22.1 with h smooth and surjective vertical arrow a. By Morphisms, Lemma 34.8 h is flat. Hence $X \to Y$ is flat by definition. □

06CP **Lemma 37.8.** *A smooth morphism of algebraic spaces is syntomic.*

Proof. Let $X \to Y$ be a smooth morphism of algebraic spaces. By definition this means there exists a diagram as in Lemma 22.1 with h smooth and surjective vertical arrow a. By Morphisms, Lemma 34.7 h is syntomic. Hence $X \to Y$ is syntomic by definition. □

0DZI **Lemma 37.9.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. There is a maximal open subspace $U \subset X$ such that $f|_U : U \to Y$ is smooth. Moreover, formation of this open commutes with base change by*

(1) *morphisms which are flat and locally of finite presentation,*
(2) *flat morphisms provided f is locally of finite presentation.*

Proof. The existence of U follows from the fact that the property of being smooth is Zariski (and even étale) local on the source, see Lemma 37.4. Moreover, this lemma allows us to translate properties (1) and (2) into the case of morphisms of schemes. The case of schemes is Morphisms, Lemma 34.15. Some details omitted. □

0AFI **Lemma 37.10.** *Let X and Y be locally Noetherian algebraic spaces over a scheme S, and let $f : X \to Y$ be a smooth morphism. For every point $x \in |X|$ with image $y \in |Y|$,*

$$\dim_x(X) = \dim_y(Y) + \dim_x(X_y)$$

where $\dim_x(X_y)$ is the relative dimension of f at x as in Definition 33.1.

Proof. By definition of the dimension of an algebraic space at a point (Properties of Spaces, Definition 9.1), this reduces to the corresponding statement for schemes (Morphisms, Lemma 34.21). □

38. Unramified morphisms

03ZG The property "unramified" (resp. "G-unramified") of morphisms of schemes is étale local on the source-and-target, see Descent, Remark 32.7. It is also stable under base change and fpqc local on the target, see Morphisms, Lemma 35.5 and Descent, Lemma 23.28. Hence, by Lemma 22.1 above, we may define the notion of an unramified morphism (resp. G-unramified morphism) of algebraic spaces as follows and it agrees with the already existing notion defined in Section 3 when the morphism is representable.

03ZH **Definition 38.1.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say f is *unramified* if the equivalent conditions of Lemma 22.1 hold with $\mathcal{P} =$ unramified.
(2) Let $x \in |X|$. We say f is *unramified at x* if there exists an open neighbourhood $X' \subset X$ of x such that $f|_{X'} : X' \to Y$ is unramified.
(3) We say f is *G-unramified* if the equivalent conditions of Lemma 22.1 hold with $\mathcal{P} =$ G-unramified.
(4) Let $x \in |X|$. We say f is *G-unramified at x* if there exists an open neighbourhood $X' \subset X$ of x such that $f|_{X'} : X' \to Y$ is G-unramified.

Because of the following lemma, from here on we will only develop theory for unramified morphisms, and whenever we want to use a G-unramified morphism we will simply say "an unramified morphism locally of finite presentation".

04G1 **Lemma 38.2.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Then f is G-unramified if and only if f is unramified and locally of finite presentation.*

Proof. Consider any diagram as in Lemma 22.1. Then all we are saying is that the morphism h is G-unramified if and only if it is unramified and locally of finite presentation. This is clear from Morphisms, Definition 35.1. □

03ZI **Lemma 38.3.** *The composition of unramified morphisms is unramified.*

Proof. See Remark 22.3 and Morphisms, Lemma 35.4. □

03ZJ **Lemma 38.4.** *The base change of an unramified morphism is unramified.*

Proof. See Remark 22.4 and Morphisms, Lemma 35.5. □

03ZK **Lemma 38.5.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is unramified,*
(2) *for every $x \in |X|$ the morphism f is unramified at x,*
(3) *for every scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is unramified,*
(4) *for every affine scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is unramified,*
(5) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is an unramified morphism,*
(6) *there exists a scheme U and a surjective étale morphism $\varphi : U \to X$ such that the composition $f \circ \varphi$ is unramified,*
(7) *for every commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where U, V are schemes and the vertical arrows are étale the top horizontal arrow is unramified,
(8) *there exists a commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where U, V are schemes, the vertical arrows are étale, and $U \to X$ is surjective such that the top horizontal arrow is unramified, and
(9) *there exist Zariski coverings $Y = \bigcup_{i \in I} Y_i$, and $f^{-1}(Y_i) = \bigcup X_{ij}$ such that each morphism $X_{ij} \to Y_i$ is unramified.*

Proof. Omitted. □

05VZ **Lemma 38.6.** *An unramified morphism of algebraic spaces is locally of finite type.*

Proof. Via a diagram as in Lemma 22.1 this translates into Morphisms, Lemma 35.9. □

05W0 **Lemma 38.7.** *If f is unramified at x then f is quasi-finite at x. In particular, an unramified morphism is locally quasi-finite.*

Proof. Via a diagram as in Lemma 22.1 this translates into Morphisms, Lemma 35.10. □

06CQ **Lemma 38.8.** *An immersion of algebraic spaces is unramified.*

Proof. Let $i : X \to Y$ be an immersion of algebraic spaces. Choose a scheme V and a surjective étale morphism $V \to Y$. Then $V \times_Y X \to V$ is an immersion of schemes, hence unramified (see Morphisms, Lemmas 35.7 and 35.8). Thus by definition i is unramified. □

05W1 **Lemma 38.9.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.*

(1) *If f is unramified, then the diagonal morphism $\Delta_{X/Y} : X \to X \times_Y X$ is an open immersion.*
(2) *If f is locally of finite type and $\Delta_{X/Y}$ is an open immersion, then f is unramified.*

Proof. We know in any case that $\Delta_{X/Y}$ is a representable monomorphism, see Lemma 4.1. Choose a scheme V and a surjective étale morphism $V \to Y$. Choose a scheme U and a surjective étale morphism $U \to X \times_Y V$. Consider the commutative diagram

$$\begin{array}{ccccc} U & \xrightarrow{\Delta_{U/V}} & U \times_V U & \longrightarrow & V \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \Delta_{V/Y}} \\ X & \xrightarrow{\Delta_{X/Y}} & X \times_Y X & \longrightarrow & V \times_Y V \end{array}$$

with cartesian right square. The left vertical arrow is surjective étale. The right vertical arrow is étale as a morphism between schemes étale over Y, see Properties of Spaces, Lemma 16.6. Hence the middle vertical arrow is étale too (but it need not be surjective).

Assume f is unramified. Then $U \to V$ is unramified, hence $\Delta_{U/V}$ is an open immersion by Morphisms, Lemma 35.13. Looking at the left square of the diagram above we conclude that $\Delta_{X/Y}$ is an étale morphism, see Properties of Spaces, Lemma 16.3. Hence $\Delta_{X/Y}$ is a representable étale monomorphism, which implies that it is an open immersion by Étale Morphisms, Theorem 14.1. (See also Spaces, Lemma 5.8 for the translation from schemes language into the language of functors.)

Assume that f is locally of finite type and that $\Delta_{X/Y}$ is an open immersion. This implies that $U \to V$ is locally of finite type too (by definition of a morphism of algebraic spaces which is locally of finite type). Looking at the displayed diagram above we conclude that $\Delta_{U/V}$ is étale as a morphism between schemes étale over $X \times_Y X$, see Properties of Spaces, Lemma 16.6. But since $\Delta_{U/V}$ is the diagonal of a morphism between schemes we see that it is in any case an immersion, see Schemes, Lemma 21.2. Hence it is an open immersion, and we conclude that $U \to V$ is unramified by Morphisms, Lemma 35.13. This in turn means that f is unramified by definition. □

05W2 **Lemma 38.10.** *Let S be a scheme. Consider a commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle p} \searrow \quad \swarrow {\scriptstyle q} & \\ & Z & \end{array}$$

of algebraic spaces over S. Assume that $X \to Z$ is locally of finite type. Then there exists an open subspace $U(f) \subset X$ such that $|U(f)| \subset |X|$ is the set of points where f is unramified. Moreover, for any morphism of algebraic spaces $Z' \to Z$, if

$f' : X' \to Y'$ is the base change of f by $Z' \to Z$, then $U(f')$ is the inverse image of $U(f)$ under the projection $X' \to X$.

Proof. This lemma is the analogue of Morphisms, Lemma 35.15 and in fact we will deduce the lemma from it. By Definition 38.1 the set $\{x \in |X| : f \text{ is unramified at } x\}$ is open in X. Hence we only need to prove the final statement. By Lemma 23.6 the morphism $X \to Y$ is locally of finite type. By Lemma 23.3 the morphism $X' \to Y'$ is locally of finite type.

Choose a scheme W and a surjective étale morphism $W \to Z$. Choose a scheme V and a surjective étale morphism $V \to W \times_Z Y$. Choose a scheme U and a surjective étale morphism $U \to V \times_Y X$. Finally, choose a scheme W' and a surjective étale morphism $W' \to W \times_Z Z'$. Set $V' = W' \times_W V$ and $U' = W' \times_W U$, so that we obtain surjective étale morphisms $V' \to Y'$ and $U' \to X'$. We will use without further mention an étale morphism of algebraic spaces induces an open map of associated topological spaces (see Properties of Spaces, Lemma 16.7). This combined with Lemma 38.5 implies that $U(f)$ is the image in $|X|$ of the set T of points in U where the morphism $U \to V$ is unramified. Similarly, $U(f')$ is the image in $|X'|$ of the set T' of points in U' where the morphism $U' \to V'$ is unramified. Now, by construction the diagram

$$\begin{array}{ccc} U' & \longrightarrow & U \\ \downarrow & & \downarrow \\ V' & \longrightarrow & V \end{array}$$

is cartesian (in the category of schemes). Hence the aforementioned Morphisms, Lemma 35.15 applies to show that T' is the inverse image of T. Since $|U'| \to |X'|$ is surjective this implies the lemma. □

06G6 **Lemma 38.11.** *Let S be a scheme. Let $X \to Y \to Z$ be morphisms of algebraic spaces over S. If $X \to Z$ is unramified, then $X \to Y$ is unramified.*

Proof. Choose a commutative diagram

$$\begin{array}{ccccc} U & \longrightarrow & V & \longrightarrow & W \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & Y & \longrightarrow & Z \end{array}$$

with vertical arrows étale and surjective. (See Spaces, Lemma 11.6.) Apply Morphisms, Lemma 35.16 to the top row. □

39. Étale morphisms

03XS The notion of an étale morphism of algebraic spaces was defined in Properties of Spaces, Definition 16.2. Here is what it means for a morphism to be étale at a point.

04RH **Definition 39.1.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $x \in |X|$. We say f is *étale at x* if there exists an open neighbourhood $X' \subset X$ of x such that $f|_{X'} : X' \to Y$ is étale.

03XT **Lemma 39.2.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is étale,*
(2) *for every $x \in |X|$ the morphism f is étale at x,*
(3) *for every scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is étale,*
(4) *for every affine scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is étale,*
(5) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is an étale morphism,*
(6) *there exists a scheme U and a surjective étale morphism $\varphi : U \to X$ such that the composition $f \circ \varphi$ is étale,*
(7) *for every commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where U, V are schemes and the vertical arrows are étale the top horizontal arrow is étale,
(8) *there exists a commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where U, V are schemes, the vertical arrows are étale, and $U \to X$ surjective such that the top horizontal arrow is étale, and
(9) *there exist Zariski coverings $Y = \bigcup Y_i$ and $f^{-1}(Y_i) = \bigcup X_{ij}$ such that each morphism $X_{ij} \to Y_i$ is étale.*

Proof. Combine Properties of Spaces, Lemmas 16.3, 16.5 and 16.4. Some details omitted. □

0465 **Lemma 39.3.** *The composition of two étale morphisms of algebraic spaces is étale.*

Proof. This is a copy of Properties of Spaces, Lemma 16.4. □

0466 **Lemma 39.4.** *The base change of an étale morphism of algebraic spaces by any morphism of algebraic spaces is étale.*

Proof. This is a copy of Properties of Spaces, Lemma 16.5. □

03XU **Lemma 39.5.** *An étale morphism of algebraic spaces is locally quasi-finite.*

Proof. Let $X \to Y$ be an étale morphism of algebraic spaces, see Properties of Spaces, Definition 16.2. By Properties of Spaces, Lemma 16.3 we see this means there exists a diagram as in Lemma 22.1 with h étale and surjective vertical arrow a. By Morphisms, Lemma 36.6 h is locally quasi-finite. Hence $X \to Y$ is locally quasi-finite by definition. □

04XX **Lemma 39.6.** *An étale morphism of algebraic spaces is smooth.*

Proof. The proof is identical to the proof of Lemma 39.5. It uses the fact that an étale morphism of schemes is smooth (by definition of an étale morphism of schemes). □

0467 **Lemma 39.7.** *An étale morphism of algebraic spaces is flat.*

Proof. The proof is identical to the proof of Lemma 39.5. It uses Morphisms, Lemma 36.12. □

0468 **Lemma 39.8.** *An étale morphism of algebraic spaces is locally of finite presentation.*

Proof. The proof is identical to the proof of Lemma 39.5. It uses Morphisms, Lemma 36.11. □

06LT **Lemma 39.9.** *An étale morphism of algebraic spaces is locally of finite type.*

Proof. An étale morphism is locally of finite presentation and a morphism locally of finite presentation is locally of finite type, see Lemmas 39.8 and 28.5. □

06CR **Lemma 39.10.** *An étale morphism of algebraic spaces is unramified.*

Proof. The proof is identical to the proof of Lemma 39.5. It uses Morphisms, Lemma 36.5. □

05W3 **Lemma 39.11.** *Let S be a scheme. Let X, Y be algebraic spaces étale over an algebraic space Z. Any morphism $X \to Y$ over Z is étale.*

Proof. This is a copy of Properties of Spaces, Lemma 16.6. □

06LU **Lemma 39.12.** *A locally finitely presented, flat, unramified morphism of algebraic spaces is étale.*

Proof. Let $X \to Y$ be a locally finitely presented, flat, unramified morphism of algebraic spaces. By Properties of Spaces, Lemma 16.3 we see this means there exists a diagram as in Lemma 22.1 with h locally finitely presented, flat, unramified and surjective vertical arrow a. By Morphisms, Lemma 36.16 h is étale. Hence $X \to Y$ is étale by definition. □

40. Proper morphisms

03ZL The notion of a proper morphism plays an important role in algebraic geometry. Here is the definition of a proper morphism of algebraic spaces.

03ZM **Definition 40.1.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. We say f is *proper* if f is separated, finite type, and universally closed.

083R **Lemma 40.2.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent*

(1) *f is proper,*
(2) *for every scheme Z and every morphism $Z \to Y$ the projection $Z \times_Y X \to Z$ is proper,*
(3) *for every affine scheme Z and every morphism $Z \to Y$ the projection $Z \times_Y X \to Z$ is proper,*
(4) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is proper, and*
(5) *there exists a Zariski covering $Y = \bigcup Y_i$ such that each of the morphisms $f^{-1}(Y_i) \to Y_i$ is proper.*

Proof. Combine Lemmas 4.12, 23.4, 8.8, and 9.5. □

04WP **Lemma 40.3.** *A base change of a proper morphism is proper.*

Proof. See Lemmas 4.4, 23.3, and 9.3. □

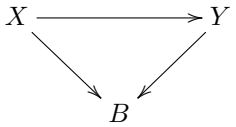
04XY **Lemma 40.4.** *A composition of proper morphisms is proper.*

Proof. See Lemmas 4.8, 23.2, and 9.4. □

04XZ **Lemma 40.5.** *A closed immersion of algebraic spaces is a proper morphism of algebraic spaces.*

Proof. As a closed immersion is by definition representable this follows from Spaces, Lemma 5.8 and the corresponding result for morphisms of schemes, see Morphisms, Lemma 41.6. □

04NX **Lemma 40.6.** *Let S be a scheme. Consider a commutative diagram of algebraic spaces*



over S.

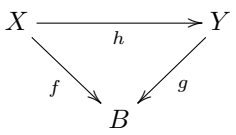
(1) *If $X \to B$ is universally closed and $Y \to B$ is separated, then the morphism $X \to Y$ is universally closed. In particular, the image of $|X|$ in $|Y|$ is closed.*
(2) *If $X \to B$ is proper and $Y \to B$ is separated, then the morphism $X \to Y$ is proper.*

Proof. Assume $X \to B$ is universally closed and $Y \to B$ is separated. We factor the morphism as $X \to X \times_B Y \to Y$. The first morphism is a closed immersion, see Lemma 4.6 hence universally closed. The projection $X \times_B Y \to Y$ is the base change of a universally closed morphism and hence universally closed, see Lemma 9.3. Thus $X \to Y$ is universally closed as the composition of universally closed morphisms, see Lemma 9.4. This proves (1). To deduce (2) combine (1) with Lemmas 4.10, 8.9, and 23.6. □

08AJ **Lemma 40.7.** *Let S be a scheme. Let B be an algebraic space over S. Let $f : X \to Y$ be a morphism of algebraic spaces over B. If X is universally closed over B and f is surjective then Y is universally closed over B. In particular, if also Y is separated and of finite type over B, then Y is proper over B.*

Proof. Assume X is universally closed and f surjective. Denote $p : X \to B$, $q : Y \to B$ the structure morphisms. Let $B' \to B$ be a morphism of algebraic spaces over S. The base change $f' : X_{B'} \to Y_{B'}$ is surjective (Lemma 5.5), and the base change $p' : X_{B'} \to B'$ is closed. If $T \subset Y_{B'}$ is closed, then $(f')^{-1}(T) \subset X_{B'}$ is closed, hence $p'((f')^{-1}(T)) = q'(T)$ is closed. So q' is closed. □

0AGD **Lemma 40.8.** *Let S be a scheme. Let*



be a commutative diagram of morphism of algebraic spaces over S. Assume

(1) *$X \to B$ is a proper morphism,*
(2) *$Y \to B$ is separated and locally of finite type,*

Then the scheme theoretic image $Z \subset Y$ of h is proper over B and $X \to Z$ is surjective.

Proof. The scheme theoretic image of h is constructed in Section 16. Observe that h is quasi-compact (Lemma 8.10) hence $|h|(|X|) \subset |Z|$ is dense (Lemma 16.3). On the other hand $|h|(|X|)$ is closed in $|Y|$ (Lemma 40.6) hence $X \to Z$ is surjective. Thus $Z \to B$ is a proper (Lemma 40.7). □

04Y0 **Lemma 40.9.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is separated,*
(2) *$\Delta_{X/Y} : X \to X \times_Y X$ is universally closed, and*
(3) *$\Delta_{X/Y} : X \to X \times_Y X$ is proper.*

Proof. The implication (1) ⇒ (3) follows from Lemma 40.5. We will use Spaces, Lemma 5.8 without further mention in the rest of the proof. Recall that $\Delta_{X/Y}$ is a representable monomorphism which is locally of finite type, see Lemma 4.1. Since proper ⇒ universally closed for morphisms of schemes we conclude that (3) implies (2). If $\Delta_{X/Y}$ is universally closed then Étale Morphisms, Lemma 7.2 implies that it is a closed immersion. Thus (2) ⇒ (1) and we win. □

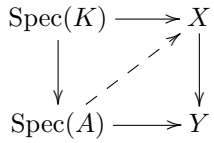
41. Valuative criteria

03IW The section introduces the basics on valuative criteria for morphisms of algebraic spaces. Here is a list of references to further results

(1) the valuative criterion for universal closedness can be found in Section 42,
(2) the valuative criterion of separatedness can be found in Section 43,
(3) the valuative criterion for properness can be found in Section 44,
(4) additional converse statements can be found in Decent Spaces, Section 16 and Decent Spaces, Lemma 17.11, and
(5) in the Noetherian case it is enough to check the criterion for discrete valuation rings as is shown in Cohomology of Spaces, Section 19 and Limits of Spaces, Section 21, and
(6) refined versions of the valuative criteria in the Noetherian case can be found in Limits of Spaces, Section 22.

We first formally state the definition and then we discuss how this differs from the case of morphisms of schemes.

03IX **Definition 41.1.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. We say *f satisfies the uniqueness part of the valuative criterion* if given any commutative solid diagram



where A is a valuation ring with field of fractions K, there exists at most one dotted arrow (without requiring existence). We say *f satisfies the existence part of the valuative criterion* if given any solid diagram as above there exists an extension K'/K of fields, a valuation ring $A' \subset K'$ dominating A and a morphism $\mathrm{Spec}(A') \to X$ such that the following diagram commutes

$$\begin{array}{ccccc} \mathrm{Spec}(K') & \longrightarrow & \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & & & & \downarrow \\ \mathrm{Spec}(A') & \longrightarrow & \mathrm{Spec}(A) & \longrightarrow & Y \end{array}$$

We say *f satisfies the valuative criterion* if f satisfies both the existence and uniqueness part.

The formulation of the existence part of the valuative criterion is slightly different for morphisms of algebraic spaces, since it may be necessary to extend the fraction field of the valuation ring. In practice this difference almost never plays a role.

(1) Checking the uniqueness part of the valuative criterion never involves any fraction field extensions, hence this is exactly the same as in the case of schemes.
(2) It is necessary to allow for field extensions in general, see Example 41.6.
(3) For morphisms of algebraic spaces it always suffices to take a finite separable extensions K'/K in the existence part of the valuative criterion, see Lemma 41.3.
(4) If $f : X \to Y$ is a separated morphism of algebraic spaces, then we can always take $K = K'$ when we check the existence part of the valuative criterion, see Lemma 41.5.
(5) For a quasi-compact and quasi-separated morphism $f : X \to Y$, we get an equivalence between "f is separated and universally closed" and "f satisfies the usual valuative criterion", see Lemma 43.3. The valuative criterion for properness is the usual one, see Lemma 44.1.

As a first step in the theory, we show that the criterion is identical to the criterion as formulated for morphisms of schemes in case the morphism of algebraic spaces is representable.

03K8 **Lemma 41.2.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is representable. The following are equivalent*

(1) *f satisfies the existence part of the valuative criterion as in Definition 41.1,*
(2) *given any commutative solid diagram*

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & Y \end{array}$$

where A is a valuation ring with field of fractions K, there exists a dotted arrow, i.e., f satisfies the existence part of the valuative criterion as in Schemes, Definition 20.3.

Proof. It suffices to show that given a commutative diagram of the form

$$\begin{array}{ccccc} \mathrm{Spec}(K') & \longrightarrow & \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & & & \overset{\varphi}{\nearrow} & \downarrow \\ \mathrm{Spec}(A') & \longrightarrow & \mathrm{Spec}(A) & \longrightarrow & Y \end{array}$$

as in Definition 41.1, then we can find a morphism $\mathrm{Spec}(A) \to X$ fitting into the diagram too. Set $X_A = \mathrm{Spec}(A) \times_Y X$. As f is representable we see that X_A is a scheme. The morphism φ gives a morphism $\varphi' : \mathrm{Spec}(A') \to X_A$. Let $x \in X_A$ be the image of the closed point of $\varphi' : \mathrm{Spec}(A') \to X_A$. Then we have the following commutative diagram of rings

$$\begin{array}{ccccc} K' & \longleftarrow & K & \longleftarrow & \mathcal{O}_{X_A,x} \\ \uparrow & & & \swarrow & \uparrow \\ A' & \longleftarrow & A & \longleftarrow & A \end{array}$$

Since A is a valuation ring, and since A' dominates A, we see that $K \cap A' = A$. Hence the ring map $\mathcal{O}_{X_A,x} \to K$ has image contained in A. Whence a morphism $\mathrm{Spec}(A) \to X_A$ (see Schemes, Section 13) as desired. □

03KH **Lemma 41.3.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent*

(1) *f satisfies the existence part of the valuative criterion as in Definition 41.1,*
(2) *f satisfies the existence part of the valuative criterion as in Definition 41.1 modified by requiring the extension K'/K to be finite separable.*

Proof. We have to show that (1) implies (2). Suppose given a diagram

$$\begin{array}{ccccc} \mathrm{Spec}(K') & \longrightarrow & \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & & & \nearrow & \downarrow \\ \mathrm{Spec}(A') & \longrightarrow & \mathrm{Spec}(A) & \longrightarrow & Y \end{array}$$

as in Definition 41.1 with $K \subset K'$ arbitrary. Choose a scheme U and a surjective étale morphism $U \to X$. Then

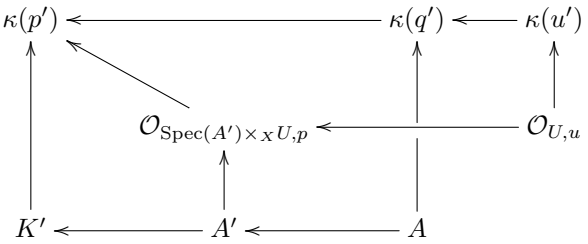
$$\mathrm{Spec}(A') \times_X U \longrightarrow \mathrm{Spec}(A')$$

is surjective étale. Let p be a point of $\mathrm{Spec}(A') \times_X U$ mapping to the closed point of $\mathrm{Spec}(A')$. Let $p' \rightsquigarrow p$ be a generalization of p mapping to the generic point of $\mathrm{Spec}(A')$. Such a generalization exists because generalizations lift along flat morphisms of schemes, see Morphisms, Lemma 25.9. Then p' corresponds to a point of the scheme $\mathrm{Spec}(K') \times_X U$. Note that

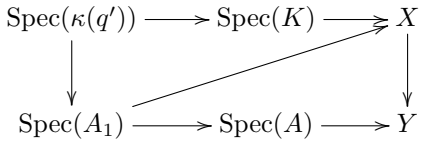
$$\mathrm{Spec}(K') \times_X U = \mathrm{Spec}(K') \times_{\mathrm{Spec}(K)} (\mathrm{Spec}(K) \times_X U)$$

Hence p' maps to a point $q' \in \mathrm{Spec}(K) \times_X U$ whose residue field is a finite separable extension of K. Finally, $p' \rightsquigarrow p$ maps to a specialization $u' \rightsquigarrow u$ on the scheme U.

With all this notation we get the following diagram of rings

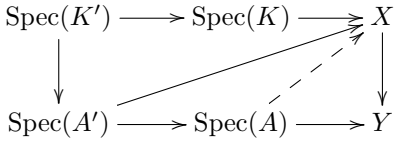


This means that the ring $B \subset \kappa(q')$ generated by the images of A and $\mathcal{O}_{U,u}$ maps to a subring of $\kappa(p')$ contained in the image B' of $\mathcal{O}_{\mathrm{Spec}(A')\times_X U,p} \to \kappa(p')$. Note that B' is a local ring. Let $\mathfrak{m} \subset B$ be the maximal ideal. By construction $A \cap \mathfrak{m}$, (resp. $\mathcal{O}_{U,u} \cap \mathfrak{m}$, resp. $A' \cap \mathfrak{m}$) is the maximal ideal of A (resp. $\mathcal{O}_{U,u}$, resp. A'). Set $\mathfrak{q} = B \cap \mathfrak{m}$. This is a prime ideal such that $A \cap \mathfrak{q}$ is the maximal ideal of A. Hence $B_\mathfrak{q} \subset \kappa(q')$ is a local ring dominating A. By Algebra, Lemma 50.2 we can find a valuation ring $A_1 \subset \kappa(q')$ with field of fractions $\kappa(q')$ dominating $B_\mathfrak{q}$. The (local) ring map $\mathcal{O}_{U,u} \to A_1$ gives a morphism $\mathrm{Spec}(A_1) \to U \to X$ such that the diagram



is commutative. Since the fraction field of A_1 is $\kappa(q')$ and since $\kappa(q')/K$ is finite separable by construction the lemma is proved. □

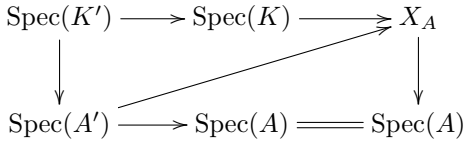
0ARH **Lemma 41.4.** *Let S be a scheme. Let $f : X \to Y$ be a separated morphism of algebraic spaces over S. Suppose given a diagram*



as in Definition 41.1 with $K \subset K'$ arbitrary. Then the dotted arrow exists making the diagram commute.

Proof. We have to show that we can find a morphism $\mathrm{Spec}(A) \to X$ fitting into the diagram.

Consider the base change $X_A = \mathrm{Spec}(A) \times_Y X$ of X. Then $X_A \to \mathrm{Spec}(A)$ is a separated morphism of algebraic spaces (Lemma 4.4). Base changing all the morphisms of the diagram above we obtain



Thus we may replace X by X_A, assume that $Y = \mathrm{Spec}(A)$ and that we have a diagram as above. We may and do replace X by a quasi-compact open subspace containing the image of $|\mathrm{Spec}(A')| \to |X|$.

The morphism $\mathrm{Spec}(A') \to X$ is quasi-compact by Lemma 8.9. Let $Z \subset X$ be the scheme theoretic image of $\mathrm{Spec}(A') \to X$. Then Z is a reduced (Lemma 16.4), quasi-compact (as a closed subspace of X), separated (as a closed subspace of X) algebraic space over A. Consider the base change

$$\mathrm{Spec}(K') = \mathrm{Spec}(A') \times_{\mathrm{Spec}(A)} \mathrm{Spec}(K) \to X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(K) = X_K$$

of the morphism $\mathrm{Spec}(A') \to X$ by the flat morphism of schemes $\mathrm{Spec}(K) \to \mathrm{Spec}(A)$. By Lemma 30.12 we see that the scheme theoretic image of this morphism is the base change Z_K of Z. On the other hand, by assumption (i.e., the commutative diagram above) this morphism factors through a morphism $\mathrm{Spec}(K) \to Z_K$ which is a section to the structure morphism $Z_K \to \mathrm{Spec}(K)$. As Z_K is separated, this section is a closed immersion (Lemma 4.7). We conclude that $Z_K = \mathrm{Spec}(K)$.

Let $V \to Z$ be a surjective étale morphism with V an affine scheme (Properties of Spaces, Lemma 6.3). Say $V = \mathrm{Spec}(B)$. Then $V \times_Z \mathrm{Spec}(A') = \mathrm{Spec}(C)$ is affine as Z is separated. Note that $B \to C$ is injective as V is the scheme theoretic image of $V \times_Z \mathrm{Spec}(A') \to V$ by Lemma 16.3. On the other hand, $A' \to C$ is étale as corresponds to the base change of $V \to Z$. Since A' is a torsion free A-module, the flatness of $A' \to C$ implies C is a torsion free A-module, hence B is a torsion free A-module. Note that being torsion free as an A-module is equivalent to being flat (More on Algebra, Lemma 22.10). Next, we write

$$V \times_Z V = \mathrm{Spec}(B')$$

Note that the two ring maps $B \to B'$ are étale as $V \to Z$ is étale. The canonical surjective map $B \otimes_A B \to B'$ becomes an isomorphism after tensoring with K over A because $Z_K = \mathrm{Spec}(K)$. However, $B \otimes_A B$ is torsion free as an A-module by our remarks above. Thus $B' = B \otimes_A B$. It follows that the base change of the ring map $A \to B$ by the faithfully flat ring map $A \to B$ is étale (note that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective as $X \to \mathrm{Spec}(A)$ is surjective). Hence $A \to B$ is étale (Descent, Lemma 23.29), in other words, $V \to X$ is étale. Since we have $V \times_Z V = V \times_{\mathrm{Spec}(A)} V$ we conclude that $Z = \mathrm{Spec}(A)$ as algebraic spaces (for example by Spaces, Lemma 9.1) and the proof is complete. □

0A3W **Lemma 41.5.** *Let S be a scheme. Let $f : X \to Y$ be a separated morphism of algebraic spaces over S. The following are equivalent*

(1) *f satisfies the existence part of the valuative criterion as in Definition 41.1,*
(2) *given any commutative solid diagram*

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & Y \end{array}$$

where A is a valuation ring with field of fractions K, there exists a dotted arrow, i.e., f satisfies the existence part of the valuative criterion as in Schemes, Definition 20.3.

Proof. We have to show that (1) implies (2). Suppose given a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & Y \end{array}$$

as in part (2). By (1) there exists a commutative diagram

$$\begin{array}{ccccc} \mathrm{Spec}(K') & \longrightarrow & \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & & & \nearrow & \downarrow \\ \mathrm{Spec}(A') & \longrightarrow & \mathrm{Spec}(A) & \longrightarrow & Y \end{array}$$

as in Definition 41.1 with $K \subset K'$ arbitrary. By Lemma 41.4 we can find a morphism $\mathrm{Spec}(A) \to X$ fitting into the diagram, i.e., (2) holds. □

03KI **Example 41.6.** Consider the algebraic space X constructed in Spaces, Example 14.2. Recall that it is Galois twist of the affine line with zero doubled. The Galois twist is with respect to a degree two Galois extension k'/k of fields. As such it comes with a morphism

$$\pi : X \longrightarrow S = \mathbf{A}^1_k$$

which is quasi-compact. We claim that π is universally closed. Namely, after base change by $\mathrm{Spec}(k') \to \mathrm{Spec}(k)$ the morphism π is identified with the morphism

$$\text{affine line with zero doubled} \longrightarrow \text{affine line}$$

which is universally closed (some details omitted). Since the morphism $\mathrm{Spec}(k') \to \mathrm{Spec}(k)$ is universally closed and surjective, a diagram chase shows that π is universally closed. On the other hand, consider the diagram

$$\begin{array}{ccc} \mathrm{Spec}(k((x))) & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \pi \\ \mathrm{Spec}(k[[x]]) & \longrightarrow & \mathbf{A}^1_k \end{array}$$

Since the unique point of X above $0 \in \mathbf{A}^1_k$ corresponds to a monomorphism $\mathrm{Spec}(k') \to X$ it is clear there cannot exist a dotted arrow! This shows that a finite separable field extension is needed in general.

03IY **Lemma 41.7.** *The base change of a morphism of algebraic spaces which satisfies the existence part of (resp. uniqueness part of) the valuative criterion by any morphism of algebraic spaces satisfies the existence part of (resp. uniqueness part of) the valuative criterion.*

Proof. Let $f : X \to Y$ be a morphism of algebraic spaces over the scheme S. Let $Z \to Y$ be any morphism of algebraic spaces over S. Consider a solid commutative diagram of the following shape

$$\begin{array}{ccccc} \mathrm{Spec}(K) & \longrightarrow & Z \times_Y X & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow & \nearrow & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & Z & \longrightarrow & Y \end{array}$$

Then the set of north-west dotted arrows making the diagram commute is in 1-1 correspondence with the set of west-north-west dotted arrows making the diagram commute. This proves the lemma in the case of "uniqueness". For the existence part, assume f satisfies the existence part of the valuative criterion. If we are given a solid commutative diagram as above, then by assumption there exists an extension K'/K of fields and a valuation ring $A' \subset K'$ dominating A and a morphism $\mathrm{Spec}(A') \to X$ fitting into the following commutative diagram

$$\begin{array}{ccccccc} \mathrm{Spec}(K') & \longrightarrow & \mathrm{Spec}(K) & \longrightarrow & Z \times_Y X & \longrightarrow & X \\ \downarrow & & & & & & \downarrow \\ \mathrm{Spec}(A') & \longrightarrow & \mathrm{Spec}(A) & \longrightarrow & Z & \longrightarrow & Y \end{array}$$

And by the remarks above the skew arrow corresponds to an arrow $\mathrm{Spec}(A') \to Z \times_Y X$ as desired. □

03IZ **Lemma 41.8.** *The composition of two morphisms of algebraic spaces which satisfy the (existence part of, resp. uniqueness part of) the valuative criterion satisfies the (existence part of, resp. uniqueness part of) the valuative criterion.*

Proof. Let $f : X \to Y$, $g : Y \to Z$ be morphisms of algebraic spaces over the scheme S. Consider a solid commutative diagram of the following shape

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow f \\ & & Y \\ & \nearrow & \downarrow g \\ \mathrm{Spec}(A) & \longrightarrow & Z \end{array}$$

If we have the uniqueness part for g, then there exists at most one north-west dotted arrow making the diagram commute. If we also have the uniqueness part for f, then we have at most one north-north-west dotted arrow making the diagram commute. The proof in the existence case comes from contemplating the following diagram

$$\begin{array}{ccccccc} \mathrm{Spec}(K'') & \longrightarrow & \mathrm{Spec}(K') & \longrightarrow & \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & \nearrow & & \nearrow & & & \downarrow f \\ & & & & & & Y \\ & & & \nearrow & & & \downarrow g \\ \mathrm{Spec}(A'') & \longrightarrow & \mathrm{Spec}(A') & \longrightarrow & \mathrm{Spec}(A) & \longrightarrow & Z \end{array}$$

Namely, the existence part for g gives us the extension K', the valuation ring A' and the arrow $\mathrm{Spec}(A') \to Y$, whereupon the existence part for f gives us the extension K'', the valuation ring A'' and the arrow $\mathrm{Spec}(A'') \to X$. □

42. Valuative criterion for universal closedness

03K9 The existence part of the valuative criterion implies universal closedness for quasi-compact morphisms, see Lemma 42.1. In the case of schemes, this is an "if and

only if" statement, but for morphisms of algebraic spaces this is wrong. Example 9.6 shows that $\mathbf{A}^1_k/\mathbf{Z} \to \mathrm{Spec}(k)$ is universally closed, but it is easy to see that the existence part of the valuative criterion fails. We revisit this topic in Decent Spaces, Section 16 and show the converse holds if the source of the morphism is a decent space (see also Decent Spaces, Lemma 17.11 for a relative version).

03KA **Lemma 42.1.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume*

(1) *f is quasi-compact, and*
(2) *f satisfies the existence part of the valuative criterion.*

Then f is universally closed.

Proof. By Lemmas 8.4 and 41.7 properties (1) and (2) are preserved under any base change. By Lemma 9.5 we only have to show that $|T \times_Y X| \to |T|$ is closed, whenever T is an affine scheme over S mapping into Y. Hence it suffices to prove: If Y is an affine scheme, $f : X \to Y$ is quasi-compact and satisfies the existence part of the valuative criterion, then $f : |X| \to |Y|$ is closed. In this situation X is a quasi-compact algebraic space. By Properties of Spaces, Lemma 6.3 there exists an affine scheme U and a surjective étale morphism $\varphi : U \to X$. Let $T \subset |X|$ closed. The inverse image $\varphi^{-1}(T) \subset U$ is closed, and hence is the set of points of an affine closed subscheme $Z \subset U$. Thus, by Algebra, Lemma 41.5 we see that $f(T) = f(\varphi(|Z|)) \subset |Y|$ is closed if it is closed under specialization.

Let $y' \rightsquigarrow y$ be a specialization in Y with $y' \in f(T)$. Choose a point $x' \in T \subset |X|$ mapping to y' under f. We may represent x' by a morphism $\mathrm{Spec}(K) \to X$ for some field K. Thus we have the following diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \xrightarrow{x'} & X \\ \downarrow & & \downarrow f \\ \mathrm{Spec}(\mathcal{O}_{Y,y}) & \longrightarrow & Y, \end{array}$$

see Schemes, Section 13 for the existence of the left vertical map. Choose a valuation ring $A \subset K$ dominating the image of the ring map $\mathcal{O}_{Y,y} \to K$ (this is possible since the image is a local ring and not a field as $y' \neq y$, see Algebra, Lemma 50.2). By assumption there exists a field extension K'/K and a valuation ring $A' \subset K'$ dominating A, and a morphism $\mathrm{Spec}(A') \to X$ fitting into the commutative diagram. Since A' dominates A, and A dominates $\mathcal{O}_{Y,y}$ we see that the closed point of $\mathrm{Spec}(A')$ maps to a point $x \in X$ with $f(x) = y$ which is a specialization of x'. Hence $x \in T$ as T is closed, and hence $y \in f(T)$ as desired. □

The following lemma will be generalized in Decent Spaces, Lemma 17.11.

0A3X **Lemma 42.2.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.*

(1) *If f is quasi-separated and universally closed, then f satisfies the existence part of the valuative criterion.*
(2) *If f is quasi-compact and quasi-separated, then f is universally closed if and only if the existence part of the valuative criterion holds.*

Proof. If (1) is true then combined with Lemma 42.1 we obtain (2). Assume f is quasi-separated and universally closed. Assume given a diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & Y \end{array}$$

as in Definition 41.1. A formal argument shows that the existence of the desired diagram

$$\begin{array}{ccccc} \mathrm{Spec}(K') & \longrightarrow & \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & & & \nearrow & \downarrow \\ \mathrm{Spec}(A') & \longrightarrow & \mathrm{Spec}(A) & \longrightarrow & Y \end{array}$$

follows from existence in the case of the morphism $X_A \to \mathrm{Spec}(A)$. Since being quasi-separated and universally closed are preserved by base change, the lemma follows from the result in the next paragraph.

Consider a solid diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \xrightarrow{x} & X \\ \downarrow & \nearrow & \downarrow f \\ \mathrm{Spec}(A) & = & \mathrm{Spec}(A) \end{array}$$

where A is a valuation ring with field of fractions K. By Lemma 8.9 and the fact that f is quasi-separated we have that the morphism x is quasi-compact. Since f is universally closed, we have in particular that $|f|(\overline{\{x\}})$ is closed in $\mathrm{Spec}(A)$. Since this image contains the generic point of $\mathrm{Spec}(A)$ there exists a point $x' \in |X|$ in the closure of x mapping to the closed point of $\mathrm{Spec}(A)$. By Lemma 16.5 we can find a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(K') & \longrightarrow & \mathrm{Spec}(K) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A') & \longrightarrow & X \end{array}$$

such that the closed point of $\mathrm{Spec}(A')$ maps to $x' \in |X|$. It follows that $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$ maps the closed point to the closed point, i.e., A' dominates A and this finishes the proof. □

0A3Y **Lemma 42.3.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is quasi-compact and separated. Then the following are equivalent*

(1) *f is universally closed,*
(2) *the existence part of the valuative criterion holds as in Definition 41.1, and*

(3) *given any commutative solid diagram*

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & Y \end{array}$$

where A is a valuation ring with field of fractions K, there exists a dotted arrow, i.e., f satisfies the existence part of the valuative criterion as in Schemes, Definition 20.3.

Proof. Since f is separated parts (2) and (3) are equivalent by Lemma 41.5. The equivalence of (3) and (1) follows from Lemma 42.2. □

089F **Lemma 42.4.** *Let S be a scheme. Let $f : X \to Y$ be a flat morphism of algebraic spaces over S. Let $\mathrm{Spec}(A) \to Y$ be a morphism where A is a valuation ring. If the closed point of $\mathrm{Spec}(A)$ maps to a point of $|Y|$ in the image of $|X| \to |Y|$, then there exists a commutative diagram*

$$\begin{array}{ccc} \mathrm{Spec}(A') & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & Y \end{array}$$

where $A \to A'$ is an extension of valuation rings (More on Algebra, Definition 123.1).

Proof. The base change $X_A \to \mathrm{Spec}(A)$ is flat (Lemma 30.4) and the closed point of $\mathrm{Spec}(A)$ is in the image of $|X_A| \to |\mathrm{Spec}(A)|$ (Properties of Spaces, Lemma 4.3). Thus we may assume $Y = \mathrm{Spec}(A)$. Let $U \to X$ be a surjective étale morphism where U is a scheme. Let $u \in U$ map to the closed point of $\mathrm{Spec}(A)$. Consider the flat local ring map $A \to B = \mathcal{O}_{U,u}$. By Algebra, Lemma 39.16 there exists a prime ideal $\mathfrak{q} \subset B$ such that $\mathfrak{q}$ lies over $(0) \subset A$. By Algebra, Lemma 50.2 we can find a valuation ring $A' \subset \kappa(\mathfrak{q})$ dominating $B/\mathfrak{q}$. The induced morphism $\mathrm{Spec}(A') \to U \to X$ is a solution to the problem posed by the lemma. □

089G **Lemma 42.5.** *Let S be a scheme. Let $f : X \to Y$ and $h : U \to X$ be morphisms of algebraic spaces over S. If*

(1) *f and h are quasi-compact,*
(2) *$|h|(|U|)$ is dense in $|X|$, and*

given any commutative solid diagram

$$\begin{array}{ccccc} \mathrm{Spec}(K) & \longrightarrow & U & \longrightarrow & X \\ \downarrow & & & \nearrow & \downarrow \\ \mathrm{Spec}(A) & & \longrightarrow & & Y \end{array}$$

where A is a valuation ring with field of fractions K

(3) *there exists at most one dotted arrow making the diagram commute, and*

(4) *there exists an extension K'/K of fields, a valuation ring $A' \subset K'$ dominating A and a morphism $\mathrm{Spec}(A') \to X$ such that the following diagram commutes*

$$\begin{array}{ccccccc} \mathrm{Spec}(K') & \longrightarrow & \mathrm{Spec}(K) & \longrightarrow & U & \longrightarrow & X \\ \downarrow & & & & & & \downarrow \\ \mathrm{Spec}(A') & \longrightarrow & \mathrm{Spec}(A) & & \longrightarrow & & Y \end{array}$$

then f is universally closed. If moreover

(5) *f is quasi-separated*

then f is separated and universally closed.

Proof. Assume (1), (2), (3), and (4). We will verify the existence part of the valuative criterion for f which will imply f is universally closed by Lemma 42.1. To do this, consider a commutative diagram

089H (42.5.1)
$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & Y \end{array}$$

where A is a valuation ring and K is the fraction field of A. Note that since valuation rings and fields are reduced, we may replace U, X, and S by their respective reductions by Properties of Spaces, Lemma 12.4. In this case the assumption that $h(U)$ is dense means that the scheme theoretic image of $h : U \to X$ is X, see Lemma 16.4.

Reduction to the case Y affine. Choose an étale morphism $\mathrm{Spec}(R) \to Y$ such that the closed point of $\mathrm{Spec}(A)$ maps to an element of $\mathrm{Im}(|\mathrm{Spec}(R)| \to |Y|)$. By Lemma 42.4 we can find a local ring map $A \to A'$ of valuation rings and a morphism $\mathrm{Spec}(A') \to \mathrm{Spec}(R)$ fitting into a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(A') & \longrightarrow & \mathrm{Spec}(R) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & Y \end{array}$$

Since in Definition 41.1 we allow for extensions of valuation rings it is clear that we may replace A by A', Y by $\mathrm{Spec}(R)$, X by $X \times_Y \mathrm{Spec}(R)$ and U by $U \times_Y \mathrm{Spec}(R)$.

From now on we assume that $Y = \mathrm{Spec}(R)$ is an affine scheme. Let $\mathrm{Spec}(B) \to X$ be an étale morphism from an affine scheme such that the morphism $\mathrm{Spec}(K) \to X$ is in the image of $|\mathrm{Spec}(B)| \to |X|$. Since we may replace K by an extension $K' \supset K$ and A by a valuation ring $A' \subset K'$ dominating A (which exists by Algebra, Lemma 50.2), we may assume the morphism $\mathrm{Spec}(K) \to X$ factors through $\mathrm{Spec}(B)$ (by definition of $|X|$). In other words, we may think of K as a B-algebra. Choose a polynomial algebra P over B and a B-algebra surjection $P \to K$. Then $\mathrm{Spec}(P) \to X$ is flat as a composition $\mathrm{Spec}(P) \to \mathrm{Spec}(B) \to X$. Hence the scheme theoretic image of the morphism $U \times_X \mathrm{Spec}(P) \to \mathrm{Spec}(P)$ is $\mathrm{Spec}(P)$ by Lemma 30.12. By

Lemma 16.5 we can find a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(K') & \longrightarrow & U \times_X \mathrm{Spec}(P) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A') & \longrightarrow & \mathrm{Spec}(P) \end{array}$$

where A' is a valuation ring and K' is the fraction field of A' such that the closed point of $\mathrm{Spec}(A')$ maps to $\mathrm{Spec}(K) \subset \mathrm{Spec}(P)$. In other words, there is a B-algebra map $\varphi : K \to A'/\mathfrak{m}_{A'}$. Choose a valuation ring $A'' \subset A'/\mathfrak{m}_{A'}$ dominating $\varphi(A)$ with field of fractions $K'' = A'/\mathfrak{m}_{A'}$ (Algebra, Lemma 50.2). We set

$$C = \{\lambda \in A' \mid \lambda \bmod \mathfrak{m}_{A'} \in A''\}.$$

which is a valuation ring by Algebra, Lemma 50.10. As C is an R-algebra with fraction field K', we obtain a solid commutative diagram

$$\begin{array}{ccccc} \mathrm{Spec}(K_1') & \dashrightarrow & \mathrm{Spec}(K') & \longrightarrow & U \longrightarrow X \\ \vdots & & \downarrow & \nearrow & \downarrow \\ \mathrm{Spec}(C_1) & \dashrightarrow & \mathrm{Spec}(C) & \longrightarrow & Y \end{array}$$

as in the statement of the lemma. Thus assumption (4) produces $C \to C_1$ and the dotted arrows making the diagram commute. Let $A_1' = (C_1)_{\mathfrak{p}}$ be the localization of C_1 at a prime $\mathfrak{p} \subset C_1$ lying over $\mathfrak{m}_{A'} \subset C$. Since $C \to C_1$ is flat by More on Algebra, Lemma 22.10 such a prime $\mathfrak{p}$ exists by Algebra, Lemmas 39.17 and 39.16. Note that A' is the localization of C at $\mathfrak{m}_{A'}$ and that A_1' is a valuation ring (Algebra, Lemma 50.9). In other words, $A' \to A_1'$ is a local ring map of valuation rings. Assumption (3) implies

$$\begin{array}{ccccc} \mathrm{Spec}(A_1') & \longrightarrow & \mathrm{Spec}(C_1) & \longrightarrow & X \\ \downarrow & & & & \uparrow \\ \mathrm{Spec}(A') & \longrightarrow & \mathrm{Spec}(P) & \longrightarrow & \mathrm{Spec}(B) \end{array}$$

commutes. Hence the restriction of the morphism $\mathrm{Spec}(C_1) \to X$ to $\mathrm{Spec}(C_1/\mathfrak{p})$ restricts to the composition

$$\mathrm{Spec}(\kappa(\mathfrak{p})) \to \mathrm{Spec}(A'/\mathfrak{m}_{A'}) = \mathrm{Spec}(K'') \to \mathrm{Spec}(K) \to X$$

on the generic point of $\mathrm{Spec}(C_1/\mathfrak{p})$. Moreover, $C_1/\mathfrak{p}$ is a valuation ring (Algebra, Lemma 50.9) dominating A'' which dominates A. Thus the morphism $\mathrm{Spec}(C_1/\mathfrak{p}) \to X$ witnesses the existence part of the valuative criterion for the diagram (42.5.1) as desired.

Next, suppose that (5) is satisfied as well, i.e., the morphism $\Delta : X \to X \times_S X$ is quasi-compact. In this case assumptions (1) – (4) hold for h and Δ. Hence the first part of the proof shows that Δ is universally closed. By Lemma 40.9 we conclude that f is separated. □

43. Valuative criterion of separatedness

03KT First we prove a converse and then we state the criterion.

03KU **Lemma 43.1.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is separated, then f satisfies the uniqueness part of the valuative criterion.*

Proof. Let a diagram as in Definition 41.1 be given. Suppose there are two distinct morphisms $a, b : \mathrm{Spec}(A) \to X$ fitting into the diagram. Let $Z \subset \mathrm{Spec}(A)$ be the equalizer of a and b. Then $Z = \mathrm{Spec}(A) \times_{(a,b),X\times_Y X,\Delta} X$. If f is separated, then Δ is a closed immersion, and this is a closed subscheme of $\mathrm{Spec}(A)$. By assumption it contains the generic point of $\mathrm{Spec}(A)$. Since A is a domain this implies $Z = \mathrm{Spec}(A)$. Hence $a = b$ as desired. □

03KV **Lemma 43.2** (Valuative criterion separatedness)**.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume*

(1) *the morphism f is quasi-separated, and*
(2) *the morphism f satisfies the uniqueness part of the valuative criterion.*

Then f is separated.

Proof. Assumption (1) means $\Delta_{X/Y}$ is quasi-compact. We claim the morphism $\Delta_{X/Y} : X \to X \times_Y X$ satisfies the existence part of the valuative criterion. Let a solid commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & X \times_Y X \end{array}$$

be given. The lower right arrow corresponds to a pair of morphisms $a, b : \mathrm{Spec}(A) \to X$ over Y. By assumption (2) we see that $a = b$. Hence using a as the dotted arrow works. Hence Lemma 42.1 applies, and we see that $\Delta_{X/Y}$ is universally closed. Since always $\Delta_{X/Y}$ is locally of finite type and separated, we conclude from More on Morphisms, Lemma 44.1 that $\Delta_{X/Y}$ is a finite morphism (also, use the general principle of Spaces, Lemma 5.8). At this point $\Delta_{X/Y}$ is a representable, finite monomorphism, hence a closed immersion by Morphisms, Lemma 44.15. □

0A3Z **Lemma 43.3.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is quasi-compact and quasi-separated. Then the following are equivalent*

(1) *f is separated and universally closed,*
(2) *the valuative criterion holds as in Definition 41.1,*
(3) *given any commutative solid diagram*

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & Y \end{array}$$

where A is a valuation ring with field of fractions K, there exists a unique dotted arrow, i.e., f satisfies the valuative criterion as in Schemes, Definition 20.3.

Proof. Since f is quasi-separated, the uniqueness part of the valutative criterion implies f is separated (Lemma 43.2). Conversely, if f is separated, then it satisfies the uniqueness part of the valuative criterion (Lemma 43.1). Having said this, we see that in each of the three cases the morphism f is separated and satisfies the uniqueness part of the valuative criterion. In this case the lemma is a formal consequence of Lemma 42.3. □

44. Valuative criterion of properness

0CKZ Here is a statement.

0A40 **Lemma 44.1** (Valuative criterion for properness)**.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is of finite type and quasi-separated. Then the following are equivalent*

(1) *f is proper,*
(2) *the valuative criterion holds as in Definition 41.1,*
(3) *given any commutative solid diagram*

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & Y \end{array}$$

where A is a valuation ring with field of fractions K, there exists a unique dotted arrow, i.e., f satisfies the valuative criterion as in Schemes, Definition 20.3.

Proof. Formal consequence of Lemma 43.3 and the definitions. □

45. Integral and finite morphisms

03ZN We have already defined in Section 3 what it means for a representable morphism of algebraic spaces to be integral (resp. finite).

03ZO **Lemma 45.1.** *Let S be a scheme. Let $f : X \to Y$ be a representable morphism of algebraic spaces over S. Then f is integral, resp. finite (in the sense of Section 3), if and only if for all affine schemes Z and morphisms $Z \to Y$ the scheme $X \times_Y Z$ is affine and integral, resp. finite, over Z.*

Proof. This follows directly from the definition of an integral (resp. finite) morphism of schemes (Morphisms, Definition 44.1). □

This clears the way for the following definition.

03ZP **Definition 45.2.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say that f is *integral* if for every affine scheme Z and morphisms $Z \to Y$ the algebraic space $X \times_Y Z$ is representable by an affine scheme integral over Z.
(2) We say that f is *finite* if for every affine scheme Z and morphisms $Z \to Y$ the algebraic space $X \times_Y Z$ is representable by an affine scheme finite over Z.

03ZQ **Lemma 45.3.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is representable and integral (resp. finite),*
(2) *f is integral (resp. finite),*
(3) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is integral (resp. finite), and*
(4) *there exists a Zariski covering $Y = \bigcup Y_i$ such that each of the morphisms $f^{-1}(Y_i) \to Y_i$ is integral (resp. finite).*

Proof. It is clear that (1) implies (2) and that (2) implies (3) by taking V to be a disjoint union of affines étale over Y, see Properties of Spaces, Lemma 6.1. Assume $V \to Y$ is as in (3). Then for every affine open W of V we see that $W \times_Y X$ is an affine open of $V \times_Y X$. Hence by Properties of Spaces, Lemma 13.1 we conclude that $V \times_Y X$ is a scheme. Moreover the morphism $V \times_Y X \to V$ is affine. This means we can apply Spaces, Lemma 11.5 because the class of integral (resp. finite) morphisms satisfies all the required properties (see Morphisms, Lemmas 44.6 and Descent, Lemmas 23.22, 23.23, and 37.1). The conclusion of applying this lemma is that f is representable and integral (resp. finite), i.e., (1) holds.

The equivalence of (1) and (4) follows from the fact that being integral (resp. finite) is Zariski local on the target (the reference above shows that being integral or finite is in fact fpqc local on the target). □

03ZR **Lemma 45.4.** *The composition of integral (resp. finite) morphisms is integral (resp. finite).*

Proof. Omitted. □

03ZS **Lemma 45.5.** *The base change of an integral (resp. finite) morphism is integral (resp. finite).*

Proof. Omitted. □

0414 **Lemma 45.6.** *A finite morphism of algebraic spaces is integral. An integral morphism of algebraic spaces which is locally of finite type is finite.*

Proof. In both cases the morphism is representable, and you can check the condition after a base change by an affine scheme mapping into Y, see Lemmas 45.3. Hence this lemma follows from the same lemma for the case of schemes, see Morphisms, Lemma 44.4. □

0415 **Lemma 45.7.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent*

(1) *f is integral, and*
(2) *f is affine and universally closed.*

Proof. In both cases the morphism is representable, and you can check the condition after a base change by an affine scheme mapping into Y, see Lemmas 45.3, 20.3, and 9.5. Hence the result follows from Morphisms, Lemma 44.7. □

04NY **Lemma 45.8.** *A finite morphism of algebraic spaces is quasi-finite.*

Proof. Let $f : X \to Y$ be a morphism of algebraic spaces. By Definition 45.2 and Lemmas 8.8 and 27.6 both properties may be checked after base change to an affine over Y, i.e., we may assume Y affine. If f is finite then X is a scheme. Hence the result follows from the corresponding result for schemes, see Morphisms, Lemma 44.10. □

04NZ **Lemma 45.9.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent*

(1) *f is finite, and*
(2) *f is affine and proper.*

Proof. In both cases the morphism is representable, and you can check the condition after base change to an affine scheme mapping into Y, see Lemmas 45.3, 20.3, and 40.2. Hence the result follows from Morphisms, Lemma 44.11. □

081Y **Lemma 45.10.** *A closed immersion is finite (and a fortiori integral).*

Proof. Omitted. □

0CZ2 **Lemma 45.11.** *Let S be a scheme. Let $X_i \to Y$, $i = 1, \ldots, n$ be finite morphisms of algebraic spaces over S. Then $X_1 \amalg \ldots \amalg X_n \to Y$ is finite too.*

Proof. Follows from the case of schemes (Morphisms, Lemma 44.13) by étale localization. □

081Z **Lemma 45.12.** *Let S be a scheme. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of algebraic spaces over S.*

(1) *If $g \circ f$ is finite and g separated then f is finite.*
(2) *If $g \circ f$ is integral and g separated then f is integral.*

Proof. Assume $g \circ f$ is finite (resp. integral) and g separated. The base change $X \times_Z Y \to Y$ is finite (resp. integral) by Lemma 45.5. The morphism $X \to X \times_Z Y$ is a closed immersion as $Y \to Z$ is separated, see Lemma 4.7. A closed immersion is finite (resp. integral), see Lemma 45.10. The composition of finite (resp. integral) morphisms is finite (resp. integral), see Lemma 45.4. Thus we win. □

46. Finite locally free morphisms

03ZT We have already defined in Section 3 what it means for a representable morphism of algebraic spaces to be finite locally free.

03ZU **Lemma 46.1.** *Let S be a scheme. Let $f : X \to Y$ be a representable morphism of algebraic spaces over S. Then f is finite locally free (in the sense of Section 3) if and only if f is affine and the sheaf $f_*\mathcal{O}_X$ is a finite locally free $\mathcal{O}_Y$-module.*

Proof. Assume f is finite locally free (as defined in Section 3). This means that for every morphism $V \to Y$ whose source is a scheme the base change $f' : V \times_Y X \to V$ is a finite locally free morphism of schemes. This in turn means (by the definition of a finite locally free morphism of schemes) that $f'_*\mathcal{O}_{V \times_Y X}$ is a finite locally free $\mathcal{O}_V$-module. We may choose $V \to Y$ to be surjective and étale. By Properties of Spaces, Lemma 26.2 we conclude the restriction of $f_*\mathcal{O}_X$ to V is finite locally free. Hence by Modules on Sites, Lemma 23.3 applied to the sheaf $f_*\mathcal{O}_X$ on $Y_{spaces,étale}$ we conclude that $f_*\mathcal{O}_X$ is finite locally free.

Conversely, assume f is affine and that $f_*\mathcal{O}_X$ is a finite locally free $\mathcal{O}_Y$-module. Let V be a scheme, and let $V \to Y$ be a surjective étale morphism. Again by Properties of Spaces, Lemma 26.2 we see that $f'_*\mathcal{O}_{V\times_Y X}$ is finite locally free. Hence $f' : V \times_Y X \to V$ is finite locally free (as it is also affine). By Spaces, Lemma 11.5 we conclude that f is finite locally free (use Morphisms, Lemma 48.4 Descent, Lemmas 23.30 and 37.1). Thus we win. □

This clears the way for the following definition.

03ZV **Definition 46.2.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. We say that f is *finite locally free* if f is affine and $f_*\mathcal{O}_X$ is a finite locally free $\mathcal{O}_Y$-module. In this case we say f is has *rank* or *degree* d if the sheaf $f_*\mathcal{O}_X$ is finite locally free of rank d.

03ZW **Lemma 46.3.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is representable and finite locally free,*
(2) *f is finite locally free,*
(3) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is finite locally free, and*
(4) *there exists a Zariski covering $Y = \bigcup Y_i$ such that each morphism $f^{-1}(Y_i) \to Y_i$ is finite locally free.*

Proof. It is clear that (1) implies (2) and that (2) implies (3) by taking V to be a disjoint union of affines étale over Y, see Properties of Spaces, Lemma 6.1. Assume $V \to Y$ is as in (3). Then for every affine open W of V we see that $W \times_Y X$ is an affine open of $V \times_Y X$. Hence by Properties of Spaces, Lemma 13.1 we conclude that $V \times_Y X$ is a scheme. Moreover the morphism $V \times_Y X \to V$ is affine. This means we can apply Spaces, Lemma 11.5 because the class of finite locally free morphisms satisfies all the required properties (see Morphisms, Lemma 48.4 Descent, Lemmas 23.30 and 37.1). The conclusion of applying this lemma is that f is representable and finite locally free, i.e., (1) holds.

The equivalence of (1) and (4) follows from the fact that being finite locally free is Zariski local on the target (the reference above shows that being finite locally free is in fact fpqc local on the target). □

03ZX **Lemma 46.4.** *The composition of finite locally free morphisms is finite locally free.*

Proof. Omitted. □

03ZY **Lemma 46.5.** *The base change of a finite locally free morphism is finite locally free.*

Proof. Omitted. □

0416 **Lemma 46.6.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is finite locally free,*
(2) *f is finite, flat, and locally of finite presentation.*

If Y is locally Noetherian these are also equivalent to

(3) *f is finite and flat.*

Proof. In each of the three cases the morphism is representable and you can check the property after base change by a surjective étale morphism $V \to Y$, see Lemmas 45.3, 46.3, 30.5, and 28.4. If Y is locally Noetherian, then V is locally Noetherian. Hence the result follows from the corresponding result in the schemes case, see Morphisms, Lemma 48.2. □

47. Rational maps

0EML This section is the analogue of Morphisms, Section 49. We will use without further mention that the intersection of dense opens of a topological space is a dense open.

0EMM **Definition 47.1.** Let S be a scheme. Let X, Y be algebraic spaces over S.

(1) Let $f : U \to Y$, $g : V \to Y$ be morphisms of algebraic spaces over S defined on dense open subspaces U, V of X. We say that f is *equivalent* to g if $f|_W = g|_W$ for some dense open subspace $W \subset U \cap V$.
(2) A *rational map from X to Y* is an equivalence class for the equivalence relation defined in (1).
(3) Given morphisms $X \to B$ and $Y \to B$ of algebraic spaces over S we say that a rational map from X to Y is a *B-rational map from X to Y* if there exists a representative $f : U \to Y$ of the equivalence class which is a morphism over B.

We say that two morphisms f, g as in (1) of the definition define the same rational map instead of saying that they are equivalent. In many cases we will consider in the future, the algebraic spaces X and Y will contain a dense open subspaces X' and Y' which are schemes. In that case a rational map from X to Y is the same as an S-rational map from X' to Y' in the sense of Morphisms, Definition 47.1. Then all of the theory developed for schemes can be brought to bear.

0EMN **Definition 47.2.** Let S be a scheme. Let X be an algebraic space over S. A *rational function on X* is a rational map from X to $\mathbf{A}^1_S$.

Looking at the discussion following Morphisms, Definition 49.3 we find that this is the same as the notion defined there in case X happens to be a scheme.

Recall that we have the canonical identification

$$\mathrm{Mor}_S(T, \mathbf{A}^1_S) = \mathrm{Mor}(T, \mathbf{A}^1_{\mathbf{Z}}) = \Gamma(T, \mathcal{O}_T)$$

for any scheme T over S, see Schemes, Example 15.2. Hence $\mathbf{A}^1_S$ is a ring-object in the category of schemes over S. In other words, addition and multiplication define morphisms

$$+ : \mathbf{A}^1_S \times_S \mathbf{A}^1_S \to \mathbf{A}^1_S \quad \text{and} \quad * : \mathbf{A}^1_S \times_S \mathbf{A}^1_S \to \mathbf{A}^1_S$$

satisfying the axioms of the addition and multiplication in a ring (commutative with 1 as always). Hence also the set of rational maps into $\mathbf{A}^1_S$ has a natural ring structure.

0EMP **Definition 47.3.** Let S be a scheme. Let X be an algebraic space over S. The *ring of rational functions on X* is the ring $R(X)$ whose elements are rational functions with addition and multiplication as just described.

We will define function fields for integral algebraic spaces later, see Spaces over Fields, Section 4.

0EMQ **Definition 47.4.** Let S be a scheme. Let φ be a rational map between two algebraic spaces X and Y over S. We say φ is *defined in a point* $x \in |X|$ if there exists a representative (U, f) of φ with $x \in |U|$. The *domain of definition* of φ is the set of all points where φ is defined.

The domain of definition is viewed as an open subspace of X via Properties of Spaces, Lemma 4.8. With this definition it isn't true in general that φ has a representative which is defined on all of the domain of definition.

0EMR **Lemma 47.5.** *Let S be a scheme. Let X and Y be algebraic spaces over S. Assume X is reduced and Y is separated over S. Let φ be a rational map from X to Y with domain of definition $U \subset X$. Then there exists a unique morphism $f : U \to Y$ of algebraic spaces representing φ.*

Proof. Let (V, g) and (V', g') be representatives of φ. Then g, g' agree on a dense open subspace $W \subset V \cap V'$. On the other hand, the equalizer E of $g|_{V\cap V'}$ and $g'|_{V\cap V'}$ is a closed subspace of $V \cap V'$ because it is the base change of $\Delta : Y \to Y \times_S Y$ by the morphism $V \cap V' \to Y \times_S Y$ given by $g|_{V\cap V'}$ and $g'|_{V\cap V'}$. Now $W \subset E$ implies that $|E| = |V \cap V'|$. As $V \cap V'$ is reduced we conclude $E = V \cap V'$ scheme theoretically, i.e., $g|_{V\cap V'} = g'|_{V\cap V'}$, see Properties of Spaces, Lemma 12.4. It follows that we can glue the representatives $g : V \to Y$ of φ to a morphism $f : U \to Y$ because $\coprod V \to U$ is a surjection of fppf sheaves and $\coprod_{V,V'} V \cap V' = (\coprod V) \times_U (\coprod V)$. □

In general it does not make sense to compose rational maps. The reason is that the image of a representative of the first rational map may have empty intersection with the domain of definition of the second. However, if we assume that our spaces are irreducible and we look at dominant rational maps, then we can compose rational maps.

0EMS **Definition 47.6.** Let S be a scheme. Let X and Y be algebraic spaces over S. Assume $|X|$ and $|Y|$ are irreducible. A rational map from X to Y is called *dominant* if any representative $f : U \to Y$ is a dominant morphism in the sense of Definition 18.1.

We can compose a dominant rational map φ between irreducible algebraic spaces X and Y with an arbitrary rational map ψ from Y to Z. Namely, choose representatives $f : U \to Y$ with $|U| \subset |X|$ open dense and $g : V \to Z$ with $|V| \subset |Y|$ open dense. Then $W = |f|^{-1}(V) \subset |X|$ is open nonempty (because the image of $|f|$ is dense and hence must meet the nonempty open V) and hence dense as $|X|$ is irreducible. We define $\psi \circ \varphi$ as the equivalence class of $g \circ f|_W : W \to Z$. We omit the verification that this is well defined.

In this way we obtain a category whose objects are irreducible algebraic spaces over S and whose morphisms are dominant rational maps.

0EMT **Definition 47.7.** Let S be a scheme. Let X and Y be algebraic spaces over S with $|X|$ and $|Y|$ irreducible. We say X and Y are *birational* if X and Y are isomorphic in the category of irreducible algebraic spaces over S and dominant rational maps.

If X and Y are birational irreducible algebraic spaces, then the set of rational maps from X to Z is bijective with the set of rational map from Y to Z for all algebraic spaces Z (functorially in Z). For "general" irreducible algebraic spaces

this is just one possible definition. Another would be to require X and Y have isomorphic rings of rational functions; sometimes these two notions are equivalent (insert future reference here).

0EMU **Lemma 47.8.** *Let S be a scheme. Let X and Y be algebraic space over S with $|X|$ and $|Y|$ irreducible. Then X and Y are birational if and only if there are nonempty open subspaces $U \subset X$ and $V \subset Y$ which are isomorphic as algebraic spaces over S.*

Proof. Assume X and Y are birational. Let $f : U \to Y$ and $g : V \to X$ define inverse dominant rational maps from X to Y and from Y to X. After shrinking U we may assume $f : U \to Y$ factors through V. As $g \circ f$ is the identity as a dominant rational map, we see that the composition $U \to V \to X$ is the identity on a dense open of U. Thus after replacing U by a smaller open we may assume that $U \to V \to X$ is the inclusion of U into X. By symmetry we find there exists an open subspace $V' \subset V$ such that $g|_{V'} : V' \to X$ factors through $U \subset X$ and such that $V' \to U \to Y$ is the identity. The inverse image of $|V'|$ by $|U| \to |V|$ is an open of $|U|$ and hence equal to $|U'|$ for some open subspace $U' \subset U$, see Properties of Spaces, Lemma 4.8. Then $U' \subset U \to V$ factors as $U' \to V'$. Similarly $V' \to U$ factors as $V' \to U'$. The reader finds that $U' \to V'$ and $V' \to U'$ are mutually inverse morphisms of algebraic spaces over S and the proof is complete. □

48. Relative normalization of algebraic spaces

0BAZ This section is the analogue of Morphisms, Section 53.

0820 **Lemma 48.1.** *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{A}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-algebras. There exists a quasi-coherent sheaf of $\mathcal{O}_X$-algebras $\mathcal{A}' \subset \mathcal{A}$ such that for any affine object U of $X_{étale}$ the ring $\mathcal{A}'(U) \subset \mathcal{A}(U)$ is the integral closure of $\mathcal{O}_X(U)$ in $\mathcal{A}(U)$.*

Proof. Let U be an object of $X_{étale}$. Then U is a scheme. Denote $\mathcal{A}|_U$ the restriction to the Zariski site. Then $\mathcal{A}|_U$ is a quasi-coherent sheaf of $\mathcal{O}_U$-algebras hence we can apply Morphisms, Lemma 53.1 to find a quasi-coherent subalgebra $\mathcal{A}'_U \subset \mathcal{A}|_U$ such that the value of $\mathcal{A}'_U$ on any affine open $W \subset U$ is as given in the statement of the lemma. If $f : U' \to U$ is a morphism in $X_{étale}$, then $\mathcal{A}|_{U'} = f^*(\mathcal{A}|_U)$ where f^* means pullback by the morphism f in the Zariski topology; this holds because $\mathcal{A}$ is quasi-coherent (see introduction to Properties of Spaces, Section 29 and the references to the discussion in the chapter on descent on schemes). Since f is étale we find that More on Morphisms, Lemma 19.1 says that we get a canonical isomorphism $f^*(\mathcal{A}'_U) = \mathcal{A}'_{U'}$. This immediately tells us that we obtain a sub presheaf $\mathcal{A}' \subset \mathcal{A}$ of $\mathcal{O}_X$-algebras over $X_{étale}$ which is a sheaf for the Zariski topology and has the right values on affine objects. But the fact that each $\mathcal{A}'_U$ is quasi-coherent on the scheme U and that for $f : U' \to U$ étale we have $\mathcal{A}'_{U'} = f^*(\mathcal{A}'_U)$ implies that $\mathcal{A}'$ is quasi-coherent on $X_{étale}$ as well (as this is a local property and we have the references above describing quasi-coherent modules on $U_{étale}$ in exactly this manner). □

0821 **Definition 48.2.** Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{A}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-algebras. The *integral closure of $\mathcal{O}_X$ in $\mathcal{A}$* is the quasi-coherent $\mathcal{O}_X$-subalgebra $\mathcal{A}' \subset \mathcal{A}$ constructed in Lemma 48.1 above.

We will apply this in particular when $\mathcal{A} = f_*\mathcal{O}_Y$ for a quasi-compact and quasi-separated morphism of algebraic spaces $f : Y \to X$ (see Lemma 11.2). We can then take the relative spectrum of the quasi-coherent $\mathcal{O}_X$-algebra (Lemma 20.7) to obtain the normalization of X in Y.

0822 **Definition 48.3.** Let S be a scheme. Let $f : Y \to X$ be a quasi-compact and quasi-separated morphism of algebraic spaces over S. Let $\mathcal{O}'$ be the integral closure of $\mathcal{O}_X$ in $f_*\mathcal{O}_Y$. The *normalization of X in Y* is the morphism of algebraic spaces

$$\nu : X' = \underline{\mathrm{Spec}}_X(\mathcal{O}') \to X$$

over S. It comes equipped with a natural factorization

$$Y \xrightarrow{f'} X' \xrightarrow{\nu} X$$

of the initial morphism f.

To get the factorization, use Remark 20.9 and functoriality of the $\underline{\mathrm{Spec}}$ construction.

0ABP **Lemma 48.4.** *Let S be a scheme. Let $f : Y \to X$ be a quasi-compact and quasi-separated morphism of algebraic spaces over S. Let $Y \to X' \to X$ be the normalization of X in Y.*

1. *If $W \to X$ is an étale morphism of algebraic spaces over S, then $W \times_X X'$ is the normalization of W in $W \times_X Y$.*
2. *If Y and X are representable, then Y' is representable and is canonically isomorphic to the normalization of the scheme X in the scheme Y as constructed in Morphisms, Section 54.*

Proof. It is immediate from the construction that the formation of the normalization of X in Y commutes with étale base change, i.e., part (1) holds. On the other hand, if X and Y are schemes, then for $U \subset X$ affine open, $f_*\mathcal{O}_Y(U) = \mathcal{O}_Y(f^{-1}(U))$ and hence $\nu^{-1}(U)$ is the spectrum of exactly the same ring as we get in the corresponding construction for schemes. □

Here is a characterization of this construction.

0823 **Lemma 48.5.** *Let S be a scheme. Let $f : Y \to X$ be a quasi-compact and quasi-separated morphism of algebraic spaces over S. The factorization $f = \nu \circ f'$, where $\nu : X' \to X$ is the normalization of X in Y is characterized by the following two properties:*

1. *the morphism ν is integral, and*
2. *for any factorization $f = \pi \circ g$, with $\pi : Z \to X$ integral, there exists a commutative diagram*

$$\begin{array}{ccc} Y & \xrightarrow{g} & Z \\ {\scriptstyle f'}\downarrow & \nearrow_{h} & \downarrow{\scriptstyle \pi} \\ X' & \xrightarrow{\nu} & X \end{array}$$

for a unique morphism $h : X' \to Z$.

Moreover, in (2) the morphism $h : X' \to Z$ is the normalization of Z in Y.

Proof. Let $\mathcal{O}' \subset f_*\mathcal{O}_Y$ be the integral closure of $\mathcal{O}_X$ as in Definition 48.3. The morphism ν is integral by construction, which proves (1). Assume given a factorization $f = \pi \circ g$ with $\pi : Z \to X$ integral as in (2). By Definition 45.2 π is affine,

and hence Z is the relative spectrum of a quasi-coherent sheaf of $\mathcal{O}_X$-algebras $\mathcal{B}$. The morphism $g : X \to Z$ corresponds to a map of $\mathcal{O}_X$-algebras $\chi : \mathcal{B} \to f_*\mathcal{O}_Y$. Since $\mathcal{B}(U)$ is integral over $\mathcal{O}_X(U)$ for every affine U étale over X (by Definition 45.2) we see from Lemma 48.1 that $\chi(\mathcal{B}) \subset \mathcal{O}'$. By the functoriality of the relative spectrum Lemma 20.7 this provides us with a unique morphism $h : X' \to Z$. We omit the verification that the diagram commutes.

It is clear that (1) and (2) characterize the factorization $f = \nu \circ f'$ since it characterizes it as an initial object in a category. The morphism h in (2) is integral by Lemma 45.12. Given a factorization $g = \pi' \circ g'$ with $\pi' : Z' \to Z$ integral, we get a factorization $f = (\pi \circ \pi') \circ g'$ and we get a morphism $h' : X' \to Z'$. Uniqueness implies that $\pi' \circ h' = h$. Hence the characterization (1), (2) applies to the morphism $h : X' \to Z$ which gives the last statement of the lemma. □

0AYF **Lemma 48.6.** *Let S be a scheme. Let $f : Y \to X$ be a quasi-compact and quasi-separated morphism of algebraic spaces over S. Let $X' \to X$ be the normalization of X in Y. If Y is reduced, so is X'.*

Proof. This follows from the fact that a subring of a reduced ring is reduced. Some details omitted. □

0AYG **Lemma 48.7.** *Let S be a scheme. Let $f : Y \to X$ be a quasi-compact and quasi-separated morphism of schemes. Let $X' \to X$ be the normalization of X in Y. If $x' \in |X'|$ is a point of codimension 0 (Properties of Spaces, Definition 10.2), then x' is the image of some $y \in |Y|$ of codimension 0.*

Proof. By Lemma 48.4 and the definitions, we may assume that $X = \mathrm{Spec}(A)$ is affine. Then $X' = \mathrm{Spec}(A')$ where A' is the integral closure of A in $\Gamma(Y, \mathcal{O}_Y)$ and x' corresponds to a minimal prime of A'. Choose a surjective étale morphism $V \to Y$ where $V = \mathrm{Spec}(B)$ is affine. Then $A' \to B$ is injective, hence every minimal prime of A' is the image of a minimal prime of B, see Algebra, Lemma 30.5. The lemma follows. □

0824 **Lemma 48.8.** *Let S be a scheme. Let $f : Y \to X$ be a quasi-compact and quasi-separated morphism of algebraic spaces over S. Suppose that $Y = Y_1 \amalg Y_2$ is a disjoint union of two algebraic spaces. Write $f_i = f|_{Y_i}$. Let X_i' be the normalization of X in Y_i. Then $X_1' \amalg X_2'$ is the normalization of X in Y.*

Proof. Omitted. □

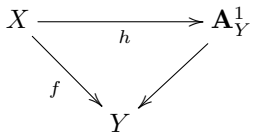
0A0Q **Lemma 48.9.** *Let S be a scheme. Let $f : X \to Y$ be a quasi-compact, quasi-separated and universally closed morphisms of algebraic spaces over S. Then $f_*\mathcal{O}_X$ is integral over $\mathcal{O}_Y$. In other words, the normalization of Y in X is equal to the factorization*

$$X \longrightarrow \underline{\mathrm{Spec}}_Y(f_*\mathcal{O}_X) \longrightarrow Y$$

of Remark 20.9.

Proof. The question is étale local on Y, hence we may reduce to the case where $Y = \mathrm{Spec}(R)$ is affine. Let $h \in \Gamma(X, \mathcal{O}_X)$. We have to show that h satisfies a monic equation over R. Think of h as a morphism as in the following commutative

diagram



Let $Z \subset \mathbf{A}^1_Y$ be the scheme theoretic image of h, see Definition 16.2. The morphism h is quasi-compact as f is quasi-compact and $\mathbf{A}^1_Y \to Y$ is separated, see Lemma 8.9. By Lemma 16.3 the morphism $X \to Z$ has dense image on underlying topological spaces. By Lemma 40.6 the morphism $X \to Z$ is closed. Hence $h(X) = Z$ (set theoretically). Thus we can use Lemma 40.7 to conclude that $Z \to Y$ is universally closed (and even proper). Since $Z \subset \mathbf{A}^1_Y$, we see that $Z \to Y$ is affine and proper, hence integral by Lemma 45.7. Writing $\mathbf{A}^1_Y = \mathrm{Spec}(R[T])$ we conclude that the ideal $I \subset R[T]$ of Z contains a monic polynomial $P(T) \in R[T]$. Hence $P(h) = 0$ and we win. □

0825 **Lemma 48.10.** *Let S be a scheme. Let $f : Y \to X$ be an integral morphism of algebraic spaces over S. Then the integral closure of X in Y is equal to Y.*

Proof. By Lemma 45.7 this is a special case of Lemma 48.9. □

0BB0 **Lemma 48.11.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume that*

(1) *Y is Nagata,*
(2) *f is quasi-separated of finite type,*
(3) *X is reduced.*

Then the normalization $\nu : Y' \to Y$ of Y in X is finite.

Proof. The question is étale local on Y, see Lemma 48.4. Thus we may assume $Y = \mathrm{Spec}(R)$ is affine. Then R is a Noetherian Nagata ring and we have to show that the integral closure of R in $\Gamma(X, \mathcal{O}_X)$ is finite over R. Since f is quasi-compact we see that X is quasi-compact. Choose an affine scheme U and a surjective étale morphism $U \to X$ (Properties of Spaces, Lemma 6.3). Then $\Gamma(X, \mathcal{O}_X) \subset \Gamma(U, \mathcal{O}_X)$. Since R is Noetherian it suffices to show that the integral closure of R in $\Gamma(U, \mathcal{O}_U)$ is finite over R. As $U \to Y$ is of finite type this follows from Morphisms, Lemma 53.15. □

49. Normalization

07U3 This section is the analogue of Morphisms, Section 54.

0BB1 **Lemma 49.1.** *Let S be a scheme. Let X be an algebraic space over S. The following are equivalent*

(1) *there is a surjective étale morphism $U \to X$ where U is a scheme such that every quasi-compact open of U has finitely many irreducible components,*
(2) *for every scheme U and every étale morphism $U \to X$ every quasi-compact open of U has finitely many irreducible components,*
(3) *for every quasi-compact algebraic space Y étale over X the set of codimension 0 points of Y (Properties of Spaces, Definition 10.2) is finite, and*
(4) *for every quasi-compact algebraic space Y étale over X the space $|Y|$ has finitely many irreducible components.*

If X is representable this means that every quasi-compact open of X has finitely many irreducible components.

Proof. The equivalence of (1) and (2) and the final statement follow from Descent, Lemma 16.3 and Properties of Spaces, Lemma 7.1. It is clear that (4) implies (1) and (2) by considering only those Y which are schemes. Similarly, (3) implies (1) and (2) since for a scheme the codimension 0 points are the generic points of its irreducible components, see for example Properties of Spaces, Lemma 11.1.

Conversely, assume (2) and let $Y \to X$ be an étale morphism of algebraic spaces with Y quasi-compact. Then we can choose an affine scheme V and a surjective étale morphism $V \to Y$ (Properties of Spaces, Lemma 6.3). Since V has finitely many irreducible components by (2) and since $|V| \to |Y|$ is surjective and continuous, we conclude that $|Y|$ has finitely many irreducible components by Topology, Lemma 8.5. Thus (4) holds. Similarly, by Properties of Spaces, Lemma 11.1 the images of the generic points of the irreducible components of V are the codimension 0 points of Y and we conclude that there are finitely many, i.e., (3) holds. □

0GMB **Lemma 49.2.** *Let S be a scheme. Let X be a locally Noetherian algebraic space over S. Then X satisfies the equivalent conditions of Lemma 49.1.*

Proof. If $U \to X$ is étale and U is a scheme, then U is a locally Noetherian scheme, see Properties of Spaces, Section 7. A locally Noetherian scheme has a locally finite set of irreducible components (Divisors, Lemma 26.1). Thus we conclude that X passes condition (2) of the lemma. □

0GMC **Lemma 49.3.** *Let S be a scheme. Let $f : X \to Y$ be a flat morphism of algebraic spaces over S. Then for $x \in |X|$ we have: x has codimension 0 in $X \Rightarrow f(x)$ has codimension 0 in Y.*

Proof. Via Properties of Spaces, Lemma 11.1 and étale localization this translates into the case of a morphism of schemes and generic points of irreducible components. Here the result follows as generalizations lift along flat morphisms of schemes, see Morphisms, Lemma 25.9. □

0GMD **Lemma 49.4.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is flat and locally of finite type and assume Y satisfies the equivalent conditions of Lemma 49.1. Then X satisfies the equivalent conditions of Lemma 49.1 and for $x \in |X|$ we have: x has codimension 0 in $X \Rightarrow f(x)$ has codimension 0 in Y.*

Proof. The last statement follows from Lemma 49.3. Choose a surjective étale morphism $V \to Y$ where V is a scheme. Choose a surjective étale morphism $U \to X \times_Y V$ where U is a scheme. It suffices to show that every quasi-compact open of U has finitely many irreducible components. We will use the results of Properties of Spaces, Lemma 11.1 without further mention. By what we've already shown, the codimension 0 points of U lie above codimension 0 points in U and these are locally finite by assumption. Hence it suffices to show that for $v \in V$ of codimension 0 the codimension 0 points of the scheme theoretic fibre $U_v = U \times_V v$ are locally finite. This is true because U_v is a scheme locally of finite type over $\kappa(v)$, hence locally Noetherian and we can apply Lemma 49.2 for example. □

07U4 **Lemma 49.5.** *Let S be a scheme. For every algebraic space X over S satisfying the equivalent conditions of Lemma 49.1 there exists a morphism of algebraic spaces*

$$\nu_X : X^\nu \longrightarrow X$$

with the following properties

(1) *if X satisfies the equivalent conditions of Lemma 49.1 then X^ν is normal and ν_X is integral,*
(2) *if X is a scheme such that every quasi-compact open has finitely many irreducible components, then $\nu_X : X^\nu \to X$ is the normalization of X constructed in Morphisms, Section 54,*
(3) *if $f : X \to Y$ is a morphism of algebraic spaces over S which both satisfy the equivalent conditions of Lemma 49.1 and every codimension 0 point of X is mapped by f to a codimension 0 point of Y, then there is a unique morphism $f^\nu : X^\nu \to Y^\nu$ of algebraic spaces over S such that $\nu_Y \circ f^\nu = f \circ \nu_X$, and*
(4) *if $f : X \to Y$ is an étale or smooth morphism of algebraic spaces and Y satisfies the equivalent conditions of Lemma 49.1, then the hypotheses of (3) hold and the morphism f^ν induces an isomorphism $X^\nu \to X \times_Y Y^\nu$.*

Proof. Consider the category $\mathcal{C}$ whose objects are the schemes U over S such that every quasi-compact open of U has finitely many irreducible components and whose morphisms are those morphisms $g : U \to V$ of schemes over S such that every generic point of an irreducible component of U is mapped to the generic point of an irreducible component of V. We have already shown that

(a) for $U \in \mathrm{Ob}(\mathcal{C})$ we have a normalization morphism $\nu_U : U^\nu \to U$ as in Morphisms, Definition 54.1,
(b) for $U \in \mathrm{Ob}(\mathcal{C})$ the morphism ν_U is integral and U^ν is a normal scheme, see Morphisms, Lemma 54.5,
(c) for every $g : U \to V \in \mathrm{Arrows}(\mathcal{C})$ there is a unique morphism $g^\nu : U^\nu \to V^\nu$ such that $\nu_V \circ g^\nu = g \circ \nu_U$, see Morphisms, Lemma 54.5 part (4) applied to the composition $X^\nu \to X \to Y$,
(d) if $V \in \mathrm{Ob}(\mathcal{C})$ and $g : U \to V$ is étale or smooth, then $U \in \mathrm{Ob}(\mathcal{C})$ and $g \in \mathrm{Arrows}(\mathcal{C})$ and the morphism g^ν induces an isomorphism $U^\nu \to U \times_V V^\nu$, see Lemma 49.4 and More on Morphisms, Lemma 19.3.

Our task is to extend this construction to the corresponding category of algebraic spaces X over S.

Let X be an algebraic space over S satisfying the equivalent conditions of Lemma 49.1. Let $U \to X$ be a surjective étale morphism where U is a scheme. Set $R = U \times_X U$ with projections $s, t : R \to U$ and $j = (t, s) : R \to U \times_S U$ so that $X = U/R$, see Spaces, Lemma 9.1. Observe that U and R are objects of $\mathcal{C}$ by our assumptions on X and that the morphisms s and t are étale morphisms of schemes over S. By (a) we have the normalization morphisms $\nu_U : U^\nu \to U$ and $\nu_R : R^\nu \to R$, by (d) we have morphisms $s^\nu : R^\nu \to U^\nu$, $t^\nu : R^\nu \to U^\nu$ which define isomorphisms $R^\nu \to R \times_{s,U} U^\nu$ and $R^\nu \to U^\nu \times_{U,t} R$. It follows that s^ν and t^ν are étale (as they are isomorphic to base changes of étale morphisms). The induced morphism $j^\nu = (t^\nu, s^\nu) : R^\nu \to U^\nu \times_S U^\nu$ is a monomorphism as it is equal to the

composition

$$\begin{aligned} R^\nu &\to (U^\nu \times_{U,t} R) \times_R (R \times_{s,U} U^\nu) \\ &= U^\nu \times_{U,t} R \times_{s,U} U^\nu \\ &\xrightarrow{j} U^\nu \times_U (U \times_S U) \times_U U^\nu \\ &= U^\nu \times_S U^\nu \end{aligned}$$

The first arrow is the diagonal morphism of ν_R. (This tells us that R^ν is a subscheme of the restriction of R to U^ν.) A formal computation with fibre products using property (d) shows that $R^\nu \times_{s^\nu, U^\nu, t^\nu} R^\nu$ is the normalization of $R \times_{s,U,t} R$. Hence the étale morphism $c : R \times_{s,U,t} R \to R$ extends uniquely to c^ν by (d). The morphism c^ν is compatible with the projection $\mathrm{pr}_{13} : U^\nu \times_S U^\nu \times_S U^\nu \to U^\nu \times_S U^\nu$. Similarly, there are morphisms $i^\nu : R^\nu \to R^\nu$ compatible with the morphism $U^\nu \times_S U^\nu \to U^\nu \times_S U^\nu$ which switches factors and there is a morphism $e^\nu : U^\nu \to R^\nu$ compatible with the diagonal morphism $U^\nu \to U^\nu \times_S U^\nu$. All in all it follows that $j^\nu : R^\nu \to U^\nu \times_S U^\nu$ is an étale equivalence relation. At this point we may and do set $X^\nu = U^\nu / R^\nu$ (Spaces, Theorem 10.5). Then we see that we have $U^\nu = X^\nu \times_X U$ by Groupoids, Lemma 20.7.

What have we shown in the previous paragraph is this: for every algebraic space X over S satisfying the equivalent conditions of Lemma 49.1 if we choose a surjective étale morphism $g : U \to X$ where U is a scheme, then we obtain a cartesian diagram

$$\begin{array}{ccc} X^\nu & \xleftarrow{g^\nu} & U^\nu \\ {\scriptstyle \nu_X}\downarrow & & \downarrow{\scriptstyle \nu_U} \\ X & \xleftarrow{g} & U \end{array}$$

of algebraic spaces. This immediately implies that X^ν is a normal algebraic space and that ν_X is a integral morphism. This gives part (1) of the lemma.

We will show below that the morphism $\nu_X : X^\nu \to X$ up to unique isomorphism is independent of the choice of g, but for now, if X is a scheme, we choose $\mathrm{id} : X \to X$ so that it is clear that we have part (2) of the lemma.

We still have to prove parts (3) and (4). Let $g : U \to X$ and $\nu_X : X^\nu \to X$ and $g^\nu : U^\nu \to X^\nu$ be as above. Let Z be a normal scheme and let $h : Z \to U$ and $a : Z \to X^\nu$ be morphisms over S such that $g \circ h = \nu_X \circ a$ and such that every irreducible compoent of Z dominates an irreducible component of U (via h). By Morphisms, Lemma 54.5 part (4) we obtain a unique morphism $h^\nu : Z \to U^\nu$ such that $h = \nu_U \circ h^\nu$. Picture:

$$\begin{array}{ccccc} & & & \overset{a}{\curvearrowleft} & \\ X^\nu & \xleftarrow{g^\nu} & U^\nu & \xleftarrow{h^\nu} & Z \\ {\scriptstyle \nu_X}\downarrow & & \downarrow{\scriptstyle \nu_U} & \swarrow{\scriptstyle h} & \\ X & \xleftarrow{g} & U & & \end{array}$$

Observe that $a = g^\nu \circ h^\nu$. Namely, since the square with corners X^ν, X, U^ν, U is cartesian, this follows immediately from the fact that h^ν is unique (given h). In other words, given $h : Z \to U$ as above (and not a) there is a unique morphism $a : Z \to X^\nu$ with $\nu_X \circ a = g \circ h$.

Let $f : X \to Y$ be as in part (3) of the statement of the lemma. Suppose we have chosen surjective étale morphisms $U \to X$ and $V \to Y$ where U and V are schemes such that f lifts to a morphism $g : U \to V$. Then $g \in \text{Arrows}(\mathcal{C})$ and we obtain a unique morphism $g^\nu : U^\nu \to V^\nu$ compatible with ν_U and ν_V. However, then the two morphisms

$$R^\nu = U^\nu \times_{X^\nu} U^\nu \to U^\nu \to V^\nu \to Y^\nu$$

must be the same by our comments in the previous paragraph (applied with Y in stead of X). Since X^ν is constructed by taking the quotient of U^ν by R^ν it follows that we obtain a (unique) morphism $f^\nu : X^\nu \to Y^\nu$ as stated in (3).

To see that the construction of X^ν is independent of the choice of $g : U \to X$ surjective étale, apply the construction in the previous paragraph to $\text{id} : X \to X$ and a morphism $U' \to U$ between étale coverings of X. This is enough because given any two étale coverings of X there is a third one which dominates both. The reader shows that the morphism between the two normalizations constructed using either $U' \to X$ or $U \to X$ becomes an isomorphism after base change to U' and hence was an isomorphism. We omit the details.

We omit the proof of (4) which is similar; hint use part (d) above. □

This leads us to the following definition.

0BB2 **Definition 49.6.** Let S be a scheme. Let X be an algebraic space over S satisfying the equivalent conditions of Lemma 49.1. We define the *normalization* of X as the morphism

$$\nu_X : X^\nu \longrightarrow X$$

constructed in Lemma 49.5.

The definition applies to locally Noetherian algebraic spaces, see Lemma 49.2. Usually the normalization is defined only for reduced algebraic spaces. With the definition above the normalization of X is the same as the normalization of the reduction X_{red} of X.

0BB3 **Lemma 49.7.** *Let S be a scheme. Let X be an algebraic space over S satisfying the equivalent conditions of Lemma 49.1. The normalization morphism ν factors through the reduction X_{red} and $X^\nu \to X_{red}$ is the normalization of X_{red}.*

Proof. We may check this étale locally on X and hence reduce to the case of schemes which is Morphisms, Lemma 54.2. Some details omitted. □

0BB4 **Lemma 49.8.** *Let S be a scheme. Let X be an algebraic space over S satisfying the equivalent conditions of Lemma 49.1.*

(1) *The normalization X^ν is normal.*
(2) *The morphism $\nu : X^\nu \to X$ is integral and surjective.*
(3) *The map $|\nu| : |X^\nu| \to |X|$ induces a bijection between the sets of points of codimension 0 (Properties of Spaces, Definition 10.2).*
(4) *Let $Z \to X$ be a morphism. Assume Z is a normal algebraic space and that for $z \in |Z|$ we have: z has codimension 0 in $Z \Rightarrow f(z)$ has codimension 0 in X. Then there exists a unique factorization $Z \to X^\nu \to X$.*

Proof. Properties (1), (2), and (3) follow from the corresponding results for schemes (Morphisms, Lemma 54.5) combined with the fact that a point of a scheme is a

generic point of an irreducible component if and only if the dimension of the local ring is zero (Properties, Lemma 10.4).

Let $Z \to X$ be a morphism as in (4). Let U be a scheme and let $U \to X$ be a surjective étale morphism. Choose a scheme V and a surjective étale morphism $V \to U \times_X Z$. The condition on codimension 0 points assures us that $V \to U$ maps generic points of irreducible components of V to generic points of irreducible components of U. Thus we obtain a unique factorization $V \to U^\nu \to U$ by Morphisms, Lemma 54.5. The uniqueness guarantees us that the two maps

$$V \times_{U \times_X Z} V \to V \to U^\nu$$

agree because these maps are the unique factorization of the map $V \times_{U \times_X Z} V \to V \to U$. Since the algebraic space $U \times_X Z$ is equal to the quotient $V/V \times_{U \times_X Z} V$ (see Spaces, Section 9) we find a canonical morphism $U \times_X Z \to U^\nu$. Picture

$$\begin{array}{ccccc} U \times_X Z & \longrightarrow & U^\nu & \longrightarrow & U \\ \downarrow & & \downarrow & & \downarrow \\ Z & \dashrightarrow & X^\nu & \longrightarrow & X \end{array}$$

To obtain the dotted arrow we note that the construction of the arrow $U \times_X Z \to U^\nu$ is functorial in the étale morphism $U \to X$ (precise formulation and proof omitted). Hence if we set $R = U \times_X U$ with projections $s, t : R \to U$, then we obtain a morphism $R \times_X Z \to R^\nu$ commuting with $s, t : R \to U$ and $s^\nu, t^\nu : R^\nu \to U^\nu$. Recall that $X^\nu = U^\nu/R^\nu$, see proof of Lemma 49.5. Since $X = U/R$ a simple sheaf theoretic argument shows that $Z = (U \times_X Z)/(R \times_X Z)$. Thus the morphisms $U \times_X Z \to U^\nu$ and $R \times_X Z \to R^\nu$ define a morphism $Z \to X^\nu$ as desired. □

0BB5 **Lemma 49.9.** *Let S be a scheme. Let X be a Nagata algebraic space over S. The normalization $\nu : X^\nu \to X$ is a finite morphism.*

Proof. Since X being Nagata is locally Noetherian, Definition 49.6 applies. By construction of X^ν in Lemma 49.5 we immediately reduce to the case of schemes which is Morphisms, Lemma 54.10. □

50. Separated, locally quasi-finite morphisms

0417 In this section we prove that an algebraic space which is locally quasi-finite and separated over a scheme, is representable. This implies that a separated and locally quasi-finite morphism is representable (see Lemma 51.1). But first... a lemma (which will be obsoleted by Proposition 50.2).

03XW **Lemma 50.1.** *Let S be a scheme. Consider a commutative diagram*

$$\begin{array}{ccccc} V' & \longrightarrow & T' \times_T X & \longrightarrow & X \\ & \searrow & \downarrow & & \downarrow \\ & & T' & \longrightarrow & T \end{array}$$

of algebraic spaces over S. Assume

(1) *$T' \to T$ is an étale morphism of affine schemes,*
(2) *$X \to T$ is a separated, locally quasi-finite morphism,*
(3) *V' is an open subspace of $T' \times_T X$, and*
(4) *$V' \to T'$ is quasi-affine.*

In this situation the image U of V' in X is a quasi-compact open subspace of X which is representable.

Proof. We first make some trivial observations. Note that V' is representable by Lemma 21.3. It is also quasi-compact (as a quasi-affine scheme over an affine scheme, see Morphisms, Lemma 13.2). Since $T' \times_T X \to X$ is étale (Properties of Spaces, Lemma 16.5) the map $|T' \times_T X| \to |X|$ is open, see Properties of Spaces, Lemma 16.7. Let $U \subset X$ be the open subspace corresponding to the image of $|V'|$, see Properties of Spaces, Lemma 4.8. As $|V'|$ is quasi-compact we see that $|U|$ is quasi-compact, hence U is a quasi-compact algebraic space, by Properties of Spaces, Lemma 5.2.

By Morphisms, Lemma 57.9 the morphism $T' \to T$ is universally bounded. Hence we can do induction on the integer n bounding the degree of the fibres of $T' \to T$, see Morphisms, Lemma 57.8 for a description of this integer in the case of an étale morphism. If $n = 1$, then $T' \to T$ is an open immersion (see Étale Morphisms, Theorem 14.1), and the result is clear. Assume $n > 1$.

Consider the affine scheme $T'' = T' \times_T T'$. As $T' \to T$ is étale we have a decomposition (into open and closed affine subschemes) $T'' = \Delta(T') \amalg T^*$. Namely $\Delta = \Delta_{T'/T}$ is open by Morphisms, Lemma 35.13 and closed because $T' \to T$ is separated as a morphism of affines. As a base change the degrees of the fibres of the second projection $\mathrm{pr}_1 : T' \times_T T' \to T'$ are bounded by n, see Morphisms, Lemma 57.5. On the other hand, $\mathrm{pr}_1|_{\Delta(T')} : \Delta(T') \to T'$ is an isomorphism and every fibre has exactly one point. Thus, on applying Morphisms, Lemma 57.8 we conclude the degrees of the fibres of the restriction $\mathrm{pr}_1|_{T^*} : T^* \to T'$ are bounded by $n - 1$. Hence the induction hypothesis applied to the diagram

$$\begin{array}{ccccc} p_0^{-1}(V') \cap X^* & \longrightarrow & X^* & \xrightarrow[p_1|_{X^*}]{} & X' \\ & \searrow & \downarrow & & \downarrow \\ & & T^* & \xrightarrow{\mathrm{pr}_1|_{T^*}} & T' \end{array}$$

gives that $p_1(p_0^{-1}(V') \cap X^*)$ is a quasi-compact scheme. Here we set $X'' = T'' \times_T X$, $X^* = T^* \times_T X$, and $X' = T' \times_T X$, and $p_0, p_1 : X'' \to X'$ are the base changes of $\mathrm{pr}_0, \mathrm{pr}_1$. Most of the hypotheses of the lemma imply by base change the corresponding hypothesis for the diagram above. For example $p_0^{-1}(V') = T'' \times_{T'} V'$ is a scheme quasi-affine over T'' as a base change. Some verifications omitted.

By Properties of Spaces, Lemma 13.1 we conclude that

$$p_1(p_0^{-1}(V')) = V' \cup p_1(p_0^{-1}(V') \cap X^*)$$

is a quasi-compact scheme. Moreover, it is clear that $p_1(p_0^{-1}(V'))$ is the inverse image of the quasi-compact open subspace $U \subset X$ discussed in the first paragraph of the proof. In other words, $T' \times_T U$ is a scheme! Note that $T' \times_T U$ is quasi-compact and separated and locally quasi-finite over T', as $T' \times_T X \to T'$ is locally quasi-finite and separated being a base change of the original morphism $X \to T$ (see Lemmas 4.4 and 27.4). This implies by More on Morphisms, Lemma 43.2 that $T' \times_T U \to T'$ is quasi-affine.

By Descent, Lemma 39.1 this gives a descent datum on $T' \times_T U/T'$ relative to the étale covering $\{T' \to W\}$, where $W \subset T$ is the image of the morphism $T' \to T$.

Because U' is quasi-affine over T' we see from Descent, Lemma 38.1 that this datum is effective, and by the last part of Descent, Lemma 39.1 this implies that U is a scheme as desired. Some minor details omitted. □

03XX **Proposition 50.2.** *Let S be a scheme. Let $f : X \to T$ be a morphism of algebraic spaces over S. Assume*

(1) *T is representable,*
(2) *f is locally quasi-finite, and*
(3) *f is separated.*

Then X is representable.

Proof. Let $T = \bigcup T_i$ be an affine open covering of the scheme T. If we can show that the open subspaces $X_i = f^{-1}(T_i)$ are representable, then X is representable, see Properties of Spaces, Lemma 13.1. Note that $X_i = T_i \times_T X$ and that locally quasi-finite and separated are both stable under base change, see Lemmas 4.4 and 27.4. Hence we may assume T is an affine scheme.

By Properties of Spaces, Lemma 6.2 there exists a Zariski covering $X = \bigcup X_i$ such that each X_i has a surjective étale covering by an affine scheme. By Properties of Spaces, Lemma 13.1 again it suffices to prove the proposition for each X_i. Hence we may assume there exists an affine scheme U and a surjective étale morphism $U \to X$. This reduces us to the situation in the next paragraph.

Assume we have

$$U \longrightarrow X \longrightarrow T$$

where U and T are affine schemes, $U \to X$ is étale surjective, and $X \to T$ is separated and locally quasi-finite. By Lemmas 39.5 and 27.3 the morphism $U \to T$ is locally quasi-finite. Since U and T are affine it is quasi-finite. Set $R = U \times_X U$. Then $X = U/R$, see Spaces, Lemma 9.1. As $X \to T$ is separated the morphism $R \to U \times_T U$ is a closed immersion, see Lemma 4.5. In particular R is an affine scheme also. As $U \to X$ is étale the projection morphisms $t, s : R \to U$ are étale as well. In particular s and t are quasi-finite, flat and of finite presentation (see Morphisms, Lemmas 36.6, 36.12 and 36.11).

Let (U, R, s, t, c) be the groupoid associated to the étale equivalence relation R on U. Let $u \in U$ be a point, and denote $p \in T$ its image. We are going to use More on Groupoids, Lemma 13.2 for the groupoid (U, R, s, t, c) over the scheme T with points p and u as above. By the discussion in the previous paragraph all the assumptions (1) – (7) of that lemma are satisfied. Hence we get an étale neighbourhood $(T', p') \to (T, p)$ and disjoint union decompositions

$$U_{T'} = U' \amalg W, \quad R_{T'} = R' \amalg W'$$

and $u' \in U'$ satisfying conclusions (a), (b), (c), (d), (e), (f), (g), and (h) of the aforementioned More on Groupoids, Lemma 13.2. We may and do assume that T' is affine (after possibly shrinking T'). Conclusion (h) implies that $R' = U' \times_{X_{T'}} U'$ with projection mappings identified with the restrictions of s' and t'. Thus $(U', R', s'|_{R'}, t'|_{R'}, c'|_{R' \times_{t', U', s'} R'})$ of conclusion (g) is an étale equivalence relation. By Spaces, Lemma 10.2 we conclude that U'/R' is an open subspace of $X_{T'}$. By conclusion (d) the schemes U', R' are affine and the morphisms $s'|_{R'}, t'|_{R'}$ are finite étale. Hence Groupoids, Proposition 23.9 kicks in and we see that U'/R' is an affine scheme.

We conclude that for every pair of points (u, p) as above we can find an étale neighbourhood $(T', p') \to (T, p)$ with $\kappa(p) = \kappa(p')$ and a point $u' \in U_{T'}$ mapping to u such that the image x' of u' in $|X_{T'}|$ has an open neighbourhood V' in $X_{T'}$ which is an affine scheme. We apply Lemma 50.1 to obtain an open subspace $W \subset X$ which is a scheme, and which contains x (the image of u in $|X|$). Since this works for every x we see that X is a scheme by Properties of Spaces, Lemma 13.1. This ends the proof. □

51. Applications

05W4 An alternative proof of the following lemma is to see it as a consequence of Zariski's main theorem for (nonrepresentable) morphisms of algebraic spaces as discussed in More on Morphisms of Spaces, Section 34. Namely, More on Morphisms of Spaces, Lemma 34.2 implies that a quasi-finite and separated morphism of algebraic spaces is quasi-affine and therefore representable.

0418 **Lemma 51.1.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is locally quasi-finite and separated, then f is representable.*

Proof. This is immediate from Proposition 50.2 and the fact that being locally quasi-finite and separated is preserved under any base change, see Lemmas 27.4 and 4.4. □

05W5 **Lemma 51.2.** *Let S be a scheme. Let $f : X \to Y$ be an étale and universally injective morphism of algebraic spaces over S. Then f is an open immersion.*

Proof. Let $T \to Y$ be a morphism from a scheme into Y. If we can show that $X \times_Y T \to T$ is an open immersion, then we are done. Since being étale and being universally injective are properties of morphisms stable under base change (see Lemmas 39.4 and 19.5) we may assume that Y is a scheme. Note that the diagonal $\Delta_{X/Y} : X \to X \times_Y X$ is étale, a monomorphism, and surjective by Lemma 19.2. Hence we see that $\Delta_{X/Y}$ is an isomorphism (see Spaces, Lemma 5.9), in particular we see that X is separated over Y. It follows that X is a scheme too, by Proposition 50.2. Finally, $X \to Y$ is an open immersion by the fundamental theorem for étale morphisms of schemes, see Étale Morphisms, Theorem 14.1. □

52. Zariski's Main Theorem (representable case)

0ABQ This is the version you can prove using that normalization commutes with étale localization. Before we can prove more powerful versions (for non-representable morphisms) we need to develop more tools. See More on Morphisms of Spaces, Section 34.

0ABR **Lemma 52.1.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is representable, of finite type, and separated. Let Y' be the normalization of Y in X. Picture:*

$$\begin{array}{ccc} X & \longrightarrow & Y' \\ & {\scriptstyle f'} \searrow \quad \swarrow {\scriptstyle \nu} & \\ & {\scriptstyle f} \searrow \quad \swarrow & \\ & Y & \end{array}$$

Then there exists an open subspace $U' \subset Y'$ such that

(1) *$(f')^{-1}(U') \to U'$ is an isomorphism, and*

(2) $(f')^{-1}(U') \subset X$ *is the set of points at which* f *is quasi-finite.*

Proof. Let $W \to Y$ be a surjective étale morphism where W is a scheme. Then $W \times_Y X$ is a scheme as well. By Lemma 48.4 the algebraic space $W \times_Y Y'$ is representable and is the normalization of the scheme W in the scheme $W \times_Y X$. Picture

$$\begin{array}{ccc} W \times_Y X & \xrightarrow{(1,f')} & W \times_Y Y' \\ & {\scriptstyle (1,f)} \searrow \quad \swarrow {\scriptstyle (1,\nu)} & \\ & W & \end{array}$$

By More on Morphisms, Lemma 43.1 the result of the lemma holds over W. Let $V' \subset W \times_Y Y'$ be the open subscheme such that

(1) $(1, f')^{-1}(V') \to V'$ is an isomorphism, and
(2) $(1, f')^{-1}(V') \subset W \times_Y X$ is the set of points at which $(1, f)$ is quasi-finite.

By Lemma 34.7 there is a maximal open set of points $U \subset X$ where f is quasi-finite and $W \times_Y U = (1, f')^{-1}(V')$. The morphism $f'|_U : U \to Y'$ is an open immersion by Lemma 12.1 as its base change to W is the isomorphism $(1, f')^{-1}(V') \to V'$ followed by the open immersion $V' \to W \times_Y Y'$. Setting $U' = \mathrm{Im}(U \to Y')$ finishes the proof (omitted: the verification that $(f')^{-1}(U') = U$). □

In the following lemma we can drop the assumption of being representable as we've shown that a locally quasi-finite separated morphism is representable.

0ABS **Lemma 52.2.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is quasi-finite and separated. Let Y' be the normalization of Y in X. Picture:*

$$\begin{array}{ccc} X & \xrightarrow{f'} & Y' \\ & {\scriptstyle f} \searrow \quad \swarrow {\scriptstyle \nu} & \\ & Y & \end{array}$$

Then f' is a quasi-compact open immersion and ν is integral. In particular f is quasi-affine.

Proof. By Lemma 51.1 the morphism f is representable. Hence we may apply Lemma 52.1. Thus there exists an open subspace $U' \subset Y'$ such that $(f')^{-1}(U') = X$ (!) and $X \to U'$ is an isomorphism! In other words, f' is an open immersion. Note that f' is quasi-compact as f is quasi-compact and $\nu : Y' \to Y$ is separated (Lemma 8.9). Hence for every affine scheme Z and morphism $Z \to Y$ the fibre product $Z \times_Y X$ is a quasi-compact open subscheme of the affine scheme $Z \times_Y Y'$. Hence f is quasi-affine by definition. □

53. Universal homeomorphisms

05Z3 The class of universal homeomorphisms of schemes is closed under composition and arbitrary base change and is fppf local on the base. See Morphisms, Lemmas 45.3 and 45.2 and Descent, Lemma 23.9. Thus, if we apply the discussion in Section 3 to this notion we see that we know what it means for a representable morphism of algebraic spaces to be a universal homeomorphism.

05Z4 **Lemma 53.1.** *Let S be a scheme. Let $f : X \to Y$ be a representable morphism of algebraic spaces over S. Then f is a universal homeomorphism (in the sense of Section 3) if and only if for every morphism of algebraic spaces $Z \to Y$ the base change map $Z \times_Y X \to Z$ induces a homeomorphism $|Z \times_Y X| \to |Z|$.*

Proof. If for every morphism of algebraic spaces $Z \to Y$ the base change map $Z \times_Y X \to Z$ induces a homeomorphism $|Z \times_Y X| \to |Z|$, then the same is true whenever Z is a scheme, which formally implies that f is a universal homeomorphism in the sense of Section 3. Conversely, if f is a universal homeomorphism in the sense of Section 3 then $X \to Y$ is integral, universally injective and surjective (by Spaces, Lemma 5.8 and Morphisms, Lemma 45.5). Hence f is universally closed, see Lemma 45.7 and universally injective and (universally) surjective, i.e., f is a universal homeomorphism. □

05Z5 **Definition 53.2.** Let S be a scheme. A morphism $f : X \to Y$ of algebraic spaces over S is called a *universal homeomorphism* if and only if for every morphism of algebraic spaces $Z \to Y$ the base change $Z \times_Y X \to Z$ induces a homeomorphism $|Z \times_Y X| \to |Z|$.

This definition does not clash with the pre-existing definition for representable morphisms of algebraic spaces by our Lemma 53.1. For morphisms of algebraic spaces it is not the case that universal homeomorphisms are always integral.

05Z6 **Example 53.3.** This is a continuation of Remark 19.4. Consider the algebraic space $X = \mathbf{A}^1_k/\{x \sim -x \mid x \neq 0\}$. There are morphisms

$$\mathbf{A}^1_k \longrightarrow X \longrightarrow \mathbf{A}^1_k$$

such that the first arrow is étale surjective, the second arrow is universally injective, and the composition is the map $x \mapsto x^2$. Hence the composition is universally closed. Thus it follows that the map $X \to \mathbf{A}^1_k$ is a universal homeomorphism, but $X \to \mathbf{A}^1_k$ is not separated.

Let S be a scheme. Let $f : X \to Y$ be a universal homeomorphism of algebraic spaces over S. Then f is universally closed, hence is quasi-compact, see Lemma 9.7. But f need not be separated (see example above), and not even quasi-separated: an example is to take infinite dimensional affine space $\mathbf{A}^\infty = \mathrm{Spec}(k[x_1, x_2, \ldots])$ modulo the equivalence relation given by flipping finitely many signs of nonzero coordinates (details omitted).

First we state the obligatory lemmas.

0CFT **Lemma 53.4.** *The base change of a universal homeomorphism of algebraic spaces by any morphism of algebraic spaces is a universal homeomorphism.*

Proof. This is immediate from the definition. □

0CFU **Lemma 53.5.** *The composition of a pair of universal homeomorphisms of algebraic spaces is a universal homeomorphism.*

Proof. Omitted. □

08AK **Lemma 53.6.** *Let S be a scheme. Let X be an algebraic space over S. The canonical closed immersion $X_{red} \to X$ (see Properties of Spaces, Definition 12.5) is a universal homeomorphism.*

Proof. Omitted. □

We put the following result here as we do not currently have a better place to put it.

0AEH **Lemma 53.7.** *Let S be a scheme. Let $f : Y \to X$ be a universally injective, integral morphism of algebraic spaces over S.*

(1) *The functor*
$$f_{small,*} : Sh(Y_{étale}) \longrightarrow Sh(X_{étale})$$
is fully faithful and its essential image is those sheaves of sets $\mathcal{F}$ on $X_{étale}$ whose restriction to $|X| \setminus f(|Y|)$ is isomorphic to $*$, and

(2) *the functor*
$$f_{small,*} : Ab(Y_{étale}) \longrightarrow Ab(X_{étale})$$
is fully faithful and its essential image is those abelian sheaves on $Y_{étale}$ whose support is contained in $f(|Y|)$.

In both cases f_{small}^{-1} is a left inverse to the functor $f_{small,*}$.

Proof. Since f is integral it is universally closed (Lemma 45.7). In particular, $f(|Y|)$ is a closed subset of $|X|$ and the statements make sense. The rest of the proof is identical to the proof of Lemma 13.5 except that we use Étale Cohomology, Proposition 47.1 instead of Étale Cohomology, Proposition 46.4. □

54. Other chapters

References

[Knu71] Donald Knutson, *Algebraic spaces*, Lecture Notes in Mathematics, vol. 203, Springer-Verlag, 1971.