MORE ON COHOMOLOGY OF SPACES

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Contents

1. Introduction

0DFS   In this chapter continues the discussion started in Cohomology of Spaces, Section 1. One can also view this chapter as the analogue for algebraic spaces of the chapter on étale cohomology for schemes, see Étale Cohomology, Section 1.

In fact, we intend this chapter to be mainly a translation of the results already proved for schemes into the language of algebraic spaces. Some of our results can be found in [Knu71].

2. Conventions

0DFT   The standing assumption is that all schemes are contained in a big fppf site Sch_{fppf}. And all rings A considered have the property that $\mathrm{Spec}(A)$ is (isomorphic) to an object of this big site.

Let S be a scheme and let X be an algebraic space over S. In this chapter and the following we will write $X \times_S X$ for the product of X with itself (in the category of algebraic spaces over S), instead of $X \times X$.

3. Transporting results from schemes

0DFU   In this section we explain briefly how results for schemes imply results for (representable) algebraic spaces and (representable) morphisms of algebraic spaces. For quasi-coherent modules more is true (because étale cohomology of a quasi-coherent module over a scheme agrees with Zariski cohomology) and this has already been discussed in Cohomology of Spaces, Section 3.

Let S be a scheme. Let X be an algebraic space over S. Now suppose that X is representable by the scheme X_0 (awkward but temporary notation; we usually just say "X is a scheme"). In this case X and X_0 have the same small étale sites:

$$X_{\text{étale}} = (X_0)_{\text{étale}}$$

This is pointed out in Properties of Spaces, Section 18. Moreover, if $f : X \to Y$ is a morphism of representable algebraic spaces over S and if $f_0 : X_0 \to Y_0$ is a morphism of schemes representing f, then the induced morphisms of small étale topoi agree:

$$
\begin{CD}
Sh(X_{\text{étale}}) @>{f_{small}}>> Sh(Y_{\text{étale}}) \\
@| @| \\
Sh((X_0)_{\text{étale}}) @>{(f_0)_{small}}>> Sh((Y_0)_{\text{étale}})
\end{CD}
$$

See Properties of Spaces, Lemma 18.8 and Topologies, Lemma 4.17.

Thus there is absolutely no difference between étale cohomology of a scheme and the étale cohomology of the corresponding algebraic space. Similarly for higher direct images along morphisms of schemes. In fact, if $f : X \to Y$ is a morphism of algebraic spaces over S which is representable (by schemes), then the higher direct images $R^i f_* \mathcal{F}$ of a sheaf $\mathcal{F}$ on $X_{\text{étale}}$ can be computed étale locally on Y (Cohomology on Sites, Lemma 7.4) hence this often reduces computations and proofs to the case where Y and X are schemes.

We will use the above without further mention in this chapter. For other topologies the same thing is true; we state it explicitly as a lemma for cohomology here.

0DFV **Lemma 3.1.** *Let S be a scheme. Let $\tau \in \{\text{étale}, fppf, ph\}$ (add more here). The inclusion functor*

$$(Sch/S)_\tau \longrightarrow (Spaces/S)_\tau$$

is a special cocontinuous functor (Sites, Definition 29.2) and hence identifies topoi.

Proof. The conditions of Sites, Lemma 29.1 are immediately verified as our functor is fully faithful and as every algebraic space has an étale covering by schemes. $\square$

4. Proper base change

0DFW The proper base change theorem for algebraic spaces follows from the proper base change theorem for schemes and Chow's lemma with a little bit of work.

0DFX **Lemma 4.1.** *Let S be a scheme. Let $f : Y \to X$ be a surjective proper morphism of algebraic spaces over S. Let $\mathcal{F}$ be a sheaf on $X_{\text{étale}}$. Then $\mathcal{F} \to f_* f^{-1} \mathcal{F}$ is injective with image the equalizer of the two maps $f_* f^{-1} \mathcal{F} \to g_* g^{-1} \mathcal{F}$ where g is the structure morphism $g : Y \times_X Y \to X$.*

Proof. For any surjective morphism $f : Y \to X$ of algebraic spaces over S, the map $\mathcal{F} \to f_* f^{-1} \mathcal{F}$ is injective. Namely, if $\overline{x}$ is a geometric point of X, then we choose a geometric point $\overline{y}$ of Y lying over $\overline{x}$ and we consider

$$\mathcal{F}_{\overline{x}} \to (f_* f^{-1} \mathcal{F})_{\overline{x}} \to (f^{-1} \mathcal{F})_{\overline{y}} = \mathcal{F}_{\overline{x}}$$

See Properties of Spaces, Lemma 19.9 for the last equality.

The second statement is local on X in the étale topology, hence we may and do assume Y is an affine scheme.

Choose a surjective proper morphism $Z \to Y$ where Z is a scheme, see Cohomology of Spaces, Lemma 18.1. The result for $Z \to X$ implies the result for $Y \to X$. Since $Z \to X$ is a surjective proper morphism of schemes and hence a ph covering (Topologies, Lemma 8.6) the result for $Z \to X$ follows from Étale Cohomology, Lemma 102.1 (in fact it is in some sense equivalent to this lemma). $\square$

0DFY **Lemma 4.2.** *Let (A, I) be a henselian pair. Let X be an algebraic space over A such that the structure morphism $f : X \to \mathrm{Spec}(A)$ is proper. Let $i : X_0 \to X$ be the inclusion of $X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A/I)$. For any sheaf $\mathcal{F}$ on $X_{\acute{e}tale}$ we have $\Gamma(X, \mathcal{F}) = \Gamma(X_0, i^{-1}\mathcal{F})$.*

Proof. Choose a surjective proper morphism $Y \to X$ where Y is a scheme, see Cohomology of Spaces, Lemma 18.1. Consider the diagram

$$
\begin{array}{ccc}
\Gamma(X_0, \mathcal{F}_0) \longrightarrow \Gamma(Y_0, \mathcal{G}_0) \rightrightarrows \Gamma((Y \times_X Y)_0, \mathcal{H}_0) \\
\uparrow \qquad\qquad\qquad \uparrow \qquad\qquad\qquad \uparrow \\
\Gamma(X, \mathcal{F}) \longrightarrow \Gamma(Y, \mathcal{G}) \rightrightarrows \Gamma(Y \times_X Y, \mathcal{H})
\end{array}
$$

Here $\mathcal{G}$, resp. $\mathcal{H}$ is the pullbackf or $\mathcal{F}$ to Y, resp. $Y \times_X Y$ and the index 0 indicates base change to $\mathrm{Spec}(A/I)$. By the case of schemes (Étale Cohomology, Lemma 91.2) we see that the middle and right vertical arrows are bijective. By Lemma 4.1 it follows that the left one is too. $\square$
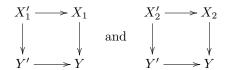
0DFZ **Lemma 4.3.** *Let A be a henselian local ring. Let X be an algebraic space over A such that $f : X \to \mathrm{Spec}(A)$ is a proper morphism. Let $X_0 \subset X$ be the fibre of f over the closed point. For any sheaf $\mathcal{F}$ on $X_{\acute{e}tale}$ we have $\Gamma(X, \mathcal{F}) = \Gamma(X_0, \mathcal{F}|_{X_0})$.*

Proof. This is a special case of Lemma 4.2. $\square$

0DG0 **Lemma 4.4.** *Let S be a scheme. Let $f : X \to Y$ and $g : Y' \to Y$ be a morphisms of algebraic spaces over S. Assume f is proper. Set $X' = Y' \times_Y X$ with projections $f' : X' \to Y'$ and $g' : X' \to X$. Let $\mathcal{F}$ be any sheaf on $X_{\acute{e}tale}$. Then $g^{-1}f_*\mathcal{F} = f'_*(g')^{-1}\mathcal{F}$.*

Proof. The question is étale local on Y'. Choose a scheme V and a surjective étale morphism $V \to Y$. Choose a scheme V' and a surjective étale morphism $V' \to V \times_Y Y'$. Then we may replace Y' by V' and Y by V. Hence we may assume Y and Y' are schemes. Then we may work Zariski locally on Y and Y' and hence we may assume Y and Y' are affine schemes.

Assume Y and Y' are affine schemes. Choose a surjective proper morphism $h_1 : X_1 \to X$ where X_1 is a scheme, see Cohomology of Spaces, Lemma 18.1. Set $X_2 = X_1 \times_X X_1$ and denote $h_2 : X_2 \to X$ the structure morphism. Observe this is a scheme. By the case of schemes (Étale Cohomology, Lemma 91.5) we know the lemma is true for the cartesian diagrams

$$
\begin{array}{ccc}
X_1' \longrightarrow X_1 \\
\downarrow \qquad\quad \downarrow \\
Y' \longrightarrow Y
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
X_2' \longrightarrow X_2 \\
\downarrow \qquad\quad \downarrow \\
Y' \longrightarrow Y
\end{array}
$$

and the sheaves $\mathcal{F}_i = (X_i \to X)^{-1}\mathcal{F}$. By Lemma 4.1 we have an exact sequence $0 \to \mathcal{F} \to h_{1,*}\mathcal{F}_1 \to h_{2,*}\mathcal{F}_2$ and similarly for $(g')^{-1}\mathcal{F}$ because $X_2' = X_1' \times_{X'} X_1'$. Hence we conlude that the lemma is true (some details omitted). $\square$

Let S be a scheme. Let $f : Y \to X$ be a morphism of algebraic spaces over S. Let $\overline{x} : \mathrm{Spec}(k) \to S$ be a geometric point. The fibre of f at $\overline{x}$ is the algebraic space $Y_{\overline{x}} = \mathrm{Spec}(k) \times_{\overline{x}, X} Y$ over $\mathrm{Spec}(k)$. If $\mathcal{F}$ is a sheaf on $Y_{\acute{e}tale}$, then denote $\mathcal{F}_{\overline{x}} = p^{-1}\mathcal{F}$ the pullback of $\mathcal{F}$ to $(Y_{\overline{x}})_{\acute{e}tale}$. Here $p : Y_{\overline{x}} \to Y$ is the projection. In the following we will consider the set $\Gamma(Y_{\overline{x}}, \mathcal{F}_{\overline{x}})$.

0DG1 **Lemma 4.5.** *Let S be a scheme. Let $f : Y \to X$ be a proper morphism of algebraic spaces over S. Let $\overline{x} \to X$ be a geometric point. For any sheaf $\mathcal{F}$ on $Y_{\acute{e}tale}$ the canonical map*

$$(f_*\mathcal{F})_{\overline{x}} \longrightarrow \Gamma(Y_{\overline{x}}, \mathcal{F}_{\overline{x}})$$

is bijective.

Proof. This is a special case of Lemma 4.4. $\square$

0DG2 **Theorem 4.6.** *Let S be a scheme. Let*

$$\begin{array}{ccc} X' & \xrightarrow{\ g'\ } & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{\ g\ } & Y \end{array}$$

be a cartesian square of algebraic spaces over S. Assume f is proper. Let $\mathcal{F}$ be an abelian torsion sheaf on $X_{\acute{e}tale}$. Then the base change map

$$g^{-1}Rf_*\mathcal{F} \longrightarrow Rf_*'(g')^{-1}\mathcal{F}$$

is an isomorphism.

Proof. This proof repeats a few of the arguments given in the proof of the proper base change theorem for schemes. See Étale Cohomology, Section 91 for more details.

The statement is étale local on Y' and Y, hence we may assume both Y and Y' are affine schemes. Observe that this in particular proves the theorem in case f is representable (we will use this below).

For every $n \geq 1$ let $\mathcal{F}[n]$ be the subsheaf of sections of $\mathcal{F}$ annihilated by n. Then $\mathcal{F} = \mathrm{colim}\,\mathcal{F}[n]$. By Cohomology of Spaces, Lemma 5.2 the functors $g^{-1}R^pf_*$ and $R^pf_*'(g')^{-1}$ commute with filtered colimits. Hence it suffices to prove the theorem if $\mathcal{F}$ is killed by n.

Let $\mathcal{F} \to \mathcal{I}^\bullet$ be a resolution by injective sheaves of $\mathbf{Z}/n\mathbf{Z}$-modules. Observe that $g^{-1}f_*\mathcal{I}^\bullet = f_*'(g')^{-1}\mathcal{I}^\bullet$ by Lemma 4.4. Applying Leray's acyclicity lemma (Derived Categories, Lemma 16.7) we conclude it suffices to prove $R^pf_*'(g')^{-1}\mathcal{I}^m = 0$ for $p > 0$ and $m \in \mathbf{Z}$.

Choose a surjective proper morphism $h : Z \to X$ where Z is a scheme, see Cohomology of Spaces, Lemma 18.1. Choose an injective map $h^{-1}\mathcal{I}^m \to \mathcal{J}$ where $\mathcal{J}$ is an injective sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules on $Z_{\acute{e}tale}$. Since h is surjective the map $\mathcal{I}^m \to h_*\mathcal{J}$ is injective (see Lemma 4.1). Since $\mathcal{I}^m$ is injective we see that $\mathcal{I}^m$ is a direct summand of $h_*\mathcal{J}$. Thus it suffices to prove the desired vanishing for $h_*\mathcal{J}$.

Denote h' the base change by g and denote $g'' : Z' \to Z$ the projection. There is a spectral sequence

$$E_2^{p,q} = R^p f'_* R^q h'_* (g'')^{-1} \mathcal{J}$$

converging to $R^{p+q}(f' \circ h')_* (g'')^{-1} \mathcal{J}$. Since h and $f \circ h$ are representable (by schemes) we know the result we want holds for them. Thus in the spectral sequence we see that $E_2^{p,q} = 0$ for $q > 0$ and $R^{p+q}(f' \circ h')_* (g'')^{-1} \mathcal{J} = 0$ for $p + q > 0$. It follows that $E_2^{p,0} = 0$ for $p > 0$. Now

$$E_2^{p,0} = R^p f'_* h'_* (g'')^{-1} \mathcal{J} = R^p f'_* (g')^{-1} h_* \mathcal{J}$$

by Lemma 4.4. This finishes the proof. $\square$

0DG3 **Lemma 4.7.** *Let S be a scheme. Let*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

be a cartesian square of algebraic spaces over S. Assume f is proper. Let $E \in D^+(X_{étale})$ have torsion cohomology sheaves. Then the base change map $g^{-1} R f_* E \to R f'_* (g')^{-1} E$ is an isomorphism.

Proof. This is a simple consequence of the proper base change theorem (Theorem 4.6) using the spectral sequences

$$E_2^{p,q} = R^p f_* H^q(E) \quad \text{and} \quad E'^{p,q}_2 = R^p f'_* (g')^{-1} H^q(E)$$

converging to $R^n f_* E$ and $R^n f'_* (g')^{-1} E$. The spectral sequences are constructed in Derived Categories, Lemma 21.3. Some details omitted. $\square$

0DG4 **Lemma 4.8.** *Let S be a scheme. Let $f : X \to Y$ be a proper morphism of algebraic spaces. Let $\overline{y} \to Y$ be a geometric point.*

(1) *For a torsion abelian sheaf $\mathcal{F}$ on $X_{étale}$ we have $(R^n f_* \mathcal{F})_{\overline{y}} = H^n_{étale}(X_{\overline{y}}, \mathcal{F}_{\overline{y}})$.*
(2) *For $E \in D^+(X_{étale})$ with torsion cohomology sheaves we have $(R^n f_* E)_{\overline{y}} = H^n_{étale}(X_{\overline{y}}, E_{\overline{y}})$.*

Proof. In the statement, $\mathcal{F}_{\overline{y}}$ denotes the pullback of $\mathcal{F}$ to $X_{\overline{y}} = \overline{y} \times_Y X$. Since pulling back by $\overline{y} \to Y$ produces the stalk of $\mathcal{F}$, the first statement of the lemma is a special case of Theorem 4.6. The second one is a special case of Lemma 4.7. $\square$

0DG5 **Lemma 4.9.** *Let k'/k be an extension of separably closed fields. Let X be a proper algebraic space over k. Let $\mathcal{F}$ be a torsion abelian sheaf on X. Then the map $H^q_{étale}(X, \mathcal{F}) \to H^q_{étale}(X_{k'}, \mathcal{F}|_{X_{k'}})$ is an isomorphism for $q \geq 0$.*

Proof. This is a special case of Theorem 4.6. $\square$

5. Comparing big and small topoi

0DG6 Let S be a scheme and let X be an algebraic space over S. In Topologies on Spaces, Lemma 4.8 we have introduced comparison morphisms $\pi_X : (Spaces/X)_{étale} \to X_{spaces,étale}$ and $i_X : Sh(X_{étale}) \to Sh((Spaces/X)_{étale})$ with $\pi_X \circ i_X = \mathrm{id}$ as morphisms of topoi and $\pi_{X,*} = i_X^{-1}$. More generally, if $f : Y \to X$ is an object of $(Spaces/X)_{étale}$, then there is a morphism $i_f : Sh(Y_{étale}) \to Sh((Spaces/X)_{étale})$ such that $f_{small} = \pi_X \circ i_f$, see Topologies on Spaces, Lemmas 4.7 and 4.11. In

Topologies on Spaces, Remark 4.14 we have extended these to a morphism of ringed sites

$$\pi_X : ((Spaces/X)_{étale}, \mathcal{O}) \to (X_{spaces,étale}, \mathcal{O}_X)$$

and morphisms of ringed topoi

$$i_X : (Sh(X_{étale}), \mathcal{O}_X) \to (Sh((Spaces/X)_{étale}), \mathcal{O})$$

and

$$i_f : (Sh(Y_{étale}), \mathcal{O}_Y) \to (Sh((Spaces/X)_{étale}, \mathcal{O}))$$

Note that the restriction $i_X^{-1} = \pi_{X,*}$ (see Topologies, Definition 4.15) transforms $\mathcal{O}$ into $\mathcal{O}_X$. Similarly, i_f^{-1} transforms $\mathcal{O}$ into $\mathcal{O}_Y$. See Topologies on Spaces, Remark 4.14. Hence $i_X^* \mathcal{F} = i_X^{-1} \mathcal{F}$ and $i_f^* \mathcal{F} = i_f^{-1} \mathcal{F}$ for any $\mathcal{O}$-module $\mathcal{F}$ on $(Spaces/X)_{étale}$. In particular i_X^* and i_f^* are exact functors. The functor i_X^* is often denoted $\mathcal{F} \mapsto \mathcal{F}|_{X_{étale}}$ (and this does not conflict with the notation in Topologies on Spaces, Definition 4.9).

0DG7 **Lemma 5.1.** *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a sheaf on $X_{étale}$. Then $\pi_X^{-1} \mathcal{F}$ is given by the rule*

$$(\pi_X^{-1} \mathcal{F})(Y) = \Gamma(Y_{étale}, f_{small}^{-1} \mathcal{F})$$

for $f : Y \to X$ in $(Spaces/X)_{étale}$. Moreover, $\pi_Y^{-1} \mathcal{F}$ satisfies the sheaf condition with respect to smooth, syntomic, fppf, fpqc, and ph coverings.

Proof. Since pullback is transitive and $f_{small} = \pi_X \circ i_f$ (see above) we see that $i_f^{-1} \pi_X^{-1} \mathcal{F} = f_{small}^{-1} \mathcal{F}$. This shows that π_X^{-1} has the description given in the lemma.

To prove that $\pi_X^{-1} \mathcal{F}$ is a sheaf for the ph topology it suffices by Topologies on Spaces, Lemma 8.7 to show that for a surjective proper morphism $V \to U$ of algebraic spaces over X we have $(\pi_X^{-1} \mathcal{F})(U)$ is the equalizer of the two maps $(\pi_X^{-1} \mathcal{F})(V) \to (\pi_X^{-1} \mathcal{F})(V \times_U V)$. This we have seen in Lemma 4.1.

The case of smooth, syntomic, fppf coverings follows from the case of ph coverings by Topologies on Spaces, Lemma 8.2.

Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be an fpqc covering of algebraic spaces over X. Let $s_i \in (\pi_X^{-1} \mathcal{F})(U_i)$ be sections which agree over $U_i \times_U U_j$. We have to prove there exists a unique $s \in (\pi_X^{-1} \mathcal{F})(U)$ restricting to s_i over U_i. Case I: U and U_i are schemes. This case follows from Étale Cohomology, Lemma 39.2. Case II: U is a scheme. Here we choose surjective étale morphisms $T_i \to U_i$ where T_i is a scheme. Then $\mathcal{T} = \{T_i \to U\}$ is an fpqc covering by schemes and by case I the result holds for $\mathcal{T}$. We omit the verification that this implies the result for $\mathcal{U}$. Case III: general case. Let $W \to U$ be a surjective étale morphism, where W is a scheme. Then $\mathcal{W} = \{U_i \times_U W \to W\}$ is an fpqc covering (by algebraic spaces) of the scheme W. By case II the result hold for $\mathcal{W}$. We omit the verification that this implies the result for $\mathcal{U}$. $\square$

0DG8 **Lemma 5.2.** *Let S be a scheme. Let $Y \to X$ be a morphism of $(Spaces/S)_{étale}$.*
  (1) *If $\mathcal{I}$ is injective in $Ab((Spaces/X)_{étale})$, then*
      (a) *$i_f^{-1} \mathcal{I}$ is injective in $Ab(Y_{étale})$,*
      (b) *$\mathcal{I}|_{X_{étale}}$ is injective in $Ab(X_{étale})$,*
  (2) *If $\mathcal{I}^\bullet$ is a K-injective complex in $Ab((Spaces/X)_{étale})$, then*
      (a) *$i_f^{-1} \mathcal{I}^\bullet$ is a K-injective complex in $Ab(Y_{étale})$,*

(b) $\mathcal{I}^\bullet|_{X_{\text{étale}}}$ *is a K-injective complex in* $Ab(X_{\text{étale}})$,

The corresponding statements for modules do not hold.

Proof. Parts (1)(b) and (2)(b) follow formally from the fact that the restriction functor $\pi_{X,*} = i_X^{-1}$ is a right adjoint of the exact functor π_X^{-1}, see Homology, Lemma 29.1 and Derived Categories, Lemma 31.9.

Parts (1)(a) and (2)(a) can be seen in two ways. First proof: We can use that i_f^{-1} is a right adjoint of the exact functor $i_{f,!}$. This functor is constructed in Topologies, Lemma 4.13 for sheaves of sets and for abelian sheaves in Modules on Sites, Lemma 16.2. It is shown in Modules on Sites, Lemma 16.3 that it is exact. Second proof. We can use that $i_f = i_Y \circ f_{big}$ as is shown in Topologies, Lemma 4.17. Since f_{big} is a localization, we see that pullback by it preserves injectives and K-injectives, see Cohomology on Sites, Lemmas 7.1 and 20.1. Then we apply the already proved parts (1)(b) and (2)(b) to the functor i_Y^{-1} to conclude.

To see a counter example for the case of modules we refer to Étale Cohomology, Lemma 99.1. $\square$

Let S be a scheme. Let $f : Y \to X$ be a morphism of algebraic spaces over S. The commutative diagram of Topologies on Spaces, Lemma 4.11 (3) leads to a commutative diagram of ringed sites

$$
\begin{array}{ccc}
(Y_{spaces,\text{étale}}, \mathcal{O}_Y) & \xleftarrow{\ \pi_Y\ } & ((Spaces/Y)_{\text{étale}}, \mathcal{O}) \\
{\scriptstyle f_{spaces,\text{étale}}} \downarrow & & \downarrow {\scriptstyle f_{big}} \\
(X_{spaces,\text{étale}}, \mathcal{O}_X) & \xleftarrow{\ \pi_X\ } & ((Spaces/X)_{\text{étale}}, \mathcal{O})
\end{array}
$$

as one easily sees by writing out the definitions of $f_{small}^\sharp$, $f_{big}^\sharp$, $\pi_X^\sharp$, and $\pi_Y^\sharp$. In particular this means that

0DG9   (5.2.1) $$(f_{big,*}\mathcal{F})|_{X_{\text{étale}}} = f_{small,*}(\mathcal{F}|_{Y_{\text{étale}}})$$

for any sheaf $\mathcal{F}$ on $(Spaces/Y)_{\text{étale}}$ and if $\mathcal{F}$ is a sheaf of $\mathcal{O}$-modules, then (5.2.1) is an isomorphism of $\mathcal{O}_X$-modules on $X_{\text{étale}}$.

0DGA   **Lemma 5.3.** *Let S be a scheme. Let $f : Y \to X$ be a morphism of algebraic spaces over S.*

     (1) *For K in $D((Spaces/Y)_{\text{étale}})$ we have $(Rf_{big,*}K)|_{X_{\text{étale}}} = Rf_{small,*}(K|_{Y_{\text{étale}}})$ in $D(X_{\text{étale}})$.*

     (2) *For K in $D((Spaces/Y)_{\text{étale}}, \mathcal{O})$ we have $(Rf_{big,*}K)|_{X_{\text{étale}}} = Rf_{small,*}(K|_{Y_{\text{étale}}})$ in $D(Mod(X_{\text{étale}}, \mathcal{O}_X))$.*

More generally, let $g : X' \to X$ be an object of $(Spaces/X)_{\text{étale}}$. Consider the fibre product

$$
\begin{array}{ccc}
Y' & \xrightarrow{\ g'\ } & Y \\
{\scriptstyle f'} \downarrow & & \downarrow {\scriptstyle f} \\
X' & \xrightarrow{\ g\ } & X
\end{array}
$$

Then

     (3) *For K in $D((Spaces/Y)_{\text{étale}})$ we have $i_g^{-1}(Rf_{big,*}K) = Rf'_{small,*}(i_{g'}^{-1}K)$ in $D(X'_{\text{étale}})$.*

(4) *For K in $D((Spaces/Y)_{étale}, \mathcal{O})$ we have $i_g^*(Rf_{big,*}K) = Rf'_{small,*}(i_{g'}^*K)$ in $D(Mod(X'_{étale}, \mathcal{O}_{X'}))$.*
(5) *For K in $D((Spaces/Y)_{étale})$ we have $g_{big}^{-1}(Rf_{big,*}K) = Rf'_{big,*}((g'_{big})^{-1}K)$ in $D((Spaces/X')_{étale})$.*
(6) *For K in $D((Spaces/Y)_{étale}, \mathcal{O})$ we have $g_{big}^*(Rf_{big,*}K) = Rf'_{big,*}((g'_{big})^*K)$ in $D(Mod(X'_{étale}, \mathcal{O}_{X'}))$.*

Proof. Part (1) follows from Lemma 5.2 and (5.2.1) on choosing a K-injective complex of abelian sheaves representing K.

Part (3) follows from Lemma 5.2 and Topologies, Lemma 4.19 on choosing a K-injective complex of abelian sheaves representing K.

Part (5) is Cohomology on Sites, Lemma 21.1.

Part (6) is Cohomology on Sites, Lemma 21.2.

Part (2) can be proved as follows. Above we have seen that $\pi_X \circ f_{big} = f_{small} \circ \pi_Y$ as morphisms of ringed sites. Hence we obtain $R\pi_{X,*} \circ Rf_{big,*} = Rf_{small,*} \circ R\pi_{Y,*}$ by Cohomology on Sites, Lemma 19.2. Since the restriction functors $\pi_{X,*}$ and $\pi_{Y,*}$ are exact, we conclude.

Part (4) follows from part (6) and part (2) applied to $f' : Y' \to X'$. $\square$

Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{H}$ be an abelian sheaf on $(Spaces/X)_{étale}$. Recall that $H_{étale}^n(U, \mathcal{H})$ denotes the cohomology of $\mathcal{H}$ over an object U of $(Spaces/X)_{étale}$.

0DGB **Lemma 5.4.** *Let S be a scheme. Let $f : Y \to X$ be a morphism of algebraic spaces over S. Then*
(1) *For K in $D(X_{étale})$ we have $H_{étale}^n(X, \pi_X^{-1}K) = H^n(X_{étale}, K)$.*
(2) *For K in $D(X_{étale}, \mathcal{O}_X)$ we have $H_{étale}^n(X, L\pi_X^*K) = H^n(X_{étale}, K)$.*
(3) *For K in $D(X_{étale})$ we have $H_{étale}^n(Y, \pi_X^{-1}K) = H^n(Y_{étale}, f_{small}^{-1}K)$.*
(4) *For K in $D(X_{étale}, \mathcal{O}_X)$ we have $H_{étale}^n(Y, L\pi_X^*K) = H^n(Y_{étale}, Lf_{small}^*K)$.*
(5) *For M in $D((Spaces/X)_{étale})$ we have $H_{étale}^n(Y, M) = H^n(Y_{étale}, i_f^{-1}M)$.*
(6) *For M in $D((Spaces/X)_{étale}, \mathcal{O})$ we have $H_{étale}^n(Y, M) = H^n(Y_{étale}, i_f^*M)$.*

Proof. To prove (5) represent M by a K-injective complex of abelian sheaves and apply Lemma 5.2 and work out the definitions. Part (3) follows from this as $i_f^{-1}\pi_X^{-1} = f_{small}^{-1}$. Part (1) is a special case of (3).

Part (6) follows from the very general Cohomology on Sites, Lemma 37.5. Then part (4) follows because $Lf_{small}^* = i_f^* \circ L\pi_X^*$. Part (2) is a special case of (4). $\square$

0DGC **Lemma 5.5.** *Let S be a scheme. Let X be an algebraic space over S. For $K \in D(X_{étale})$ the map*
$$K \longrightarrow R\pi_{X,*}\pi_X^{-1}K$$
is an isomorphism where $\pi_X : Sh((Spaces/X)_{étale}) \to Sh(X_{étale})$ is as above.

Proof. This is true because both π_X^{-1} and $\pi_{X,*} = i_X^{-1}$ are exact functors and the composition $\pi_{X,*} \circ \pi_X^{-1}$ is the identity functor. $\square$

0DGD **Lemma 5.6.** *Let S be a scheme. Let $f : Y \to X$ be a proper morphism of algebraic spaces over S. Then we have*
(1) *$\pi_X^{-1} \circ f_{small,*} = f_{big,*} \circ \pi_Y^{-1}$ as functors $Sh(Y_{étale}) \to Sh((Spaces/X)_{étale})$,*

(2) $\pi_X^{-1} R f_{small,*} K = R f_{big,*} \pi_Y^{-1} K$ for K in $D^+(Y_{étale})$ whose cohomology sheaves are torsion, and

(3) $\pi_X^{-1} R f_{small,*} K = R f_{big,*} \pi_Y^{-1} K$ for all K in $D(Y_{étale})$ if f is finite.

Proof. Proof of (1). Let $\mathcal{F}$ be a sheaf on $Y_{étale}$. Let $g : X' \to X$ be an object of $(Spaces/X)_{étale}$. Consider the fibre product

$$
\begin{array}{ccc}
Y' & \xrightarrow{\;f'\;} & X' \\
{\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\
Y & \xrightarrow{\;f\;} & X
\end{array}
$$

Then we have

$$(f_{big,*} \pi_Y^{-1} \mathcal{F})(X') = (\pi_Y^{-1} \mathcal{F})(Y') = ((g'_{small})^{-1} \mathcal{F})(Y') = (f'_{small,*} (g'_{small})^{-1} \mathcal{F})(X')$$

the second equality by Lemma 5.1. On the other hand

$$(\pi_X^{-1} f_{small,*} \mathcal{F})(X') = (g_{small}^{-1} f_{small,*} \mathcal{F})(X')$$

again by Lemma 5.1. Hence by proper base change for sheaves of sets (Lemma 4.4) we conclude the two sets are canonically isomorphic. The isomorphism is compatible with restriction mappings and defines an isomorphism $\pi_X^{-1} f_{small,*} \mathcal{F} = f_{big,*} \pi_Y^{-1} \mathcal{F}$. Thus an isomorphism of functors $\pi_X^{-1} \circ f_{small,*} = f_{big,*} \circ \pi_Y^{-1}$.

Proof of (2). There is a canonical base change map $\pi_X^{-1} R f_{small,*} K \to R f_{big,*} \pi_Y^{-1} K$ for any K in $D(Y_{étale})$, see Cohomology on Sites, Remark 19.3. To prove it is an isomorphism, it suffices to prove the pull back of the base change map by $i_g : Sh(X'_{étale}) \to Sh((Sch/X)_{étale})$ is an isomorphism for any object $g : X' \to X$ of $(Sch/X)_{étale}$. Let T', g', f' be as in the previous paragraph. The pullback of the base change map is

$$
\begin{aligned}
g_{small}^{-1} R f_{small,*} K &= i_g^{-1} \pi_X^{-1} R f_{small,*} K \\
&\to i_g^{-1} R f_{big,*} \pi_Y^{-1} K \\
&= R f'_{small,*} (i_{g'}^{-1} \pi_Y^{-1} K) \\
&= R f'_{small,*} ((g'_{small})^{-1} K)
\end{aligned}
$$

where we have used $\pi_X \circ i_g = g_{small}$, $\pi_Y \circ i_{g'} = g'_{small}$, and Lemma 5.3. This map is an isomorphism by the proper base change theorem (Lemma 4.7) provided K is bounded below and the cohomology sheaves of K are torsion.

Proof of (3). If f is finite, then the functors $f_{small,*}$ and $f_{big,*}$ are exact. This follows from Cohomology of Spaces, Lemma 4.1 for f_{small}. Since any base change f' of f is finite too, we conclude from Lemma 5.3 part (3) that $f_{big,*}$ is exact too (as the higher derived functors are zero). Thus this case follows from part (1). $\square$

6. Comparing fppf and étale topologies

0DGE This section is the analogue of Étale Cohomology, Section 100.

Let S be a scheme. Let X be an algebraic space over S. On the category $Spaces/X$ we consider the fppf and étale topologies. The identity functor $(Spaces/X)_{étale} \to (Spaces/X)_{fppf}$ is continuous and defines a morphism of sites

$$\epsilon_X : (Spaces/X)_{fppf} \longrightarrow (Spaces/X)_{étale}$$

by an application of Sites, Proposition 14.7. Please note that $\epsilon_{X,*}$ is the identity functor on underlying presheaves and that ϵ_X^{-1} associates to an étale sheaf the fppf sheafification. Consider the morphism of sites

$$\pi_X : (Spaces/X)_{\text{étale}} \longrightarrow X_{spaces,\text{étale}}$$

comparing big and small étale sites, see Section 5. The composition determines a morphism of sites

$$a_X = \pi_X \circ \epsilon_X : (Spaces/X)_{fppf} \longrightarrow X_{spaces,\text{étale}}$$

If $\mathcal{H}$ is an abelian sheaf on $(Spaces/X)_{fppf}$, then we will write $H^n_{fppf}(U, \mathcal{H})$ for the cohomology of $\mathcal{H}$ over an object U of $(Spaces/X)_{fppf}$.

0DGF **Lemma 6.1.** *Let S be a scheme. Let X be an algebraic space over S.*

(1) *For $\mathcal{F} \in Sh(X_{\text{étale}})$ we have $\epsilon_{X,*}a_X^{-1}\mathcal{F} = \pi_X^{-1}\mathcal{F}$ and $a_{X,*}a_X^{-1}\mathcal{F} = \mathcal{F}$.*
(2) *For $\mathcal{F} \in Ab(X_{\text{étale}})$ we have $R^i\epsilon_{X,*}(a_X^{-1}\mathcal{F}) = 0$ for $i > 0$.*

Proof. We have $a_X^{-1}\mathcal{F} = \epsilon_X^{-1}\pi_X^{-1}\mathcal{F}$. By Lemma 5.1 the étale sheaf $\pi_X^{-1}\mathcal{F}$ is a sheaf for the fppf topology and therefore is equal to $a_X^{-1}\mathcal{F}$ (as pulling back by ϵ_X is given by fppf sheafification). Recall moreover that $\epsilon_{X,*}$ is the identity on underlying presheaves. Now part (1) is immediate from the explicit description of π_X^{-1} in Lemma 5.1.

We will prove part (2) by reducing it to the case of schemes – see part (1) of Étale Cohomology, Lemma 100.6. This will "clearly work" as every algebraic space is étale locally a scheme. The details are given below but we urge the reader to skip the proof.

For an abelian sheaf $\mathcal{H}$ on $(Spaces/X)_{fppf}$ the higher direct image $R^p\epsilon_{X,*}\mathcal{H}$ is the sheaf associated to the presheaf $U \mapsto H^p_{fppf}(U, \mathcal{H})$ on $(Spaces/X)_{\text{étale}}$. See Cohomology on Sites, Lemma 7.4. Since every object of $(Spaces/X)_{\text{étale}}$ has a covering by schemes, it suffices to prove that given U/X a scheme and $\xi \in H^p_{fppf}(U, a_X^{-1}\mathcal{F})$ we can find an étale covering $\{U_i \to U\}$ such that ξ restricts to zero on U_i. We have

$$\begin{aligned} H^p_{fppf}(U, a_X^{-1}\mathcal{F}) &= H^p((Spaces/U)_{fppf}, (a_X^{-1}\mathcal{F})|_{Spaces/U}) \\ &= H^p((Sch/U)_{fppf}, (a_X^{-1}\mathcal{F})|_{Sch/U}) \end{aligned}$$

where the second identification is Lemma 3.1 and the first is a general fact about restriction (Cohomology on Sites, Lemma 7.1). Looking at the first paragraph and the corresponding result in the case of schemes (Étale Cohomology, Lemma 100.1) we conclude that the sheaf $(a_X^{-1}\mathcal{F})|_{Sch/U}$ matches the pullback by the "schemes version of a_U". Therefore we can find an étale covering $\{U_i \to U\}$ such that our class dies in $H^p((Sch/U_i)_{fppf}, (a_X^{-1}\mathcal{F})|_{Sch/U_i})$ for each i, see Étale Cohomology, Lemma 100.6 (the precise statement one should use here is that V_n holds for all n which is the statement of part (2) for the case of schemes). Transporting back (using the same formulas as above but now for U_i) we conclude ξ restricts to zero over U_i as desired. $\square$

The hard work done in the case of schemes now tells us that étale and fppf cohomology agree for sheaves coming from the small étale site.

0DGG **Lemma 6.2.** *Let S be a scheme. Let X be an algebraic space over S. For $K \in D^+(X_{\acute{e}tale})$ the maps*

$$\pi_X^{-1}K \longrightarrow R\epsilon_{X,*}a_X^{-1}K \quad and \quad K \longrightarrow Ra_{X,*}a_X^{-1}K$$

are isomorphisms with $a_X : Sh((Spaces/X)_{fppf}) \to Sh(X_{\acute{e}tale})$ as above.

Proof. We only prove the second statement; the first is easier and proved in exactly the same manner. There is an immediate reduction to the case where K is given by a single abelian sheaf. Namely, represent K by a bounded below complex $\mathcal{F}^\bullet$. By the case of a sheaf we see that $\mathcal{F}^n = a_{X,*}a_X^{-1}\mathcal{F}^n$ and that the sheaves $R^q a_{X,*}a_X^{-1}\mathcal{F}^n$ are zero for $q > 0$. By Leray's acyclicity lemma (Derived Categories, Lemma 16.7) applied to $a_X^{-1}\mathcal{F}^\bullet$ and the functor $a_{X,*}$ we conclude. From now on assume $K = \mathcal{F}$.

By Lemma 6.1 we have $a_{X,*}a_X^{-1}\mathcal{F} = \mathcal{F}$. Thus it suffices to show that $R^q a_{X,*}a_X^{-1}\mathcal{F} = 0$ for $q > 0$. For this we can use $a_X = \epsilon_X \circ \pi_X$ and the Leray spectral sequence (Cohomology on Sites, Lemma 14.7). By Lemma 6.1 we have $R^i\epsilon_{X,*}(a_X^{-1}\mathcal{F}) = 0$ for $i > 0$. We have $\epsilon_{X,*}a_X^{-1}\mathcal{F} = \pi_X^{-1}\mathcal{F}$ and by Lemma 5.5 we have $R^j\pi_{X,*}(\pi_X^{-1}\mathcal{F}) = 0$ for $j > 0$. This concludes the proof. $\square$

0DGH **Lemma 6.3.** *Let S be a scheme and let X be an algebraic space over S. With $a_X : Sh((Spaces/X)_{fppf}) \to Sh(X_{\acute{e}tale})$ as above:*

    (1) $H^q(X_{\acute{e}tale}, \mathcal{F}) = H^q_{fppf}(X, a_X^{-1}\mathcal{F})$ *for an abelian sheaf $\mathcal{F}$ on $X_{\acute{e}tale}$,*

    (2) $H^q(X_{\acute{e}tale}, K) = H^q_{fppf}(X, a_X^{-1}K)$ *for $K \in D^+(X_{\acute{e}tale})$.*

Example: if A is an abelian group, then $H^q_{\acute{e}tale}(X, \underline{A}) = H^q_{fppf}(X, \underline{A})$.

Proof. This follows from Lemma 6.2 by Cohomology on Sites, Remark 14.4. $\square$

0DGI **Lemma 6.4.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Then there are commutative diagrams of topoi*

$$
\begin{array}{ccc}
Sh((Spaces/X)_{fppf}) & \xrightarrow{\ f_{big,fppf}\ } & Sh((Spaces/Y)_{fppf}) \\
{\scriptstyle \epsilon_X} \downarrow & & \downarrow {\scriptstyle \epsilon_Y} \\
Sh((Spaces/X)_{\acute{e}tale}) & \xrightarrow{\ f_{big,\acute{e}tale}\ } & Sh((Spaces/Y)_{\acute{e}tale})
\end{array}
$$

and

$$
\begin{array}{ccc}
Sh((Spaces/X)_{fppf}) & \xrightarrow{\ f_{big,fppf}\ } & Sh((Spaces/Y)_{fppf}) \\
{\scriptstyle a_X} \downarrow & & \downarrow {\scriptstyle a_Y} \\
Sh(X_{\acute{e}tale}) & \xrightarrow{\ f_{small}\ } & Sh(Y_{\acute{e}tale})
\end{array}
$$

with $a_X = \pi_X \circ \epsilon_X$ and $a_Y = \pi_X \circ \epsilon_X$.

Proof. This follows immediately from working out the definitions of the morphisms involved, see Topologies on Spaces, Section 7 and Section 5. $\square$

0DGJ **Lemma 6.5.** *In Lemma 6.4 if f is proper, then we have*

    (1) $a_Y^{-1} \circ f_{small,*} = f_{big,fppf,*} \circ a_X^{-1}$, *and*

    (2) $a_Y^{-1}(Rf_{small,*}K) = Rf_{big,fppf,*}(a_X^{-1}K)$ *for K in $D^+(X_{\acute{e}tale})$ with torsion cohomology sheaves.*

Proof. Proof of (1). You can prove this by repeating the proof of Lemma 5.6 part (1); we will instead deduce the result from this. As $\epsilon_{Y,*}$ is the identity functor on underlying presheaves, it reflects isomorphisms. Lemma 6.1 shows that $\epsilon_{Y,*} \circ a_Y^{-1} = \pi_Y^{-1}$ and similarly for X. To show that the canonical map $a_Y^{-1} f_{small,*} \mathcal{F} \to f_{big,fppf,*} a_X^{-1} \mathcal{F}$ is an isomorphism, it suffices to show that

$$\pi_Y^{-1} f_{small,*} \mathcal{F} = \epsilon_{Y,*} a_Y^{-1} f_{small,*} \mathcal{F}$$
$$\to \epsilon_{Y,*} f_{big,fppf,*} a_X^{-1} \mathcal{F}$$
$$= f_{big,\acute{e}tale,*} \epsilon_{X,*} a_X^{-1} \mathcal{F}$$
$$= f_{big,\acute{e}tale,*} \pi_X^{-1} \mathcal{F}$$

is an isomorphism. This is part (1) of Lemma 5.6.

To see (2) we use that

$$R\epsilon_{Y,*} R f_{big,fppf,*} a_X^{-1} K = R f_{big,\acute{e}tale,*} R\epsilon_{X,*} a_X^{-1} K$$
$$= R f_{big,\acute{e}tale,*} \pi_X^{-1} K$$
$$= \pi_Y^{-1} R f_{small,*} K$$
$$= R\epsilon_{Y,*} a_Y^{-1} R f_{small,*} K$$

The first equality by the commutative diagram in Lemma 6.4 and Cohomology on Sites, Lemma 19.2. Then second equality is Lemma 6.2. The third is Lemma 5.6 part (2). The fourth is Lemma 6.2 again. Thus the base change map $a_Y^{-1}(R f_{small,*} K) \to R f_{big,fppf,*}(a_X^{-1} K)$ induces an isomorphism

$$R\epsilon_{Y,*} a_Y^{-1} R f_{small,*} K \to R\epsilon_{Y,*} R f_{big,fppf,*} a_X^{-1} K$$

The proof is finished by the following remark: a map $\alpha : a_Y^{-1} L \to M$ with L in $D^+(Y_{\acute{e}tale})$ and M in $D^+((Spaces/Y)_{fppf})$ such that $R\epsilon_{Y,*}\alpha$ is an isomorphism, is an isomorphism. Namely, we show by induction on i that $H^i(\alpha)$ is an isomorphism. This is true for all sufficiently small i. If it holds for $i \leq i_0$, then we see that $R^j \epsilon_{Y,*} H^i(M) = 0$ for $j > 0$ and $i \leq i_0$ by Lemma 6.1 because $H^i(M) = a_Y^{-1} H^i(L)$ in this range. Hence $\epsilon_{Y,*} H^{i_0+1}(M) = H^{i_0+1}(R\epsilon_{Y,*} M)$ by a spectral sequence argument. Thus $\epsilon_{Y,*} H^{i_0+1}(M) = \pi_Y^{-1} H^{i_0+1}(L) = \epsilon_{Y,*} a_Y^{-1} H^{i_0+1}(L)$. This implies $H^{i_0+1}(\alpha)$ is an isomorphism (because $\epsilon_{Y,*}$ reflects isomorphisms as it is the identity on underlying presheaves) as desired. $\square$

0DGK **Lemma 6.6.** *In Lemma 6.4 if f is finite, then $a_Y^{-1}(R f_{small,*} K) = R f_{big,fppf,*}(a_X^{-1} K)$ for K in $D^+(X_{\acute{e}tale})$.*

Proof. Let $V \to Y$ be a surjective étale morphism where V is a scheme. It suffices to prove the base change map is an isomorphism after restricting to V. Hence we may assume that Y is a scheme. As the morphism is finite, hence representable, we conclude that we may assume both X and Y are schemes. In this case the result follows from the case of schemes (Étale Cohomology, Lemma 100.6 part (2)) using the comparison of topoi discussed in Section 3 and in particular given in Lemma 3.1. Some details omitted. $\square$

0DGL **Lemma 6.7.** *In Lemma 6.4 assume f is flat, locally of finite presentation, and surjective. Then the functor*

$$Sh(Y_{\acute{e}tale}) \longrightarrow \left\{ (\mathcal{G}, \mathcal{H}, \alpha) \middle| \begin{array}{l} \mathcal{G} \in Sh(X_{\acute{e}tale}),\ \mathcal{H} \in Sh((Sch/Y)_{fppf}), \\ \alpha : a_X^{-1}\mathcal{G} \to f_{big,fppf}^{-1}\mathcal{H}\ an\ isomorphism \end{array} \right\}$$

sending $\mathcal{F}$ to $(f_{small}^{-1}\mathcal{F}, a_Y^{-1}\mathcal{F}, can)$ is an equivalence.

Proof. The functor a_X^{-1} is fully faithful (as $a_{X,*}a_X^{-1} = \mathrm{id}$ by Lemma 6.1). Hence the forgetful functor $(\mathcal{G}, \mathcal{H}, \alpha) \mapsto \mathcal{H}$ identifies the category of triples with a full subcategory of $Sh((Sch/Y)_{fppf})$. Moreover, the functor a_Y^{-1} is fully faithful, hence the functor in the lemma is fully faithful as well.

Suppose that we have an étale covering $\{Y_i \to Y\}$. Let $f_i : X_i \to Y_i$ be the base change of f. Denote $f_{ij} = f_i \times f_j : X_i \times_X X_j \to Y_i \times_Y Y_j$. Claim: if the lemma is true for f_i and f_{ij} for all i, j, then the lemma is true for f. To see this, note that the given étale covering determines an étale covering of the final object in each of the four sites $Y_{\acute{e}tale}, X_{\acute{e}tale}, (Sch/Y)_{fppf}, (Sch/X)_{fppf}$. Thus the category of sheaves is equivalent to the category of glueing data for this covering (Sites, Lemma 26.5) in each of the four cases. A huge commutative diagram of categories then finishes the proof of the claim. We omit the details. The claim shows that we may work étale locally on Y. In particular, we may assume Y is a scheme.

Assume Y is a scheme. Choose a scheme X' and a surjective étale morphism $s : X' \to X$. Set $f' = f \circ s : X' \to Y$ and observe that f' is surjective, locally of finite presentation, and flat. Claim: if the lemma is true for f', then it is true for f. Namely, given a triple $(\mathcal{G}, \mathcal{H}, \alpha)$ for f, we can pullback by s to get a triple $(s_{small}^{-1}\mathcal{G}, \mathcal{H}, s_{big,fppf}^{-1}\alpha)$ for f'. A solution for this triple gives a sheaf $\mathcal{F}$ on $Y_{\acute{e}tale}$ with $a_Y^{-1}\mathcal{F} = \mathcal{H}$. By the first paragraph of the proof this means the triple is in the essential image. This reduces us to the case where both X and Y are schemes. This case follows from Étale Cohomology, Lemma 100.4 via the discussion in Section 3 and in particular Lemma 3.1. $\square$

7. Comparing fppf and étale topologies: modules

0DGM We continue the discussion in Section 6 but in this section we briefly discuss what happens for sheaves of modules.

Let S be a scheme. Let X be an algebraic space over S. The morphisms of sites ϵ_X, π_X, and their composition a_X introduced in Section 6 have natural enhancements to morphisms of ringed sites. The first is written as

$$\epsilon_X : ((Spaces/X)_{fppf}, \mathcal{O}) \longrightarrow ((Spaces/X)_{\acute{e}tale}, \mathcal{O})$$

Note that we can use the same symbol for the structure sheaf as indeed the sheaves have the same underlying presheaf. The second is

$$\pi_X : ((Spaces/X)_{\acute{e}tale}, \mathcal{O}) \longrightarrow (X_{\acute{e}tale}, \mathcal{O}_X)$$

The third is the morphism

$$a_X : ((Spaces/X)_{fppf}, \mathcal{O}) \longrightarrow (X_{\acute{e}tale}, \mathcal{O}_X)$$

Let us review what we already know about quasi-coherent modules on these sites.

0DGN **Lemma 7.1.** *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module.*

(1) *The rule*

$$\mathcal{F}^a : (Spaces/X)_{\acute{e}tale} \longrightarrow Ab, \quad (f : Y \to X) \longmapsto \Gamma(Y, f^*\mathcal{F})$$

   satisfies the sheaf condition for fpqc and a fortiori fppf and étale coverings,
(2) $\mathcal{F}^a = \pi_X^*\mathcal{F}$ *on* $(Spaces/X)_{\acute{e}tale}$,
(3) $\mathcal{F}^a = a_X^*\mathcal{F}$ *on* $(Spaces/X)_{fppf}$,
(4) *the rule* $\mathcal{F} \mapsto \mathcal{F}^a$ *defines an equivalence between quasi-coherent* $\mathcal{O}_X$-modules *and quasi-coherent modules on* $((Spaces/X)_{\acute{e}tale}, \mathcal{O})$,
(5) *the rule* $\mathcal{F} \mapsto \mathcal{F}^a$ *defines an equivalence between quasi-coherent* $\mathcal{O}_X$-modules *and quasi-coherent modules on* $((Spaces/X)_{fppf}, \mathcal{O})$,
(6) *we have* $\epsilon_{X,*}a_X^*\mathcal{F} = \pi_X^*\mathcal{F}$ *and* $a_{X,*}a_X^*\mathcal{F} = \mathcal{F}$,
(7) *we have* $R^i\epsilon_{X,*}(a_X^*\mathcal{F}) = 0$ *and* $R^i a_{X,*}(a_X^*\mathcal{F}) = 0$ *for* $i > 0$.

Proof. Part (1) is a consequence of fppf descent of quasi-coherent modules. Namely, suppose that $\{f_i : U_i \to U\}$ is an fpqc covering in $(Spaces/X)_{\acute{e}tale}$. Denote $g : U \to X$ the structure morphism. Suppose that we have a family of sections $s_i \in \Gamma(U_i, f_i^*g^*\mathcal{F})$ such that $s_i|_{U_i \times_U U_j} = s_j|_{U_i \times_U U_j}$. We have to find the correspond section $s \in \Gamma(U, g^*\mathcal{F})$. We can reinterpret the s_i as a family of maps $\varphi_i : f_i^*\mathcal{O}_U = \mathcal{O}_{U_i} \to f_i^*g^*\mathcal{F}$ compatible with the canonical descent data associated to the quasi-coherent sheaves $\mathcal{O}_U$ and $g^*\mathcal{F}$ on U. Hence by Descent on Spaces, Proposition 4.1 we see that we may (uniquely) descend these to a map $\mathcal{O}_U \to g^*\mathcal{F}$ which gives us our section s.

We will deduce (2) – (7) from the corresponding statement for schemes. Choose an étale covering $\{X_i \to X\}_{i \in I}$ where each X_i is a scheme. Observe that $X_i \times_X X_j$ is a scheme too. This covering induces a covering of the final object in each of the three sites $(Spaces/X)_{fppf}$, $(Spaces/X)_{\acute{e}tale}$, and $X_{\acute{e}tale}$. Hence we see that the category of sheaves on these sites are equivalent to descent data for these coverings, see Sites, Lemma 26.5. Parts (2), (3) are local (because we have the glueing statement). Being quasi-coherent is a local property, hence parts (4), (5) are local. Clearly (6) and (7) are local. It follows that it suffices to prove parts (2) – (7) of the lemma when X is a scheme.

Assume X is a scheme. The embeddings $(Sch/X)_{\acute{e}tale} \subset (Spaces/X)_{\acute{e}tale}$ and $(Sch/X)_{fppf} \subset (Spaces/X)_{fppf}$ determine equivalences of ringed topoi by Lemma 3.1. We conclude that (2) – (7) follows from the case of schemes. Étale Cohomology, Lemma 101.1. To transport the property of being quasi-coherent via this equivalence use that being quasi-coherent is an intrinsic property of modules as explained in Modules on Sites, Section 23. Some minor details omitted. $\square$

0DGP **Lemma 7.2.** *Let S be a scheme. Let X be an algebraic space over S. For $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module the maps*

$$\pi_X^*\mathcal{F} \longrightarrow R\epsilon_{X,*}(a_X^*\mathcal{F}) \quad and \quad \mathcal{F} \longrightarrow Ra_{X,*}(a_X^*\mathcal{F})$$

are isomorphisms.

Proof. This is an immediate consequence of parts (6) and (7) of Lemma 7.1. $\square$

0DGQ **Lemma 7.3.** *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3$ be a complex of quasi-coherent $\mathcal{O}_X$-modules. Set*

$$\mathcal{H}_{\acute{e}tale} = \operatorname{Ker}(\pi_X^*\mathcal{F}_2 \to \pi_X^*\mathcal{F}_3)/\operatorname{Im}(\pi_X^*\mathcal{F}_1 \to \pi_X^*\mathcal{F}_2)$$

on $(Spaces/X)_{étale}$ *and set*

$$\mathcal{H}_{fppf} = \text{Ker}(a_X^* \mathcal{F}_2 \to a_X^* \mathcal{F}_3)/\text{Im}(a_X^* \mathcal{F}_1 \to a_X^* \mathcal{F}_2)$$

on $(Spaces/X)_{fppf}$. *Then* $\mathcal{H}_{étale} = \epsilon_{X,*}\mathcal{H}_{fppf}$ *and*

$$H^p_{étale}(U, \mathcal{H}_{étale}) = H^p_{fppf}(U, \mathcal{H}_{fppf}) = 0$$

for $p > 0$ *and any affine object* U *of* $(Spaces/X)_{étale}$.

More is true, namely the collection of modules on $(Spaces/X)_{fppf}$ which fppf locally look like those in the lemma are called adquate modules. They form a weak Serre subcategory of the category of all $\mathcal{O}$-modules and their cohomology is studied in Adequate Modules, Section 5.

Proof. For any object $f : U \to X$ of $(Spaces/X)_{étale}$ consider the restriction $\mathcal{H}_{étale}|_{U_{étale}}$ of $\mathcal{H}_{étale}$ to $U_{étale}$ via the functor $i_f^* = i_f^{-1}$ discussed in Section 5. The sheaf $\mathcal{H}_{étale}|_{U_{étale}}$ is equal to the homology of complex $f^*\mathcal{F}_\bullet$ in degree 1. This is true because $i_f \circ \pi_X = f$ as morphisms of ringed sites $U_{étale} \to X_{étale}$. In particular we see that $\mathcal{H}_{étale}|_{U_{étale}}$ is a quasi-coherent $\mathcal{O}_U$-module. Next, let $g : V \to U$ be a flat morphism in $(Spaces/X)_{étale}$. Since

$$i_{f \circ g}^* \circ \pi_X^* = (f \circ g)^* = g^* \circ f^*$$

as morphisms of sites $V_{étale} \to X_{étale}$ and since g is flat hence g^* is exact, we obtain

$$\mathcal{H}_{étale}|_{V_{étale}} = g^* \left(\mathcal{H}_{étale}|_{U_{étale}} \right)$$

With these preparations we are ready to prove the lemma.

Let $\mathcal{U} = \{g_i : U_i \to U\}_{i \in I}$ be an fppf covering with $f : U \to X$ as above. The sheaf propery holds for $\mathcal{H}_{étale}$ and the covering $\mathcal{U}$ by (1) of Lemma 7.1 applied to $\mathcal{H}_{étale}|_{U_{étale}}$ and the above. Therefore we see that $\mathcal{H}_{étale}$ is already an fppf sheaf and this means that $\mathcal{H}_{fppf}$ is equal to $\mathcal{H}_{étale}$ as a presheaf. In particular $\mathcal{H}_{étale} = \epsilon_{X,*}\mathcal{H}_{fppf}$.

Finally, to prove the vanishing, we use Cohomology on Sites, Lemma 10.9. We let $\mathcal{B}$ be the affine objects of $(Spaces/X)_{fppf}$ and we let Cov be the set of finite fppf coverings $\mathcal{U} = \{U_i \to U\}_{i=1,\dots,n}$ with U, U_i affine. We have

$$\check{H}^p(\mathcal{U}, \mathcal{H}_{étale}) = \check{H}^p(\mathcal{U}, (\mathcal{H}_{étale}|_{U_{étale}})^a)$$

because the values of $\mathcal{H}_{étale}$ on the affine schemes $U_{i_0} \times_U \dots \times_U U_{i_p}$ flat over U agree with the values of the pullback of the quasi-coherent module $\mathcal{H}_{étale}|_{U_{étale}}$ by the first paragraph. Hence we obtain vanishing by Descent, Lemma 9.2. This finishes the proof. $\square$

0DGR   **Lemma 7.4.** *Let* S *be a scheme. Let* X *be an algebraic space over* S. *For* $K \in D_{QCoh}(\mathcal{O}_X)$ *the maps*

$$L\pi_X^* K \longrightarrow R\epsilon_{X,*}(La_X^* K) \quad and \quad K \longrightarrow Ra_{X,*}(La_X^* K)$$

are isomorphisms. Here $a_X : Sh((Spaces/X)_{fppf}) \to Sh(X_{étale})$ *is as above.*

Proof. The question is étale local on X hence we may assume X is affine. Say $X = \text{Spec}(A)$. Then we have $D_{QCoh}(\mathcal{O}_X) = D(A)$ by Derived Categories of Spaces, Lemma 4.2 and Derived Categories of Schemes, Lemma 3.5. Hence we can choose an K-flat complex of A-modules $K^\bullet$ whose corresponding complex $\mathcal{K}^\bullet$ of quasi-coherent $\mathcal{O}_X$-modules represents K. We claim that $\mathcal{K}^\bullet$ is a K-flat complex of $\mathcal{O}_X$-modules.

Proof of the claim. By Derived Categories of Schemes, Lemma 3.6 we see that $\widetilde{K}^\bullet$ is K-flat on the scheme $(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)})$. Next, note that $\mathcal{K}^\bullet = \epsilon^* \widetilde{K}^\bullet$ where ϵ is as in Derived Categories of Spaces, Lemma 4.2 whence $\mathcal{K}^\bullet$ is K-flat by Cohomology on Sites, Lemma 18.7 and the fact that the étale site of a scheme has enough points (Étale Cohomology, Remarks 29.11).

By the claim we see that $La_X^* K = a_X^* \mathcal{K}^\bullet$ and $L\pi_X^* K = \pi_X^* \mathcal{K}^\bullet$. Since the first part of the proof shows that the pullback $a_X^* \mathcal{K}^n$ of the quasi-coherent module is acyclic for $\epsilon_{X,*}$, resp. $a_{X,*}$, surely the proof is done by Leray's acyclicity lemma? Actually..., no because Leray's acyclicity lemma only applies to bounded below complexes. However, in the next paragraph we will show the result does follow from the bounded below case because our complex is the derived limit of bounded below complexes of quasi-coherent modules.

The cohomology sheaves of $\pi_X^* \mathcal{K}^\bullet$ and $a_X^* \mathcal{K}^\bullet$ have vanishing higher cohomology groups over affine objects of $(\mathit{Spaces}/X)_{\text{étale}}$ by Lemma 7.3. Therefore we have

$$L\pi_X^* K = R\lim \tau_{\geq -n}(L\pi_X^* K) \quad \text{and} \quad La_X^* K = R\lim \tau_{\geq -n}(La_X^* K)$$

by Cohomology on Sites, Lemma 23.10.

Proof of $L\pi_X^* K = R\epsilon_{X,*}(La_X^* \mathcal{F})$. By the above we have

$$R\epsilon_{X,*} La_X^* K = R\lim R\epsilon_{X,*}(\tau_{\geq -n}(La_X^* K))$$

by Cohomology on Sites, Lemma 23.3. Note that $\tau_{\geq -n}(La_X^* K)$ is represented by $\tau_{\geq -n}(a_X^* \mathcal{K}^\bullet)$ which may not be the same as $a_X^*(\tau_{\geq -n} \mathcal{K}^\bullet)$. But clearly the systems

$$\{\tau_{\geq -n}(a_X^* \mathcal{K}^\bullet)\}_{n \geq 1} \quad \text{and} \quad \{a_X^*(\tau_{\geq -n} \mathcal{K}^\bullet)\}_{n \geq 1}$$

are isomorphic as pro-systems. By Leray's acyclicity lemma (Derived Categories, Lemma 16.7) and the first part of the lemma we see that

$$R\epsilon_{X,*}(a_X^*(\tau_{\geq -n} \mathcal{K}^\bullet)) = \pi_X^*(\tau_{\geq -n} \mathcal{K}^\bullet)$$

Then we can use that the systems

$$\{\tau_{\geq -n}(\pi_X^* \mathcal{K}^\bullet)\}_{n \geq 1} \quad \text{and} \quad \{\pi_X^*(\tau_{\geq -n} \mathcal{K}^\bullet)\}_{n \geq 1}$$

are isomorphic as pro-systems. Finally, we put everything together as follows

$$\begin{aligned}
R\epsilon_{X,*} La_X^* K &= R\epsilon_{X,*}(R\lim \tau_{\geq -n}(La_X^* K)) \\
&= R\lim R\epsilon_{X,*}(\tau_{\geq -n}(La_X^* K)) \\
&= R\lim R\epsilon_{X,*}(\tau_{\geq -n}(a_X^* \mathcal{K}^\bullet)) \\
&= R\lim R\epsilon_{X,*}(a_X^*(\tau_{\geq -n} \mathcal{K}^\bullet)) \\
&= R\lim \pi_X^*(\tau_{\geq -n} \mathcal{K}^\bullet) \\
&= R\lim \tau_{\geq -n}(\pi_X^* \mathcal{K}^\bullet) \\
&= R\lim \tau_{\geq -n}(L\pi_X^* K) \\
&= L\pi_X^* K
\end{aligned}$$

Here in equalities four and six we have used that isomorphic pro-systems have the same $R\lim$ (small detail omitted). You can avoid this step by using more about cohomology of the terms of the complex $\tau_{\geq -n} a_X^* \mathcal{K}^\bullet$ proved in Lemma 7.3 as this will prove directly that $R\epsilon_{X,*}(\tau_{\geq -n}(a_X^* \mathcal{K}^\bullet)) = \tau_{\geq -n}(\pi_X^* \mathcal{K}^\bullet)$.

The equality $K = Ra_{X,*}(La_X^* \mathcal{F})$ is proved in exactly the same way using in the final step that $K = R\lim \tau_{\geq -n} K$ by Derived Categories of Spaces, Lemma 5.7. $\square$

8. Comparing ph and étale topologies

0DGS This section is the analogue of Étale Cohomology, Section 102.

Let S be a scheme. Let X be an algebraic space over S. On the category $Spaces/X$ we consider the ph and étale topologies. The identity functor $(Spaces/X)_{étale} \to (Spaces/X)_{ph}$ is continuous as every étale covering is a ph covering by Topologies on Spaces, Lemma 8.2. Hence it defines a morphism of sites

$$\epsilon_X : (Spaces/X)_{ph} \longrightarrow (Spaces/X)_{étale}$$

by an application of Sites, Proposition 14.7. Please note that $\epsilon_{X,*}$ is the identity functor on underlying presheaves and that ϵ_X^{-1} associates to an étale sheaf the ph sheafification. Consider the morphism of sites

$$\pi_X : (Spaces/X)_{étale} \longrightarrow X_{spaces,étale}$$

comparing big and small étale sites, see Section 5. The composition determines a morphism of sites

$$a_X = \pi_X \circ \epsilon_X : (Spaces/X)_{ph} \longrightarrow X_{spaces,étale}$$

If $\mathcal{H}$ is an abelian sheaf on $(Spaces/X)_{ph}$, then we will write $H_{ph}^n(U, \mathcal{H})$ for the cohomology of $\mathcal{H}$ over an object U of $(Spaces/X)_{ph}$.

0DGT **Lemma 8.1.** *Let S be a scheme. Let X be an algebraic space over S.*

(1) *For $\mathcal{F} \in Sh(X_{étale})$ we have $\epsilon_{X,*} a_X^{-1} \mathcal{F} = \pi_X^{-1} \mathcal{F}$ and $a_{X,*} a_X^{-1} \mathcal{F} = \mathcal{F}$.*

(2) *For $\mathcal{F} \in Ab(X_{étale})$ torsion we have $R^i \epsilon_{X,*} (a_X^{-1} \mathcal{F}) = 0$ for $i > 0$.*

Proof. We have $a_X^{-1} \mathcal{F} = \epsilon_X^{-1} \pi_X^{-1} \mathcal{F}$. By Lemma 5.1 the étale sheaf $\pi_X^{-1} \mathcal{F}$ is a sheaf for the ph topology and therefore is equal to $a_X^{-1} \mathcal{F}$ (as pulling back by ϵ_X is given by ph sheafification). Recall moreover that $\epsilon_{X,*}$ is the identity on underlying presheaves. Now part (1) is immediate from the explicit description of π_X^{-1} in Lemma 5.1.

We will prove part (2) by reducing it to the case of schemes – see part (1) of Étale Cohomology, Lemma 102.5. This will "clearly work" as every algebraic space is étale locally a scheme. The details are given below but we urge the reader to skip the proof.

For an abelian sheaf $\mathcal{H}$ on $(Spaces/X)_{ph}$ the higher direct image $R^p \epsilon_{X,*} \mathcal{H}$ is the sheaf associated to the presheaf $U \mapsto H_{ph}^p(U, \mathcal{H})$ on $(Spaces/X)_{étale}$. See Cohomology on Sites, Lemma 7.4. Since every object of $(Spaces/X)_{étale}$ has a covering by schemes, it suffices to prove that given U/X a scheme and $\xi \in H_{ph}^p(U, a_X^{-1} \mathcal{F})$ we can find an étale covering $\{U_i \to U\}$ such that ξ restricts to zero on U_i. We have

$$H_{ph}^p(U, a_X^{-1} \mathcal{F}) = H^p((Spaces/U)_{ph}, (a_X^{-1} \mathcal{F})|_{Spaces/U})$$
$$= H^p((Sch/U)_{ph}, (a_X^{-1} \mathcal{F})|_{Sch/U})$$

where the second identification is Lemma 3.1 and the first is a general fact about restriction (Cohomology on Sites, Lemma 7.1). Looking at the first paragraph and the corresponding result in the case of schemes (Étale Cohomology, Lemma 102.1) we conclude that the sheaf $(a_X^{-1} \mathcal{F})|_{Sch/U}$ matches the pullback by the "schemes version of a_U". Therefore we can find an étale covering $\{U_i \to U\}$ such that our class dies in $H^p((Sch/U_i)_{ph}, (a_X^{-1} \mathcal{F})|_{Sch/U_i})$ for each i, see Étale Cohomology, Lemma 102.5 (the precise statement one should use here is that V_n holds for all

n which is the statement of part (2) for the case of schemes). Transporting back (using the same formulas as above but now for U_i) we conclude ξ restricts to zero over U_i as desired. $\qquad\square$

The hard work done in the case of schemes now tells us that étale and ph cohomology agree for torsion abelian sheaves coming from the small étale site.

0DGU **Lemma 8.2.** *Let S be a scheme. Let X be an algebraic space over S. For $K \in D^+(X_{étale})$ with torsion cohomology sheaves the maps*

$$\pi_X^{-1}K \longrightarrow R\epsilon_{X,*}a_X^{-1}K \quad and \quad K \longrightarrow Ra_{X,*}a_X^{-1}K$$

are isomorphisms with $a_X : Sh((Spaces/X)_{ph}) \to Sh(X_{étale})$ as above.

Proof. We only prove the second statement; the first is easier and proved in exactly the same manner. There is a reduction to the case where K is given by a single torsion abelian sheaf. Namely, represent K by a bounded below complex $\mathcal{F}^\bullet$ of torsion abelian sheaves. This is possible by Cohomology on Sites, Lemma 19.8. By the case of a sheaf we see that $\mathcal{F}^n = a_{X,*}a_X^{-1}\mathcal{F}^n$ and that the sheaves $R^q a_{X,*}a_X^{-1}\mathcal{F}^n$ are zero for $q > 0$. By Leray's acyclicity lemma (Derived Categories, Lemma 16.7) applied to $a_X^{-1}\mathcal{F}^\bullet$ and the functor $a_{X,*}$ we conclude. From now on assume $K = \mathcal{F}$ where $\mathcal{F}$ is a torsion abelian sheaf.

By Lemma 8.1 we have $a_{X,*}a_X^{-1}\mathcal{F} = \mathcal{F}$. Thus it suffices to show that $R^q a_{X,*}a_X^{-1}\mathcal{F} = 0$ for $q > 0$. For this we can use $a_X = \epsilon_X \circ \pi_X$ and the Leray spectral sequence (Cohomology on Sites, Lemma 14.7). By Lemma 8.1 we have $R^i\epsilon_{X,*}(a_X^{-1}\mathcal{F}) = 0$ for $i > 0$. We have $\epsilon_{X,*}a_X^{-1}\mathcal{F} = \pi_X^{-1}\mathcal{F}$ and by Lemma 5.5 we have $R^j\pi_{X,*}(\pi_X^{-1}\mathcal{F}) = 0$ for $j > 0$. This concludes the proof. $\qquad\square$

0DGV **Lemma 8.3.** *Let S be a scheme and let X be an algebraic space over S. With $a_X : Sh((Spaces/X)_{ph}) \to Sh(X_{étale})$ as above:*

(1) $H^q(X_{étale}, \mathcal{F}) = H^q_{ph}(X, a_X^{-1}\mathcal{F})$ *for a torsion abelian sheaf $\mathcal{F}$ on $X_{étale}$,*

(2) $H^q(X_{étale}, K) = H^q_{ph}(X, a_X^{-1}K)$ *for $K \in D^+(X_{étale})$ with torsion cohomology sheaves*

Example: if A is a torsion abelian group, then $H^q_{étale}(X, \underline{A}) = H^q_{ph}(X, \underline{A})$.

Proof. This follows from Lemma 8.2 by Cohomology on Sites, Remark 14.4. $\qquad\square$

0DGW **Lemma 8.4.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Then there are commutative diagrams of topoi*

$$
\begin{array}{ccc}
Sh((Spaces/X)_{ph}) & \xrightarrow{\;f_{big,ph}\;} & Sh((Spaces/Y)_{ph}) \\
\epsilon_X \downarrow & & \downarrow \epsilon_Y \\
Sh((Spaces/X)_{étale}) & \xrightarrow{\;f_{big,étale}\;} & Sh((Spaces/Y)_{étale})
\end{array}
$$

and

$$
\begin{array}{ccc}
Sh((Spaces/X)_{ph}) & \xrightarrow{\;f_{big,ph}\;} & Sh((Spaces/Y)_{ph}) \\
a_X \downarrow & & \downarrow a_Y \\
Sh(X_{étale}) & \xrightarrow{\;f_{small}\;} & Sh(Y_{étale})
\end{array}
$$

with $a_X = \pi_X \circ \epsilon_X$ and $a_Y = \pi_X \circ \epsilon_X$.

Proof. This follows immediately from working out the definitions of the morphisms involved, see Topologies on Spaces, Section 8 and Section 5. $\square$

0DGX **Lemma 8.5.** *In Lemma 8.4 if f is proper, then we have*

(1) $a_Y^{-1} \circ f_{small,*} = f_{big,ph,*} \circ a_X^{-1}$, *and*
(2) $a_Y^{-1}(Rf_{small,*}K) = Rf_{big,ph,*}(a_X^{-1}K)$ *for K in $D^+(X_{étale})$ with torsion cohomology sheaves.*

Proof. Proof of (1). You can prove this by repeating the proof of Lemma 5.6 part (1); we will instead deduce the result from this. As $\epsilon_{Y,*}$ is the identity functor on underlying presheaves, it reflects isomorphisms. Lemma 8.1 shows that $\epsilon_{Y,*} \circ a_Y^{-1} = \pi_Y^{-1}$ and similarly for X. To show that the canonical map $a_Y^{-1}f_{small,*}\mathcal{F} \to f_{big,ph,*}a_X^{-1}\mathcal{F}$ is an isomorphism, it suffices to show that

$$\pi_Y^{-1}f_{small,*}\mathcal{F} = \epsilon_{Y,*}a_Y^{-1}f_{small,*}\mathcal{F}$$
$$\to \epsilon_{Y,*}f_{big,ph,*}a_X^{-1}\mathcal{F}$$
$$= f_{big,étale,*}\epsilon_{X,*}a_X^{-1}\mathcal{F}$$
$$= f_{big,étale,*}\pi_X^{-1}\mathcal{F}$$

is an isomorphism. This is part (1) of Lemma 5.6.

To see (2) we use that

$$R\epsilon_{Y,*}Rf_{big,ph,*}a_X^{-1}K = Rf_{big,étale,*}R\epsilon_{X,*}a_X^{-1}K$$
$$= Rf_{big,étale,*}\pi_X^{-1}K$$
$$= \pi_Y^{-1}Rf_{small,*}K$$
$$= R\epsilon_{Y,*}a_Y^{-1}Rf_{small,*}K$$

The first equality by the commutative diagram in Lemma 8.4 and Cohomology on Sites, Lemma 19.2. Then second equality is Lemma 8.2. The third is Lemma 5.6 part (2). The fourth is Lemma 8.2 again. Thus the base change map $a_Y^{-1}(Rf_{small,*}K) \to Rf_{big,ph,*}(a_X^{-1}K)$ induces an isomorphism

$$R\epsilon_{Y,*}a_Y^{-1}Rf_{small,*}K \to R\epsilon_{Y,*}Rf_{big,ph,*}a_X^{-1}K$$

The proof is finished by the following remark: consider a map $\alpha : a_Y^{-1}L \to M$ with L in $D^+(Y_{étale})$ having torsion cohomology sheaves and M in $D^+((Spaces/Y)_{ph})$. If $R\epsilon_{Y,*}\alpha$ is an isomorphism, then α is an isomorphism. Namely, we show by induction on i that $H^i(\alpha)$ is an isomorphism. This is true for all sufficiently small i. If it holds for $i \le i_0$, then we see that $R^j\epsilon_{Y,*}H^i(M) = 0$ for $j > 0$ and $i \le i_0$ by Lemma 8.1 because $H^i(M) = a_Y^{-1}H^i(L)$ in this range. Hence $\epsilon_{Y,*}H^{i_0+1}(M) = H^{i_0+1}(R\epsilon_{Y,*}M)$ by a spectral sequence argument. Thus $\epsilon_{Y,*}H^{i_0+1}(M) = \pi_Y^{-1}H^{i_0+1}(L) = \epsilon_{Y,*}a_Y^{-1}H^{i_0+1}(L)$. This implies $H^{i_0+1}(\alpha)$ is an isomorphism (because $\epsilon_{Y,*}$ reflects isomorphisms as it is the identity on underlying presheaves) as desired. $\square$

9. Other chapters

Preliminaries

References

[Knu71] Donald Knutson, *Algebraic spaces*, Lecture Notes in Mathematics, vol. 203, Springer-Verlag, 1971.