SITES AND SHEAVES

00UZ

Contents

This is a chapter of the Stacks Project, version 74af77a7, compiled on Jun 27, 2023.

1. Introduction

00V0 The notion of a site was introduced by Grothendieck to be able to study sheaves in the étale topology of schemes. The basic reference for this notion is perhaps [AGV71]. Our notion of a site differs from that in [AGV71]; what we call a site is called a category endowed with a pretopology in [AGV71, Exposé II, Définition 1.3]. The reason we do this is that in algebraic geometry it is often convenient to work with a given class of coverings, for example when defining when a property of schemes is local in a given topology, see Descent, Section 15. Our exposition will closely follow [Art62]. We will not use universes.

2. Presheaves

00V1 Let $\mathcal{C}$ be a category. A *presheaf of sets* is a contravariant functor $\mathcal{F}$ from $\mathcal{C}$ to *Sets* (see Categories, Remark 2.11). So for every object U of $\mathcal{C}$ we have a set $\mathcal{F}(U)$. The elements of this set are called the *sections* of $\mathcal{F}$ over U. For every morphism $f : V \to U$ the map $\mathcal{F}(f) : \mathcal{F}(U) \to \mathcal{F}(V)$ is called the *restriction map* and is often denoted $f^* : \mathcal{F}(U) \to \mathcal{F}(V)$. Another way of expressing this is to say that $f^*(s)$ is the *pullback* of s via f. Functoriality means that $g^*f^*(s) = (f \circ g)^*(s)$. Sometimes we use the notation $s|_V := f^*(s)$. This notation is consistent with the notion of restriction of functions from topology because if $W \to V \to U$ are morphisms in $\mathcal{C}$ and s is a section of $\mathcal{F}$ over U then $s|_W = (s|_V)|_W$ by the functorial nature of $\mathcal{F}$. Of course we have to be careful since it may very well happen that there is more than one morphism $V \to U$ and it is certainly not going to be the case that the corresponding pullback maps are equal.

00V2 **Definition 2.1.** A *presheaf of sets* on $\mathcal{C}$ is a contravariant functor from $\mathcal{C}$ to *Sets*. *Morphisms of presheaves* are transformations of functors. The category of presheaves of sets is denoted $PSh(\mathcal{C})$.

Note that for any object U of $\mathcal{C}$ the functor of points h_U, see Categories, Example 3.4 is a presheaf. These are called the *representable presheaves*. These presheaves

have the pleasing property that for any presheaf $\mathcal{F}$ we have

090F (2.1.1) $$\mathrm{Mor}_{PSh(\mathcal{C})}(h_U, \mathcal{F}) = \mathcal{F}(U).$$

This is the Yoneda lemma (Categories, Lemma 3.5).

Similarly, we can define the notion of a presheaf of abelian groups, rings, etc. More generally we may define a presheaf with values in a category.

00V3 **Definition 2.2.** Let $\mathcal{C}$, $\mathcal{A}$ be categories. A *presheaf* $\mathcal{F}$ on $\mathcal{C}$ with values in $\mathcal{A}$ is a contravariant functor from $\mathcal{C}$ to $\mathcal{A}$, i.e., $\mathcal{F} : \mathcal{C}^{opp} \to \mathcal{A}$. A *morphism* of presheaves $\mathcal{F} \to \mathcal{G}$ on $\mathcal{C}$ with values in $\mathcal{A}$ is a transformation of functors from $\mathcal{F}$ to $\mathcal{G}$.

These form the objects and morphisms of the category of presheaves on $\mathcal{C}$ with values in $\mathcal{A}$.

00V4 **Remark 2.3.** As already pointed out we may consider the category of presheaves with values in any of the "big" categories listed in Categories, Remark 2.2. These will be "big" categories as well and they will be listed in the above mentioned remark as we go along.

3. Injective and surjective maps of presheaves

00V5

00V6 **Definition 3.1.** Let $\mathcal{C}$ be a category, and let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of presheaves of sets.

(1) We say that φ is *injective* if for every object U of $\mathcal{C}$ the map $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective.
(2) We say that φ is *surjective* if for every object U of $\mathcal{C}$ the map $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is surjective.

00V7 **Lemma 3.2.** *The injective (resp. surjective) maps defined above are exactly the monomorphisms (resp. epimorphisms) of $PSh(\mathcal{C})$. A map is an isomorphism if and only if it is both injective and surjective.*

Proof. We shall show that $\varphi : \mathcal{F} \to \mathcal{G}$ is injective if and only if it is a monomorphism of $PSh(\mathcal{C})$. Indeed, the "only if" direction is straightforward, so let us show the "if" direction. Assume that φ is a monomorphism. Let $U \in \mathrm{Ob}(\mathcal{C})$; we need to show that φ_U is injective. So let $a, b \in \mathcal{F}(U)$ be such that $\varphi_U(a) = \varphi_U(b)$; we need to check that $a = b$. Under the isomorphism (2.1.1), the elements a and b of $\mathcal{F}(U)$ correspond to two natural transformations $a', b' \in \mathrm{Mor}_{PSh(\mathcal{C})}(h_U, \mathcal{F})$. Similarly, under the analogous isomorphism $\mathrm{Mor}_{PSh(\mathcal{C})}(h_U, \mathcal{G}) = \mathcal{G}(U)$, the two equal elements $\varphi_U(a)$ and $\varphi_U(b)$ of $\mathcal{G}(U)$ correspond to the two natural transformations $\varphi \circ a', \varphi \circ b' \in \mathrm{Mor}_{PSh(\mathcal{C})}(h_U, \mathcal{G})$, which therefore must also be equal. So $\varphi \circ a' = \varphi \circ b'$, and thus $a' = b'$ (since φ is monic), whence $a = b$. This finishes (1).

We shall show that $\varphi : \mathcal{F} \to \mathcal{G}$ is surjective if and only if it is an epimorphism of $PSh(\mathcal{C})$. Indeed, the "only if" direction is straightforward, so let us show the "if" direction. Assume that φ is an epimorphism.

For any two morphisms $f : A \to B$ and $g : A \to C$ in the category *Sets*, we let $\mathrm{inl}_{f,g}$ and $\mathrm{inr}_{f,g}$ denote the two canonical maps from B and C to $B \coprod_A C$. (Here, the pushout is evaluated in *Sets*.)

Now, we define a presheaf $\mathcal{H}$ of sets on $\mathcal{C}$ by setting $\mathcal{H}(U) = \mathcal{G}(U) \coprod_{\mathcal{F}(U)} \mathcal{G}(U)$ (where the pushout is evaluated in *Sets* and induced by the map $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$)

for every $U \in \mathrm{Ob}(\mathcal{C})$; its action on morphisms is defined in the obvious way (by the functoriality of pushout). Then, there are two natural transformations $i_1 : \mathcal{G} \to \mathcal{H}$ and $i_2 : \mathcal{G} \to \mathcal{H}$ whose components at an object $U \in \mathrm{Ob}(\mathcal{C})$ are given by the maps $\mathrm{inl}_{\varphi_U,\varphi_U}$ and $\mathrm{inr}_{\varphi_U,\varphi_U}$, respectively. The definition of a pushout shows that $i_1 \circ \varphi = i_2 \circ \varphi$, whence $i_1 = i_2$ (since φ is an epimorphism). Thus, for every $U \in \mathrm{Ob}(\mathcal{C})$, we have $\mathrm{inl}_{\varphi_U,\varphi_U} = \mathrm{inr}_{\varphi_U,\varphi_U}$. Thus, φ_U must be surjective (since a simple combinatorial argument shows that if $f : A \to B$ is a morphism in *Sets*, then $\mathrm{inl}_{f,f} = \mathrm{inr}_{f,f}$ if and only if f is surjective). In other words, φ is surjective, and (2) is proven.

We shall show that $\varphi : \mathcal{F} \to \mathcal{G}$ is both injective and surjective if and only if it is an isomorphism of $PSh(\mathcal{C})$. This time, the "if" direction is straightforward. To prove the "only if" direction, it suffices to observe that if φ is both injective and surjective, then φ_U is an invertible map for every $U \in \mathrm{Ob}(\mathcal{C})$, and the inverses of these maps for all U can be combined to a natural transformation $\mathcal{G} \to \mathcal{F}$ which is an inverse to φ. □

00V8 **Definition 3.3.** We say $\mathcal{F}$ is a *subpresheaf* of $\mathcal{G}$ if for every object $U \in \mathrm{Ob}(\mathcal{C})$ the set $\mathcal{F}(U)$ is a subset of $\mathcal{G}(U)$, compatibly with the restriction mappings.

In other words, the inclusion maps $\mathcal{F}(U) \to \mathcal{G}(U)$ glue together to give an (injective) morphism of presheaves $\mathcal{F} \to \mathcal{G}$.

00V9 **Lemma 3.4.** *Let $\mathcal{C}$ be a category. Suppose that $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves of sets on $\mathcal{C}$. There exists a unique subpresheaf $\mathcal{G}' \subset \mathcal{G}$ such that φ factors as $\mathcal{F} \to \mathcal{G}' \to \mathcal{G}$ and such that the first map is surjective.*

Proof. To prove existence, just set $\mathcal{G}'(U) = \varphi_U(\mathcal{F}(U))$ for every $U \in \mathrm{Ob}(C)$ (and inherit the action on morphisms from $\mathcal{G}$), and prove that this defines a subpresheaf of $\mathcal{G}$ and that φ factors as $\mathcal{F} \to \mathcal{G}' \to \mathcal{G}$ with the first map being surjective. Uniqueness is straightforward. □

00VA **Definition 3.5.** Notation as in Lemma 3.4. We say that $\mathcal{G}'$ is the *image of* φ.

4. Limits and colimits of presheaves

00VB Let $\mathcal{C}$ be a category. Limits and colimits exist in the category $PSh(\mathcal{C})$. In addition, for any $U \in \mathrm{Ob}(\mathcal{C})$ the functor

$$PSh(\mathcal{C}) \longrightarrow Sets, \quad \mathcal{F} \longmapsto \mathcal{F}(U)$$

commutes with limits and colimits. Perhaps the easiest way to prove these statements is the following. Given a diagram $\mathcal{F} : \mathcal{I} \to PSh(\mathcal{C})$ define presheaves

$$\mathcal{F}_{\lim} : U \longmapsto \lim_{i \in \mathcal{I}} \mathcal{F}_i(U) \text{ and } \mathcal{F}_{\mathrm{colim}} : U \longmapsto \mathrm{colim}_{i \in \mathcal{I}} \mathcal{F}_i(U)$$

There are clearly projection maps $\mathcal{F}_{\lim} \to \mathcal{F}_i$ and canonical maps $\mathcal{F}_i \to \mathcal{F}_{\mathrm{colim}}$. These maps satisfy the requirements of the maps of a limit (resp. colimit) of Categories, Definition 14.1 (resp. Categories, Definition 14.2). Indeed, they clearly form a cone, resp. a cocone, over $\mathcal{F}$. Furthermore, if $(\mathcal{G}, q_i : \mathcal{G} \to \mathcal{F}_i)$ is another system (as in the definition of a limit), then we get for every U a system of maps $\mathcal{G}(U) \to \mathcal{F}_i(U)$ with suitable functoriality requirements. And thus a unique map $\mathcal{G}(U) \to \mathcal{F}_{\lim}(U)$. It is easy to verify these are compatible as we vary U and arise from the desired map $\mathcal{G} \to \mathcal{F}_{\lim}$. A similar argument works in the case of the colimit.

5. Functoriality of categories of presheaves

00VC Let $u : \mathcal{C} \to \mathcal{D}$ be a functor between categories. In this case we denote

$$u^p : PSh(\mathcal{D}) \longrightarrow PSh(\mathcal{C})$$

the functor that associates to $\mathcal{G}$ on $\mathcal{D}$ the presheaf $u^p\mathcal{G} = \mathcal{G} \circ u$. Note that by the previous section this functor commutes with all limits.

For $V \in \mathrm{Ob}(\mathcal{D})$ let $\mathcal{I}_V^u$ denote the category with

053L (5.0.1)
$$\begin{array}{rcl} \mathrm{Ob}(\mathcal{I}_V^u) & = & \{(U, \phi) \mid U \in \mathrm{Ob}(\mathcal{C}), \phi : V \to u(U)\} \\ \mathrm{Mor}_{\mathcal{I}_V^u}((U, \phi), (U', \phi')) & = & \{f : U \to U' \text{ in } \mathcal{C} \mid u(f) \circ \phi = \phi'\} \end{array}$$

We sometimes drop the subscript ^{u} from the notation and we simply write $\mathcal{I}_V$. We will use these categories to define a left adjoint to the functor u^p. Before we do so we prove a few technical lemmas.

00X4 **Lemma 5.1.** *Let $u : \mathcal{C} \to \mathcal{D}$ be a functor between categories. Suppose that $\mathcal{C}$ has fibre products and equalizers, and that u commutes with them. Then the categories $(\mathcal{I}_V)^{opp}$ satisfy the hypotheses of Categories, Lemma 19.8.*

Proof. There are two conditions to check.

First, suppose we are given three objects $\phi : V \to u(U)$, $\phi' : V \to u(U')$, and $\phi'' : V \to u(U'')$ and morphisms $a : U' \to U$, $b : U'' \to U$ such that $u(a) \circ \phi' = \phi$ and $u(b) \circ \phi'' = \phi$. We have to show there exists another object $\phi''' : V \to u(U''')$ and morphisms $c : U''' \to U'$ and $d : U''' \to U''$ such that $u(c) \circ \phi''' = \phi'$, $u(d) \circ \phi''' = \phi''$ and $a \circ c = b \circ d$. We take $U''' = U' \times_U U''$ with c and d the projection morphisms. This works as u commutes with fibre products; we omit the verification.

Second, suppose we are given two objects $\phi : V \to u(U)$ and $\phi' : V \to u(U')$ and morphisms $a, b : (U, \phi) \to (U', \phi')$. We have to find a morphism $c : (U'', \phi'') \to (U, \phi)$ which equalizes a and b. Let $c : U'' \to U$ be the equalizer of a and b in the category $\mathcal{C}$. As u commutes with equalizers and since $u(a) \circ \phi = u(b) \circ \phi = \phi'$ we obtain a morphism $\phi'' : V \to u(U'')$. □

00X3 **Lemma 5.2.** *Let $u : \mathcal{C} \to \mathcal{D}$ be a functor between categories. Assume*

1. *the category $\mathcal{C}$ has a final object X and $u(X)$ is a final object of $\mathcal{D}$, and*
2. *the category $\mathcal{C}$ has fibre products and u commutes with them.*

Then the index categories $(\mathcal{I}_V^u)^{opp}$ are filtered (see Categories, Definition 19.1).

Proof. The assumptions imply that the assumptions of Lemma 5.1 are satisfied (see the discussion in Categories, Section 18). By Categories, Lemma 19.8 we see that $\mathcal{I}_V$ is a (possibly empty) disjoint union of directed categories. Hence it suffices to show that $\mathcal{I}_V$ is connected.

First, we show that $\mathcal{I}_V$ is nonempty. Namely, let X be the final object of $\mathcal{C}$, which exists by assumption. Let $V \to u(X)$ be the morphism coming from the fact that $u(X)$ is final in $\mathcal{D}$ by assumption. This gives an object of $\mathcal{I}_V$.

Second, we show that $\mathcal{I}_V$ is connected. Let $\phi_1 : V \to u(U_1)$ and $\phi_2 : V \to u(U_2)$ be in $\mathrm{Ob}(\mathcal{I}_V)$. By assumption $U_1 \times U_2$ exists and $u(U_1 \times U_2) = u(U_1) \times u(U_2)$. Consider the morphism $\phi : V \to u(U_1 \times U_2)$ corresponding to (ϕ_1, ϕ_2) by the universal property of products. Clearly the object $\phi : V \to u(U_1 \times U_2)$ maps to both $\phi_1 : V \to u(U_1)$ and $\phi_2 : V \to u(U_2)$. □

Given $g : V' \to V$ in $\mathcal{D}$ we get a functor $\overline{g} : \mathcal{I}_V \to \mathcal{I}_{V'}$ by setting $\overline{g}(U, \phi) = (U, \phi \circ g)$ on objects. Given a presheaf $\mathcal{F}$ on $\mathcal{C}$ we obtain a functor

$$\mathcal{F}_V : \mathcal{I}_V^{opp} \longrightarrow Sets, \quad (U, \phi) \longmapsto \mathcal{F}(U).$$

In other words, $\mathcal{F}_V$ is a presheaf of sets on $\mathcal{I}_V$. Note that we have $\mathcal{F}_{V'} \circ \overline{g} = \mathcal{F}_V$. We define

$$u_p\mathcal{F}(V) = \mathrm{colim}_{\mathcal{I}_V^{opp}} \mathcal{F}_V$$

As a colimit we obtain for each $(U, \phi) \in \mathrm{Ob}(\mathcal{I}_V)$ a canonical map $\mathcal{F}(U) \xrightarrow{c(\phi)} u_p\mathcal{F}(V)$. For $g : V' \to V$ as above there is a canonical restriction map $g^* : u_p\mathcal{F}(V) \to u_p\mathcal{F}(V')$ compatible with $\mathcal{F}_{V'} \circ \overline{g} = \mathcal{F}_V$ by Categories, Lemma 14.8. It is the unique map so that for all $(U, \phi) \in \mathrm{Ob}(\mathcal{I}_V)$ the diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{c(\phi)} & u_p\mathcal{F}(V) \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle g^*} \\ \mathcal{F}(U) & \xrightarrow{c(\phi \circ g)} & u_p\mathcal{F}(V') \end{array}$$

commutes. The uniqueness of these maps implies that we obtain a presheaf. This presheaf will be denoted $u_p\mathcal{F}$.

00VD **Lemma 5.3.** *There is a canonical map $\mathcal{F}(U) \to u_p\mathcal{F}(u(U))$, which is compatible with restriction maps (on $\mathcal{F}$ and on $u_p\mathcal{F}$).*

Proof. This is just the map $c(\mathrm{id}_{u(U)})$ introduced above. □

Note that any map of presheaves $\mathcal{F} \to \mathcal{F}'$ gives rise to compatible systems of maps between functors $\mathcal{F}_V \to \mathcal{F}'_V$, and hence to a map of presheaves $u_p\mathcal{F} \to u_p\mathcal{F}'$. In other words, we have defined a functor

$$u_p : PSh(\mathcal{C}) \longrightarrow PSh(\mathcal{D})$$

00VE **Lemma 5.4.** *The functor u_p is a left adjoint to the functor u^p. In other words the formula*

$$\mathrm{Mor}_{PSh(\mathcal{C})}(\mathcal{F}, u^p\mathcal{G}) = \mathrm{Mor}_{PSh(\mathcal{D})}(u_p\mathcal{F}, \mathcal{G})$$

holds bifunctorially in $\mathcal{F}$ and $\mathcal{G}$.

Proof. Let $\mathcal{G}$ be a presheaf on $\mathcal{D}$ and let $\mathcal{F}$ be a presheaf on $\mathcal{C}$. We will show that the displayed formula holds by constructing maps either way. We will leave it to the reader to verify they are each others inverse.

Given a map $\alpha : u_p\mathcal{F} \to \mathcal{G}$ we get $u^p\alpha : u^pu_p\mathcal{F} \to u^p\mathcal{G}$. Lemma 5.3 says that there is a map $\mathcal{F} \to u^pu_p\mathcal{F}$. The composition of the two gives the desired map. (The good thing about this construction is that it is clearly functorial in everything in sight.)

Conversely, given a map $\beta : \mathcal{F} \to u^p\mathcal{G}$ we get a map $u_p\beta : u_p\mathcal{F} \to u_pu^p\mathcal{G}$. We claim that the functor $u^p\mathcal{G}_Y$ on $\mathcal{I}_Y$ has a canonical map to the constant functor with value $\mathcal{G}(Y)$. Namely, for every object (X, ϕ) of $\mathcal{I}_Y$, the value of $u^p\mathcal{G}_Y$ on this object is $\mathcal{G}(u(X))$ which maps to $\mathcal{G}(Y)$ by $\mathcal{G}(\phi) = \phi^*$. This is a transformation of functors because $\mathcal{G}$ is a functor itself. This leads to a map $u_pu^p\mathcal{G}(Y) \to \mathcal{G}(Y)$. Another trivial verification shows that this is functorial in Y leading to a map of presheaves $u_pu^p\mathcal{G} \to \mathcal{G}$. The composition $u_p\mathcal{F} \to u_pu^p\mathcal{G} \to \mathcal{G}$ is the desired map. □

00VF **Remark 5.5.** Suppose that $\mathcal{A}$ is a category such that any diagram $\mathcal{I}_Y \to \mathcal{A}$ has a colimit in $\mathcal{A}$. In this case it is clear that there are functors u^p and u_p, defined in exactly the same way as above, on the categories of presheaves with values in $\mathcal{A}$. Moreover, the adjointness of the pair u^p and u_p continues to hold in this setting.

04D2 **Lemma 5.6.** *Let $u : \mathcal{C} \to \mathcal{D}$ be a functor between categories. For any object U of $\mathcal{C}$ we have $u_p h_U = h_{u(U)}$.*

Proof. By adjointness of u_p and u^p we have

$$\mathrm{Mor}_{PSh(\mathcal{D})}(u_p h_U, \mathcal{G}) = \mathrm{Mor}_{PSh(\mathcal{C})}(h_U, u^p\mathcal{G}) = u^p\mathcal{G}(U) = \mathcal{G}(u(U))$$

and hence by Yoneda's lemma we see that $u_p h_U = h_{u(U)}$ as presheaves. □

6. Sites

00VG Our notion of a site uses the following type of structures.

0396 **Definition 6.1.** Let $\mathcal{C}$ be a category, see Conventions, Section 3. A *family of morphisms with fixed target* in $\mathcal{C}$ is given by an object $U \in \mathrm{Ob}(\mathcal{C})$, a set I and for each $i \in I$ a morphism $U_i \to U$ of $\mathcal{C}$ with target U. We use the notation $\{U_i \to U\}_{i\in I}$ to indicate this.

It can happen that the set I is empty! This notation is meant to suggest an open covering as in topology.

00VH **Definition 6.2.** A *site*[1] is given by a category $\mathcal{C}$ and a set $\mathrm{Cov}(\mathcal{C})$ of families of morphisms with fixed target $\{U_i \to U\}_{i\in I}$, called *coverings of* $\mathcal{C}$, satisfying the following axioms

(1) If $V \to U$ is an isomorphism then $\{V \to U\} \in \mathrm{Cov}(\mathcal{C})$.
(2) If $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})$ and for each i we have $\{V_{ij} \to U_i\}_{j\in J_i} \in \mathrm{Cov}(\mathcal{C})$, then $\{V_{ij} \to U\}_{i\in I, j\in J_i} \in \mathrm{Cov}(\mathcal{C})$.
(3) If $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})$ and $V \to U$ is a morphism of $\mathcal{C}$ then $U_i \times_U V$ exists for all i and $\{U_i \times_U V \to V\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})$.

Clarifications. In axiom (1) we require there should be a covering $\{U_i \to U\}_{i\in I}$ of $\mathcal{C}$ such that $I = \{i\}$ is a singleton set and such that the morphism $U_i \to U$ is equal to the morphism $V \to U$ given in (1). In the following we often denote $\{V \to U\}$ a family of morphisms with fixed target whose index set is a singleton. In axiom (3) we require the existence of the covering for some choice of the fibre products $U_i \times_U V$ for $i \in I$.

00VI **Remark 6.3.** (On set theoretic issues – skip on a first reading.) The main reason for introducing sites is to study the category of sheaves on a site, because it is the generalization of the category of sheaves on a topological space that has been so important in algebraic geometry. In order to avoid thinking about things like "classes of classes" and so on, we will not allow sites to be "big" categories, in contrast to what we do for categories and 2-categories.

Suppose that $\mathcal{C}$ is a category and that $\mathrm{Cov}(\mathcal{C})$ is a proper class of coverings satisfying (1), (2) and (3) above. We will not allow this as a site either, mainly because we are going to take limits over coverings. However, there are several natural ways to replace $\mathrm{Cov}(\mathcal{C})$ by a set of coverings or a slightly different structure that give rise to the same category of sheaves. For example:

[1] This notation differs from that of [AGV71], as explained in the introduction.

(1) In Sets, Section 11 we show how to pick a suitable set of coverings that gives the same category of sheaves.
(2) Another thing we can do is to take the associated topology (see Definition 48.2). The resulting topology on $\mathcal{C}$ has the same category of sheaves. Two topologies have the same categories of sheaves if and only if they are equal, see Theorem 50.2. A topology on a category is given by a choice of sieves on objects. The collection of all possible sieves and even all possible topologies on $\mathcal{C}$ is a set.
(3) We could also slightly modify the notion of a site, see Remark 48.4 below, and end up with a canonical set of coverings.

Each of these solutions has some minor drawback. For the first, one has to check that constructions later on do not depend on the choice of the set of coverings. For the second, one has to learn about topologies and redo many of the arguments for sites. For the third, see the last sentence of Remark 48.4.

Our approach will be to work with sites as in Definition 6.2 above. Given a category $\mathcal{C}$ with a proper class of coverings as above, we will replace this by a set of coverings producing a site using Sets, Lemma 11.1. It is shown in Lemma 8.8 below that the resulting category of sheaves (the topos) is independent of this choice. We leave it to the reader to use one of the other two strategies to deal with these issues if he/she so desires.

00VJ **Example 6.4.** Let X be a topological space. Let X_{Zar} be the category whose objects consist of all the open sets U in X and whose morphisms are just the inclusion maps. That is, there is at most one morphism between any two objects in X_{Zar}. Now define $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(X_{Zar})$ if and only if $\bigcup U_i = U$. Conditions (1) and (2) above are clear, and (3) is also clear once we realize that in X_{Zar} we have $U \times V = U \cap V$. Note that in particular the empty set has to be an element of X_{Zar} since otherwise this would not work in general. Furthermore, it is equally important, as we will see later, to allow the *empty covering of the empty set as a covering*! We turn X_{Zar} into a site by choosing a suitable set of coverings $\mathrm{Cov}(X_{Zar})_{\kappa,\alpha}$ as in Sets, Lemma 11.1. Presheaves and sheaves (as defined below) on the site X_{Zar} agree exactly with the usual notion of a presheaves and sheaves on a topological space, as defined in Sheaves, Section 1.

00VK **Example 6.5.** Let G be a group. Consider the category G-*Sets* whose objects are sets X with a left G-action, with G-equivariant maps as the morphisms. An important example is ${}_GG$ which is the G-set whose underlying set is G and action given by left multiplication. This category has fiber products, see Categories, Section 7. We declare $\{\varphi_i : U_i \to U\}_{i\in I}$ to be a covering if $\bigcup_{i\in I} \varphi_i(U_i) = U$. This gives a class of coverings on G-*Sets* which is easily seen to satisfy conditions (1), (2), and (3) of Definition 6.2. The result is not a site since both the collection of objects of the underlying category and the collection of coverings form a proper class. We first replace by G-*Sets* by a full subcategory G-$Sets_\alpha$ as in Sets, Lemma 10.1. After this the site $(G\text{-}Sets_\alpha, \mathrm{Cov}_{\kappa,\alpha'}(G\text{-}Sets_\alpha))$ gotten by suitably restricting the collection of coverings as in Sets, Lemma 11.1 will be denoted $\mathcal{T}_G$.

As a special case, if the group G is countable, then we can let $\mathcal{T}_G$ be the category of countable G-sets and coverings those jointly surjective families of morphisms $\{\varphi_i : U_i \to U\}_{i\in I}$ such that I is countable.

07GE **Example 6.6.** Let $\mathcal{C}$ be a category. There is a canonical way to turn this into a site where $\{f : V \to U \mid f \text{ is an isomorphism}\}$ are the coverings of U. Sheaves on this site are the presheaves on $\mathcal{C}$. This corresponding topology is called the *chaotic* or *indiscrete topology*.

7. Sheaves

00VL Let $\mathcal{C}$ be a site. Before we introduce the notion of a sheaf with values in a category we explain what it means for a presheaf of sets to be a sheaf. Let $\mathcal{F}$ be a presheaf of sets on $\mathcal{C}$ and let $\{U_i \to U\}_{i\in I}$ be an element of $\mathrm{Cov}(\mathcal{C})$. By assumption all the fibre products $U_i \times_U U_j$ exist in $\mathcal{C}$. There are two natural maps

$$\prod_{i\in I} \mathcal{F}(U_i) \xrightarrow[\mathrm{pr}_1^*]{\mathrm{pr}_0^*} \prod_{(i_0,i_1)\in I\times I} \mathcal{F}(U_{i_0} \times_U U_{i_1})$$

which we will denote pr_i^*, $i = 0, 1$ as indicated in the displayed equation. Namely, an element of the left hand side corresponds to a family $(s_i)_{i\in I}$, where each s_i is a section of $\mathcal{F}$ over U_i. For each pair $(i_0, i_1) \in I \times I$ we have the projection morphisms

$$\mathrm{pr}_{i_0}^{(i_0,i_1)} : U_{i_0} \times_U U_{i_1} \longrightarrow U_{i_0} \text{ and } \mathrm{pr}_{i_1}^{(i_0,i_1)} : U_{i_0} \times_U U_{i_1} \longrightarrow U_{i_1}.$$

Thus we may pull back either the section s_{i_0} via the first of these maps or the section s_{i_1} via the second. Explicitly the maps we referred to above are

$$\mathrm{pr}_0^* : (s_i)_{i\in I} \longmapsto \left(\mathrm{pr}_{i_0}^{(i_0,i_1),*}(s_{i_0})\right)_{(i_0,i_1)\in I\times I}$$

and

$$\mathrm{pr}_1^* : (s_i)_{i\in I} \longmapsto \left(\mathrm{pr}_{i_1}^{(i_0,i_1),*}(s_{i_1})\right)_{(i_0,i_1)\in I\times I}.$$

Finally consider the natural map

$$\mathcal{F}(U) \longrightarrow \prod\nolimits_{i\in I} \mathcal{F}(U_i), \quad s \longmapsto (s|_{U_i})_{i\in I}$$

where we have used the notation $s|_{U_i}$ to indicate the pullback of s via the map $U_i \to U$. It is clear from the functorial nature of $\mathcal{F}$ and the commutativity of the fibre product diagrams that $\mathrm{pr}_0^*((s|_{U_i})_{i\in I}) = \mathrm{pr}_1^*((s|_{U_i})_{i\in I})$.

00VM **Definition 7.1.** Let $\mathcal{C}$ be a site, and let $\mathcal{F}$ be a presheaf of sets on $\mathcal{C}$. We say $\mathcal{F}$ is a *sheaf* if for every covering $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})$ the diagram

00VN (7.1.1)
$$\mathcal{F}(U) \longrightarrow \prod_{i\in I} \mathcal{F}(U_i) \xrightarrow[\mathrm{pr}_1^*]{\mathrm{pr}_0^*} \prod_{(i_0,i_1)\in I\times I} \mathcal{F}(U_{i_0} \times_U U_{i_1})$$

represents the first arrow as the equalizer of pr_0^* and pr_1^*.

Loosely speaking this means that given sections $s_i \in \mathcal{F}(U_i)$ such that

$$s_i|_{U_i\times_U U_j} = s_j|_{U_i\times_U U_j}$$

in $\mathcal{F}(U_i \times_U U_j)$ for all pairs $(i, j) \in I \times I$ then there exists a unique $s \in \mathcal{F}(U)$ such that $s_i = s|_{U_i}$.

04B3 **Remark 7.2.** If the covering $\{U_i \to U\}_{i\in I}$ is the empty family (this means that $I = \emptyset$), then the sheaf condition signifies that $\mathcal{F}(U) = \{*\}$ is a singleton set. This is because in (7.1.1) the second and third sets are empty products in the category of sets, which are final objects in the category of sets, hence singletons.

00VO **Example 7.3.** Let X be a topological space. Let X_{Zar} be the site constructed in Example 6.4. The notion of a sheaf on X_{Zar} coincides with the notion of a sheaf on X introduced in Sheaves, Definition 7.1.

00VP **Example 7.4.** Let X be a topological space. Let us consider the site X'_{Zar} which is the same as the site X_{Zar} of Example 6.4 except that we disallow the empty covering of the empty set. In other words, we do allow the covering $\{\emptyset \to \emptyset\}$ but we do not allow the covering whose index set is empty. It is easy to show that this still defines a site. However, we claim that the sheaves on X'_{Zar} are different from the sheaves on X_{Zar}. For example, as an extreme case consider the situation where $X = \{p\}$ is a singleton. Then the objects of X'_{Zar} are $\emptyset, X$ and every covering of $\emptyset$ can be refined by $\{\emptyset \to \emptyset\}$ and every covering of X by $\{X \to X\}$. Clearly, a sheaf on this is given by any choice of a set $\mathcal{F}(\emptyset)$ and any choice of a set $\mathcal{F}(X)$, together with any restriction map $\mathcal{F}(X) \to \mathcal{F}(\emptyset)$. Thus sheaves on X'_{Zar} are the same as usual sheaves on the two point space $\{\eta, p\}$ with open sets $\{\emptyset, \{\eta\}, \{p, \eta\}\}$. In general sheaves on X'_{Zar} are the same as sheaves on the space $X \amalg \{\eta\}$, with opens given by the empty set and any set of the form $U \cup \{\eta\}$ for $U \subset X$ open.

00VQ **Definition 7.5.** The category $Sh(\mathcal{C})$ of sheaves of sets is the full subcategory of the category $PSh(\mathcal{C})$ whose objects are the sheaves of sets.

Let $\mathcal{A}$ be a category. If products indexed by I, and $I \times I$ exist in $\mathcal{A}$ for any I that occurs as an index set for covering families then Definition 7.1 above makes sense, and defines a notion of a sheaf on $\mathcal{C}$ with values in $\mathcal{A}$. Note that the diagram in $\mathcal{A}$

$$\mathcal{F}(U) \longrightarrow \prod_{i \in I} \mathcal{F}(U_i) \underset{\mathrm{pr}_1^*}{\overset{\mathrm{pr}_0^*}{\rightrightarrows}} \prod_{(i_0,i_1) \in I \times I} \mathcal{F}(U_{i_0} \times_U U_{i_1})$$

is an equalizer diagram if and only if for every object X of $\mathcal{A}$ the diagram of sets

$$\mathrm{Mor}_{\mathcal{A}}(X, \mathcal{F}(U)) \longrightarrow \prod \mathrm{Mor}_{\mathcal{A}}(X, \mathcal{F}(U_i)) \underset{\mathrm{pr}_1^*}{\overset{\mathrm{pr}_0^*}{\rightrightarrows}} \prod \mathrm{Mor}_{\mathcal{A}}(X, \mathcal{F}(U_{i_0} \times_U U_{i_1}))$$

is an equalizer diagram.

Suppose $\mathcal{A}$ is arbitrary. Let $\mathcal{F}$ be a presheaf with values in $\mathcal{A}$. Choose any object $X \in \mathrm{Ob}(\mathcal{A})$. Then we get a presheaf of sets $\mathcal{F}_X$ defined by the rule

$$\mathcal{F}_X(U) = \mathrm{Mor}_{\mathcal{A}}(X, \mathcal{F}(U)).$$

From the above it follows that a good definition is obtained by requiring all the presheaves $\mathcal{F}_X$ to be sheaves of sets.

00VR **Definition 7.6.** Let $\mathcal{C}$ be a site, let $\mathcal{A}$ be a category and let $\mathcal{F}$ be a presheaf on $\mathcal{C}$ with values in $\mathcal{A}$. We say that $\mathcal{F}$ is a *sheaf* if for all objects X of $\mathcal{A}$ the presheaf of sets $\mathcal{F}_X$ (defined above) is a sheaf.

8. Families of morphisms with fixed target

00VS This section is meant to introduce some notions regarding families of morphisms with the same target.

00VT **Definition 8.1.** Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be a family of morphisms of $\mathcal{C}$ with fixed target. Let $\mathcal{V} = \{V_j \to V\}_{j \in J}$ be another.

(1) A *morphism of families of maps with fixed target of $\mathcal{C}$ from $\mathcal{U}$ to $\mathcal{V}$*, or simply a *morphism from $\mathcal{U}$ to $\mathcal{V}$* is given by a morphism $U \to V$, a map of sets $\alpha : I \to J$ and for each $i \in I$ a morphism $U_i \to V_{\alpha(i)}$ such that the diagram

$$\begin{array}{ccc} U_i & \longrightarrow & V_{\alpha(i)} \\ \downarrow & & \downarrow \\ U & \longrightarrow & V \end{array}$$

is commutative.

(2) In the special case that $U = V$ and $U \to V$ is the identity we call $\mathcal{U}$ a *refinement* of the family $\mathcal{V}$.

A trivial but important remark is that if $\mathcal{V} = \{V_j \to V\}_{j \in J}$ is the *empty family of maps*, i.e., if $J = \emptyset$, then no family $\mathcal{U} = \{U_i \to V\}_{i \in I}$ with $I \neq \emptyset$ can refine $\mathcal{V}$!

00VU **Definition 8.2.** Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{\varphi_i : U_i \to U\}_{i \in I}$, and $\mathcal{V} = \{\psi_j : V_j \to U\}_{j \in J}$ be two families of morphisms with fixed target.

(1) We say $\mathcal{U}$ and $\mathcal{V}$ are *combinatorially equivalent* if there exist maps $\alpha : I \to J$ and $\beta : J \to I$ such that $\varphi_i = \psi_{\alpha(i)}$ and $\psi_j = \varphi_{\beta(j)}$.

(2) We say $\mathcal{U}$ and $\mathcal{V}$ are *tautologically equivalent* if there exist maps $\alpha : I \to J$ and $\beta : J \to I$ and for all $i \in I$ and $j \in J$ commutative diagrams

$$\begin{array}{ccc} U_i & \longrightarrow & V_{\alpha(i)} \\ & \searrow \quad \swarrow & \\ & U & \end{array} \qquad \begin{array}{ccc} V_j & \longrightarrow & U_{\beta(j)} \\ & \searrow \quad \swarrow & \\ & U & \end{array}$$

with isomorphisms as horizontal arrows.

00VV **Lemma 8.3.** *Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{\varphi_i : U_i \to U\}_{i \in I}$, and $\mathcal{V} = \{\psi_j : V_j \to U\}_{j \in J}$ be two families of morphisms with the same fixed target.*

(1) *If $\mathcal{U}$ and $\mathcal{V}$ are combinatorially equivalent then they are tautologically equivalent.*

(2) *If $\mathcal{U}$ and $\mathcal{V}$ are tautologically equivalent then $\mathcal{U}$ is a refinement of $\mathcal{V}$ and $\mathcal{V}$ is a refinement of $\mathcal{U}$.*

(3) *The relation "being combinatorially equivalent" is an equivalence relation on all families of morphisms with fixed target.*

(4) *The relation "being tautologically equivalent" is an equivalence relation on all families of morphisms with fixed target.*

(5) *The relation "$\mathcal{U}$ refines $\mathcal{V}$ and $\mathcal{V}$ refines $\mathcal{U}$" is an equivalence relation on all families of morphisms with fixed target.*

Proof. Omitted. □

In the following lemma, given a category $\mathcal{C}$, a presheaf $\mathcal{F}$ on $\mathcal{C}$, a family $\mathcal{U} = \{U_i \to U\}_{i \in I}$ such that all fibre products $U_i \times_U U_{i'}$ exist, we say that *the sheaf condition for $\mathcal{F}$ with respect to $\mathcal{U}$* holds if the diagram (7.1.1) is an equalizer diagram.

00VW **Lemma 8.4.** *Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{\varphi_i : U_i \to U\}_{i \in I}$, and $\mathcal{V} = \{\psi_j : V_j \to U\}_{j \in J}$ be two families of morphisms with the same fixed target. Assume that the fibre products $U_i \times_U U_{i'}$ and $V_j \times_U V_{j'}$ exist. If $\mathcal{U}$ and $\mathcal{V}$ are tautologically equivalent, then for any presheaf $\mathcal{F}$ on $\mathcal{C}$ the sheaf condition for $\mathcal{F}$ with respect to $\mathcal{U}$ is equivalent to the sheaf condition for $\mathcal{F}$ with respect to $\mathcal{V}$.*

Proof. First, note that if $\varphi : A \to B$ is an isomorphism in the category $\mathcal{C}$, then $\varphi^* : \mathcal{F}(B) \to \mathcal{F}(A)$ is an isomorphism. Let $\beta : J \to I$ be a map and let $\chi_j : V_j \to U_{\beta(j)}$ be isomorphisms over U which are assumed to exist by hypothesis. Let us show that the sheaf condition for $\mathcal{V}$ implies the sheaf condition for $\mathcal{U}$. Suppose given sections $s_i \in \mathcal{F}(U_i)$ such that

$$s_i|_{U_i \times_U U_{i'}} = s_{i'}|_{U_i \times_U U_{i'}}$$

in $\mathcal{F}(U_i \times_U U_{i'})$ for all pairs $(i, i') \in I \times I$. Then we can define $s_j = \chi_j^* s_{\beta(j)}$. For any pair $(j, j') \in J \times J$ the morphism $\chi_j \times_{\mathrm{id}_U} \chi_{j'} : V_j \times_U V_{j'} \to U_{\beta(j)} \times_U U_{\beta(j')}$ is an isomorphism as well. Hence by transport of structure we see that

$$s_j|_{V_j \times_U V_{j'}} = s_{j'}|_{V_j \times_U V_{j'}}$$

as well. The sheaf condition w.r.t. $\mathcal{V}$ implies there exists a unique s such that $s|_{V_j} = s_j$ for all $j \in J$. By the first remark of the proof this implies that $s|_{U_i} = s_i$ for all $i \in \mathrm{Im}(\beta)$ as well. Suppose that $i \in I$, $i \notin \mathrm{Im}(\beta)$. For such an i we have isomorphisms $U_i \to V_{\alpha(i)} \to U_{\beta(\alpha(i))}$ over U. This gives a morphism $U_i \to U_i \times_U U_{\beta(\alpha(i))}$ which is a section of the projection. Because s_i and $s_{\beta(\alpha(i))}$ restrict to the same element on the fibre product we conclude that $s_{\beta(\alpha(i))}$ pulls back to s_i via $U_i \to U_{\beta(\alpha(i))}$. Thus we see that also $s_i = s|_{U_i}$ as desired. □

0G1K **Lemma 8.5.** *Let $\mathcal{C}$ be a category. Let $\mathcal{V} = \{V_j \to U\}_{j \in J} \to \mathcal{U} = \{U_i \to U\}_{i \in I}$ be a morphism of families of maps with fixed target of $\mathcal{C}$ given by $id : U \to U$, $\alpha : J \to I$ and $f_j : V_j \to U_{\alpha(j)}$. Let $\mathcal{F}$ be a presheaf on $\mathcal{C}$. If $\mathcal{F}(U) \to \prod_{j \in J} \mathcal{F}(V_j)$ is injective then $\mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i)$ is injective.*

Proof. Omitted. □

0G1L **Lemma 8.6.** *Let $\mathcal{C}$ be a category. Let $\mathcal{V} = \{V_j \to U\}_{j \in J} \to \mathcal{U} = \{U_i \to U\}_{i \in I}$ be a morphism of families of maps with fixed target of $\mathcal{C}$ given by $id : U \to U$, $\alpha : J \to I$ and $f_j : V_j \to U_{\alpha(j)}$. Let $\mathcal{F}$ be a presheaf on $\mathcal{C}$. If*

1. *the fibre products $U_i \times_U U_{i'}$, $U_i \times_U V_j$, $V_j \times_U V_{j'}$ exist,*
2. *$\mathcal{F}$ satisfies the sheaf condition with respect to $\mathcal{V}$, and*
3. *for every $i \in I$ the map $\mathcal{F}(U_i) \to \prod_{j \in J} \mathcal{F}(V_j \times_U U_i)$ is injective.*

Then $\mathcal{F}$ satisfies the sheaf condition with respect to $\mathcal{U}$.

Proof. By Lemma 8.5 the map $\mathcal{F}(U) \to \prod \mathcal{F}(U_i)$ is injective. Suppose given $s_i \in \mathcal{F}(U_i)$ such that $s_i|_{U_i \times_U U_{i'}} = s_{i'}|_{U_i \times_U U_{i'}}$ for all $i, i' \in I$. Set $s_j = f_j^*(s_{\alpha(j)}) \in \mathcal{F}(V_j)$. Since the morphisms f_j are morphisms over U we obtain induced morphisms $f_{jj'} : V_j \times_U V_{j'} \to U_{\alpha(i)} \times_U U_{\alpha(i')}$ compatible with the $f_j, f_{j'}$ via the projection maps. It follows that

$$s_j|_{V_j \times_U V_{j'}} = f_{jj'}^*(s_{\alpha(j)}|_{U_{\alpha(j)} \times_U U_{\alpha(j')}}) = f_{jj'}^*(s_{\alpha(j')}|_{U_{\alpha(j)} \times_U U_{\alpha(j')}}) = s_{j'}|_{V_j \times_U V_{j'}}$$

for all $j, j' \in J$. Hence, by the sheaf condition for $\mathcal{F}$ with respect to $\mathcal{V}$, we get a section $s \in \mathcal{F}(U)$ which restricts to s_j on each V_j. We are done if we show s restricts to s_i on U_i for any $i \in I$. Since $\mathcal{F}$ satisfies (3) it suffices to show that s and s_i restrict to the same element over $U_i \times_U V_j$ for all $j \in J$. To see this we use

$$s|_{U_i \times_U V_j} = s_j|_{U_i \times_U V_j} = (\mathrm{id} \times f_j)^* s_{\alpha(j)}|_{U_i \times_U U_{\alpha(j)}} = (\mathrm{id} \times f_j)^* s_i|_{U_i \times_U U_{\alpha(j)}} = s_i|_{U_i \times_U V_j}$$

as desired. □

00VX **Lemma 8.7.** *Let $\mathcal{C}$ be a category. Let Cov_i, $i = 1, 2$ be two sets of families of morphisms with fixed target which each define the structure of a site on $\mathcal{C}$.*

(1) *If every $\mathcal{U} \in Cov_1$ is tautologically equivalent to some $\mathcal{V} \in Cov_2$, then $Sh(\mathcal{C}, Cov_2) \subset Sh(\mathcal{C}, Cov_1)$. If also, every $\mathcal{U} \in Cov_2$ is tautologically equivalent to some $\mathcal{V} \in Cov_1$ then the category of sheaves are equal.*
(2) *Suppose that for each $\mathcal{U} \in Cov_1$ there exists a $\mathcal{V} \in Cov_2$ such that $\mathcal{V}$ refines $\mathcal{U}$. In this case $Sh(\mathcal{C}, Cov_2) \subset Sh(\mathcal{C}, Cov_1)$. If also for every $\mathcal{U} \in Cov_2$ there exists a $\mathcal{V} \in Cov_1$ such that $\mathcal{V}$ refines $\mathcal{U}$, then the categories of sheaves are equal.*

Proof. Part (1) follows directly from Lemma 8.4 and the definitions.

Proof of (2). Let $\mathcal{F}$ be a sheaf of sets for the site $(\mathcal{C}, \mathrm{Cov}_2)$. Let $\mathcal{U} \in \mathrm{Cov}_1$, say $\mathcal{U} = \{U_i \to U\}_{i \in I}$. By assumption we may choose a refinement $\mathcal{V} \in \mathrm{Cov}_2$ of $\mathcal{U}$, say $\mathcal{V} = \{V_j \to U\}_{j \in J}$ and refinement given by $\alpha : J \to I$ and $f_j : V_j \to U_{\alpha(j)}$. Observe that $\mathcal{F}$ satisfies the sheaf condition for $\mathcal{V}$ and for the coverings $\{V_j \times_U U_i \to U_i\}_{j \in J}$ as these are in Cov_2. Hence $\mathcal{F}$ satisfies the sheaf condition for $\mathcal{U}$ by Lemma 8.6. □

00VY **Lemma 8.8.** *Let $\mathcal{C}$ be a category. Let $Cov(\mathcal{C})$ be a proper class of coverings satisfying conditions (1), (2) and (3) of Definition 6.2. Let $Cov_1, Cov_2 \subset Cov(\mathcal{C})$ be two subsets of $Cov(\mathcal{C})$ which endow $\mathcal{C}$ with the structure of a site. If every covering $\mathcal{U} \in Cov(\mathcal{C})$ is combinatorially equivalent to a covering in Cov_1 and combinatorially equivalent to a covering in Cov_2, then $Sh(\mathcal{C}, Cov_1) = Sh(\mathcal{C}, Cov_2)$.*

Proof. This is clear from Lemmas 8.7 and 8.3 above as the hypothesis implies that every covering $\mathcal{U} \in \mathrm{Cov}_1 \subset \mathrm{Cov}(\mathcal{C})$ is combinatorially equivalent to an element of Cov_2, and similarly with the roles of Cov_1 and Cov_2 reversed. □

9. The example of G-sets

00VZ As an example, consider the site $\mathcal{T}_G$ of Example 6.5. We will describe the category of sheaves on $\mathcal{T}_G$. The answer will turn out to be independent of the choices made in defining $\mathcal{T}_G$. In fact, during the proof we will need only the following properties of the site $\mathcal{T}_G$:

(a) $\mathcal{T}_G$ is a full subcategory of *G-Sets*,
(b) $\mathcal{T}_G$ contains the G-set ${}_GG$,
(c) $\mathcal{T}_G$ has fibre products and they are the same as in *G-Sets*,
(d) given $U \in \mathrm{Ob}(\mathcal{T}_G)$ and a G-invariant subset $O \subset U$, there exists an object of $\mathcal{T}_G$ isomorphic to O, and
(e) any surjective family of maps $\{U_i \to U\}_{i \in I}$, with $U, U_i \in \mathrm{Ob}(\mathcal{T}_G)$ is combinatorially equivalent to a covering of $\mathcal{T}_G$.

These properties hold by Sets, Lemmas 10.2 and 11.1.

Remark that the map

$$\mathrm{Hom}_G({}_GG, {}_GG) \longrightarrow G^{opp}, \varphi \longmapsto \varphi(1)$$

is an isomorphism of groups. The inverse map sends $g \in G$ to the map $R_g : s \mapsto sg$ (i.e. right multiplication). Note that $R_{g_1 g_2} = R_{g_2} \circ R_{g_1}$ so the opposite is necessary.

This implies that for every presheaf $\mathcal{F}$ on $\mathcal{T}_G$ the value $\mathcal{F}({}_GG)$ inherits the structure of a G-set as follows: $g \cdot s$ for $g \in G$ and $s \in \mathcal{F}({}_GG)$ defined by $\mathcal{F}(R_g)(s)$. This is a left action because

$$(g_1 g_2) \cdot s = \mathcal{F}(R_{g_1 g_2})(s) = \mathcal{F}(R_{g_2} \circ R_{g_1})(s) = \mathcal{F}(R_{g_1})(\mathcal{F}(R_{g_2})(s)) = g_1 \cdot (g_2 \cdot s).$$

Here we've used that $\mathcal{F}$ is contravariant. Note that if $\mathcal{F} \to \mathcal{G}$ is a morphism of presheaves of sets on $\mathcal{T}_G$ then we get a map $\mathcal{F}({}_GG) \to \mathcal{G}({}_GG)$ which is compatible with the G-actions we have just defined. All in all we have constructed a functor

$$PSh(\mathcal{T}_G) \longrightarrow G\text{-}Sets, \quad \mathcal{F} \longmapsto \mathcal{F}({}_GG).$$

We leave it to the reader to verify that this construction has the pleasing property that the representable presheaf h_U is mapped to something canonically isomorphic to U. In a formula $h_U({}_GG) = \mathrm{Hom}_G({}_GG, U) \cong U$.

Suppose that S is a G-set. We define a presheaf $\mathcal{F}_S$ by the formula[2]

$$\mathcal{F}_S(U) = \mathrm{Mor}_{G\text{-}Sets}(U, S).$$

This is clearly a presheaf. On the other hand, suppose that $\{U_i \to U\}_{i \in I}$ is a covering in $\mathcal{T}_G$. This implies that $\coprod_i U_i \to U$ is surjective. Thus it is clear that the map

$$\mathcal{F}_S(U) = \mathrm{Mor}_{G\text{-}Sets}(U, S) \longrightarrow \prod \mathcal{F}_S(U_i) = \prod \mathrm{Mor}_{G\text{-}Sets}(U_i, S)$$

is injective. And, given a family of G-equivariant maps $s_i : U_i \to S$, such that all the diagrams

$$\begin{array}{ccc} U_i \times_U U_j & \longrightarrow & U_j \\ \downarrow & & \downarrow{\scriptstyle s_j} \\ U_i & \xrightarrow{s_i} & S \end{array}$$

commute, there is a unique G-equivariant map $s : U \to S$ such that s_i is the composition $U_i \to U \to S$. Namely, we just define $s(u) = s_i(u_i)$ where $i \in I$ is any index such that there exists some $u_i \in U_i$ mapping to u under the map $U_i \to U$. The commutativity of the diagrams above implies exactly that this construction is well defined. All in all we have constructed a functor

$$G\text{-}Sets \longrightarrow Sh(\mathcal{T}_G), \quad S \longmapsto \mathcal{F}_S.$$

We now have the following diagram of categories and functors

$$\begin{array}{ccc} PSh(\mathcal{T}_G) & \xrightarrow{\mathcal{F} \mapsto \mathcal{F}({}_GG)} & G\text{-}Sets \\ & \nwarrow \quad \swarrow_{S \mapsto \mathcal{F}_S} & \\ & Sh(\mathcal{T}_G) & \end{array}$$

It is immediate from the definitions that $\mathcal{F}_S({}_GG) = \mathrm{Mor}_G({}_GG, S) = S$, the last equality by evaluation at 1. This almost proves the following.

00W0 **Proposition 9.1.** *The functors $\mathcal{F} \mapsto \mathcal{F}({}_GG)$ and $S \mapsto \mathcal{F}_S$ define quasi-inverse equivalences between $Sh(\mathcal{T}_G)$ and G-Sets.*

Proof. We have already seen that composing the functors one way around is isomorphic to the identity functor. In the other direction, for any sheaf $\mathcal{H}$ there is a natural map of sheaves

$$can : \mathcal{H} \longrightarrow \mathcal{F}_{\mathcal{H}({}_GG)}.$$

[2]It may appear this is the representable presheaf defined by S. This may not be the case because S may not be an object of $\mathcal{T}_G$ which was chosen to be a sufficiently large set of G-sets.

Namely, for any object U of $\mathcal{T}_G$ we let can_U be the map

$$\begin{array}{ccc} \mathcal{H}(U) & \longrightarrow & \mathcal{F}_{\mathcal{H}({}_GG)}(U) = \mathrm{Mor}_G(U, \mathcal{H}({}_GG)) \\ s & \longmapsto & (u \mapsto \alpha_u^* s). \end{array}$$

Here $\alpha_u : {}_GG \to U$ is the map $\alpha_u(g) = gu$ and $\alpha_u^* : \mathcal{H}(U) \to \mathcal{H}({}_GG)$ is the pullback map. A trivial but confusing verification shows that this is indeed a map of presheaves. We have to show that *can* is an isomorphism. We do this by showing can_U is an isomorphism for all $U \in \mathrm{Ob}(\mathcal{T}_G)$. We leave the (important but easy) case that $U = {}_GG$ to the reader. A general object U of $\mathcal{T}_G$ is a disjoint union of G-orbits: $U = \coprod_{i \in I} O_i$. The family of maps $\{O_i \to U\}_{i \in I}$ is tautologically equivalent to a covering in $\mathcal{T}_G$ (by the properties of $\mathcal{T}_G$ listed at the beginning of this section). Hence by Lemma 8.4 the sheaf $\mathcal{H}$ satisfies the sheaf property with respect to $\{O_i \to U\}_{i \in I}$. The sheaf property for this covering implies $\mathcal{H}(U) = \prod_i \mathcal{H}(O_i)$. Hence it suffices to show that can_U is an isomorphism when U consists of a single G-orbit. Let $u \in U$ and let $H \subset G$ be its stabilizer. Clearly, $\mathrm{Mor}_G(U, \mathcal{H}({}_GG)) = \mathcal{H}({}_GG)^H$ equals the subset of H-invariant elements. On the other hand consider the covering $\{{}_GG \to U\}$ given by $g \mapsto gu$ (again it is just combinatorially equivalent to some covering of $\mathcal{T}_G$, and again this doesn't matter). Note that the fibre product $({}_GG) \times_U ({}_GG)$ is equal to $\{(g, gh), g \in G, h \in H\} \cong \coprod_{h \in H} {}_GG$. Hence the sheaf property for this covering reads as

$$\mathcal{H}(U) \longrightarrow \mathcal{H}({}_GG) \underset{\mathrm{pr}_1^*}{\overset{\mathrm{pr}_0^*}{\rightrightarrows}} \prod_{h \in H} \mathcal{H}({}_GG).$$

The two maps pr_i^* into the factor $\mathcal{H}({}_GG)$ differ by multiplication by h. Now the result follows from this and the fact that *can* is an isomorphism for $U = {}_GG$. □

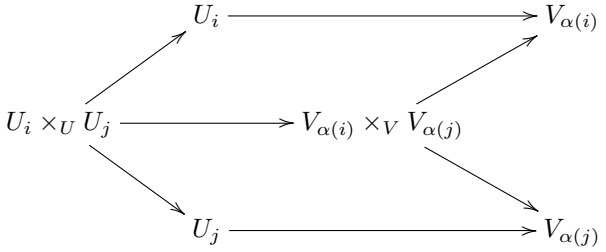
10. Sheafification

00W1 In order to define the sheafification we study the zeroth Čech cohomology group of a covering and its functoriality properties.

Let $\mathcal{F}$ be a presheaf of sets on $\mathcal{C}$, and let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be a covering of $\mathcal{C}$. Let us use the notation $\mathcal{F}(\mathcal{U})$ to indicate the equalizer

$$H^0(\mathcal{U}, \mathcal{F}) = \{(s_i)_{i \in I} \in \prod\nolimits_i \mathcal{F}(U_i) \mid s_i|_{U_i \times_U U_j} = s_j|_{U_i \times_U U_j} \ \forall i, j \in I\}.$$

As we will see later, this is the zeroth Čech cohomology of $\mathcal{F}$ over U with respect to the covering $\mathcal{U}$. A small remark is that we can define $H^0(\mathcal{U}, \mathcal{F})$ as soon as all the morphisms $U_i \to U$ are representable, i.e., $\mathcal{U}$ need not be a covering of the site. There is a canonical map $\mathcal{F}(U) \to H^0(\mathcal{U}, \mathcal{F})$. It is clear that a morphism of coverings $\mathcal{U} \to \mathcal{V}$ induces commutative diagrams



This in turn produces a map $H^0(\mathcal{V}, \mathcal{F}) \to H^0(\mathcal{U}, \mathcal{F})$, compatible with the map $\mathcal{F}(V) \to \mathcal{F}(U)$.

By construction, a presheaf $\mathcal{F}$ is a sheaf if and only if for every covering $\mathcal{U}$ of $\mathcal{C}$ the natural map $\mathcal{F}(U) \to H^0(\mathcal{U}, \mathcal{F})$ is bijective. We will use this notion to prove the following simple lemma about limits of sheaves.

00W2 **Lemma 10.1.** *Let $\mathcal{F} : \mathcal{I} \to Sh(\mathcal{C})$ be a diagram. Then $\lim_{\mathcal{I}} \mathcal{F}$ exists and is equal to the limit in the category of presheaves.*

Proof. Let $\lim_i \mathcal{F}_i$ be the limit as a presheaf. We will show that this is a sheaf and then it will trivially follow that it is a limit in the category of sheaves. To prove the sheaf property, let $\mathcal{V} = \{V_j \to V\}_{j \in J}$ be a covering. Let $(s_j)_{j \in J}$ be an element of $H^0(\mathcal{V}, \lim_i \mathcal{F}_i)$. Using the projection maps we get elements $(s_{j,i})_{j \in J}$ in $H^0(\mathcal{V}, \mathcal{F}_i)$. By the sheaf property for $\mathcal{F}_i$ we see that there is a unique $s_i \in \mathcal{F}_i(V)$ such that $s_{j,i} = s_i|_{V_j}$. Let $\phi : i \to i'$ be a morphism of the index category. We would like to show that $\mathcal{F}(\phi) : \mathcal{F}_i \to \mathcal{F}_{i'}$ maps s_i to $s_{i'}$. We know this is true for the sections $s_{i,j}$ and $s_{i',j}$ for all j and hence by the sheaf property for $\mathcal{F}_{i'}$ this is true. At this point we have an element $s = (s_i)_{i \in \mathrm{Ob}(\mathcal{I})}$ of $(\lim_i \mathcal{F}_i)(V)$. We leave it to the reader to see this element has the required property that $s_j = s|_{V_j}$. □

00W3 **Example 10.2.** A particular example is the limit over the empty diagram. This gives the final object in the category of (pre)sheaves. It is the presheaf that associates to each object U of $\mathcal{C}$ a singleton set, with unique restriction mappings and moreover this presheaf is a sheaf. We often denote this sheaf by $*$.

Let $\mathcal{J}_U$ be the category of all coverings of U. In other words, the objects of $\mathcal{J}_U$ are the coverings of U in $\mathcal{C}$, and the morphisms are the refinements. By our conventions on sites this is indeed a category, i.e., the collection of objects and morphisms forms a set. Note that $\mathrm{Ob}(\mathcal{J}_U)$ is not empty since $\{\mathrm{id}_U\}$ is an object of it. According to the remarks above the construction $\mathcal{U} \mapsto H^0(\mathcal{U}, \mathcal{F})$ is a contravariant functor on $\mathcal{J}_U$. We define

$$\mathcal{F}^+(U) = \mathrm{colim}_{\mathcal{J}_U^{opp}} H^0(\mathcal{U}, \mathcal{F})$$

See Categories, Section 14 for a discussion of limits and colimits. We point out that later we will see that $\mathcal{F}^+(U)$ is the zeroth Čech cohomology of $\mathcal{F}$ over U.

Before we say more about the structure of the colimit, we turn the collection of sets $\mathcal{F}^+(U)$, $U \in \mathrm{Ob}(\mathcal{C})$ into a presheaf. Namely, let $V \to U$ be a morphism of $\mathcal{C}$. By the axioms of a site there is a functor[3]

$$\mathcal{J}_U \longrightarrow \mathcal{J}_V, \quad \{U_i \to U\} \longmapsto \{U_i \times_U V \to V\}.$$

Note that the projection maps furnish a functorial morphism of coverings $\{U_i \times_U V \to V\} \to \{U_i \to U\}$ and hence, by the construction above, a functorial map of sets $H^0(\{U_i \to U\}, \mathcal{F}) \to H^0(\{U_i \times_U V \to V\}, \mathcal{F})$. In other words, there is a transformation of functors from $H^0(-, \mathcal{F}) : \mathcal{J}_U^{opp} \to Sets$ to the composition $\mathcal{J}_U^{opp} \to \mathcal{J}_V^{opp} \xrightarrow{H^0(-,\mathcal{F})} Sets$. Hence by generalities of colimits we obtain a canonical map $\mathcal{F}^+(U) \to \mathcal{F}^+(V)$. In terms of the description of the set $\mathcal{F}^+(U)$ above, it just takes the element associated with $s = (s_i) \in H^0(\{U_i \to U\}, \mathcal{F})$ to the element associated with $(s_i|_{V \times_U U_i}) \in H^0(\{U_i \times_U V \to V\}, \mathcal{F})$.

[3]This construction actually involves a choice of the fibre products $U_i \times_U V$ and hence the axiom of choice. The resulting map does not depend on the choices made, see below.

00W4 **Lemma 10.3.** *The constructions above define a presheaf $\mathcal{F}^+$ together with a canonical map of presheaves $\mathcal{F} \to \mathcal{F}^+$.*

Proof. All we have to do is to show that given morphisms $W \to V \to U$ the composition $\mathcal{F}^+(U) \to \mathcal{F}^+(V) \to \mathcal{F}^+(W)$ equals the map $\mathcal{F}^+(U) \to \mathcal{F}^+(W)$. This can be shown directly by verifying that, given a covering $\{U_i \to U\}$ and $s = (s_i) \in H^0(\{U_i \to U\}, \mathcal{F})$, we have canonically $W \times_U U_i \cong W \times_V (V \times_U U_i)$, and $s_i|_{W \times_U U_i}$ corresponds to $(s_i|_{V \times_U U_i})|_{W \times_V (V \times_U U_i)}$ via this isomorphism. □

More indirectly, the result of Lemma 10.6 shows that we may pullback an element s as above via any morphism from any covering of W to $\{U_i \to U\}$ and we will always end up with the same element in $\mathcal{F}^+(W)$.

00W5 **Lemma 10.4.** *The association $\mathcal{F} \mapsto (\mathcal{F} \to \mathcal{F}^+)$ is a functor.*

Proof. Instead of proving this we state exactly what needs to be proven. Let $\mathcal{F} \to \mathcal{G}$ be a map of presheaves. Prove the commutativity of:

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & \mathcal{F}^+ \\ \downarrow & & \downarrow \\ \mathcal{G} & \longrightarrow & \mathcal{G}^+ \end{array}$$

□

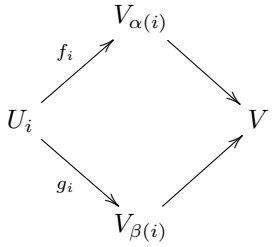
The next two lemmas imply that the colimits above are colimits over a directed set.

00W6 **Lemma 10.5.** *Given a pair of coverings $\{U_i \to U\}$ and $\{V_j \to U\}$ of a given object U of the site $\mathcal{C}$, there exists a covering which is a common refinement.*

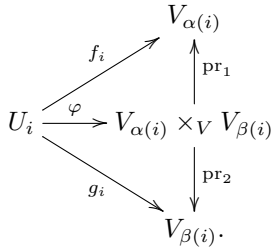
Proof. Since $\mathcal{C}$ is a site we have that for every i the family $\{V_j \times_U U_i \to U_i\}_j$ is a covering. And, then another axiom implies that $\{V_j \times_U U_i \to U\}_{i,j}$ is a covering of U. Clearly this covering refines both given coverings. □

00W7 **Lemma 10.6.** *Any two morphisms $f, g : \mathcal{U} \to \mathcal{V}$ of coverings inducing the same morphism $U \to V$ induce the same map $H^0(\mathcal{V}, \mathcal{F}) \to H^0(\mathcal{U}, \mathcal{F})$.*

Proof. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ and $\mathcal{V} = \{V_j \to V\}_{j \in J}$. The morphism f consists of a map $U \to V$, a map $\alpha : I \to J$ and maps $f_i : U_i \to V_{\alpha(i)}$. Likewise, g determines a map $\beta : I \to J$ and maps $g_i : U_i \to V_{\beta(i)}$. As f and g induce the same map $U \to V$, the diagram



is commutative for every $i \in I$. Hence f and g factor through the fibre product



Now let $s = (s_j)_j \in H^0(\mathcal{V}, \mathcal{F})$. Then for all $i \in I$:

$$(f^*s)_i = f_i^*(s_{\alpha(i)}) = \varphi^*\mathrm{pr}_1^*(s_{\alpha(i)}) = \varphi^*\mathrm{pr}_2^*(s_{\beta(i)}) = g_i^*(s_{\beta(i)}) = (g^*s)_i,$$

where the middle equality is given by the definition of $H^0(\mathcal{V}, \mathcal{F})$. This shows that the maps $H^0(\mathcal{V}, \mathcal{F}) \to H^0(\mathcal{U}, \mathcal{F})$ induced by f and g are equal. □

00W8 **Remark 10.7.** In particular this lemma shows that if $\mathcal{U}$ is a refinement of $\mathcal{V}$, and if $\mathcal{V}$ is a refinement of $\mathcal{U}$, then there is a canonical identification $H^0(\mathcal{U}, \mathcal{F}) = H^0(\mathcal{V}, \mathcal{F})$.

From these two lemmas, and the fact that $\mathcal{J}_U$ is nonempty, it follows that the diagram $H^0(-, \mathcal{F}) : \mathcal{J}_U^{opp} \to \mathit{Sets}$ is filtered, see Categories, Definition 19.1. Hence, by Categories, Section 19 the colimit $\mathcal{F}^+(U)$ may be described in the following straightforward manner. Namely, every element in the set $\mathcal{F}^+(U)$ arises from an element $s \in H^0(\mathcal{U}, \mathcal{F})$ for some covering $\mathcal{U}$ of U. Given a second element $s' \in H^0(\mathcal{U}', \mathcal{F})$ then s and s' determine the same element of the colimit if and only if there exists a covering $\mathcal{V}$ of U and refinements $f : \mathcal{V} \to \mathcal{U}$ and $f' : \mathcal{V} \to \mathcal{U}'$ such that $f^*s = (f')^*s'$ in $H^0(\mathcal{V}, \mathcal{F})$. Since the trivial covering $\{\mathrm{id}_U\}$ is an object of $\mathcal{J}_U$ we get a canonical map $\mathcal{F}(U) \to \mathcal{F}^+(U)$.

00W9 **Lemma 10.8.** *The map $\theta : \mathcal{F} \to \mathcal{F}^+$ has the following property: For every object U of $\mathcal{C}$ and every section $s \in \mathcal{F}^+(U)$ there exists a covering $\{U_i \to U\}$ such that $s|_{U_i}$ is in the image of $\theta : \mathcal{F}(U_i) \to \mathcal{F}^+(U_i)$.*

Proof. Namely, let $\{U_i \to U\}$ be a covering such that s arises from the element $(s_i) \in H^0(\{U_i \to U\}, \mathcal{F})$. According to Lemma 10.6 we may consider the covering $\{U_i \to U_i\}$ and the (obvious) morphism of coverings $\{U_i \to U_i\} \to \{U_i \to U\}$ to compute the pullback of s to an element of $\mathcal{F}^+(U_i)$. And indeed, using this covering we get exactly $\theta(s_i)$ for the restriction of s to U_i. □

00WA **Definition 10.9.** We say that a presheaf of sets $\mathcal{F}$ on a site $\mathcal{C}$ is *separated* if, for all coverings of $\{U_i \to U\}$, the map $\mathcal{F}(U) \to \prod \mathcal{F}(U_i)$ is injective.

00WB **Theorem 10.10.** *With $\mathcal{F}$ as above*

00WC (1) *The presheaf $\mathcal{F}^+$ is separated.*

00WD (2) *If $\mathcal{F}$ is separated, then $\mathcal{F}^+$ is a sheaf and the map of presheaves $\mathcal{F} \to \mathcal{F}^+$ is injective.*

00WE (3) *If $\mathcal{F}$ is a sheaf, then $\mathcal{F} \to \mathcal{F}^+$ is an isomorphism.*

00WF (4) *The presheaf $\mathcal{F}^{++}$ is always a sheaf.*

Proof. Proof of (1). Suppose that $s, s' \in \mathcal{F}^+(U)$ and suppose that there exists some covering $\{U_i \to U\}$ such that $s|_{U_i} = s'|_{U_i}$ for all i. We now have three coverings of U: the covering $\{U_i \to U\}$ above, a covering $\mathcal{U}$ for s as in Lemma

10.8, and a similar covering $\mathcal{U}'$ for s'. By Lemma 10.5, we can find a common refinement, say $\{W_j \to U\}$. This means we have $s_j, s'_j \in \mathcal{F}(W_j)$ such that $s|_{W_j} = \theta(s_j)$, similarly for $s'|_{W_j}$, and such that $\theta(s_j) = \theta(s'_j)$. This last equality means that there exists some covering $\{W_{jk} \to W_j\}$ such that $s_j|_{W_{jk}} = s'_j|_{W_{jk}}$. Then since $\{W_{jk} \to U\}$ is a covering we see that s, s' map to the same element of $H^0(\{W_{jk} \to U\}, \mathcal{F})$ as desired.

Proof of (2). It is clear that $\mathcal{F} \to \mathcal{F}^+$ is injective because all the maps $\mathcal{F}(U) \to H^0(\mathcal{U}, \mathcal{F})$ are injective. It is also clear that, if $\mathcal{U} \to \mathcal{U}'$ is a refinement, then $H^0(\mathcal{U}', \mathcal{F}) \to H^0(\mathcal{U}, \mathcal{F})$ is injective. Now, suppose that $\{U_i \to U\}$ is a covering, and let (s_i) be a family of elements of $\mathcal{F}^+(U_i)$ satisfying the sheaf condition $s_i|_{U_i \times_U U_{i'}} = s_{i'}|_{U_i \times_U U_{i'}}$ for all $i, i' \in I$. Choose coverings (as in Lemma 10.8) $\{U_{ij} \to U_i\}$ such that $s_i|_{U_{ij}}$ is the image of the (unique) element $s_{ij} \in \mathcal{F}(U_{ij})$. The sheaf condition implies that s_{ij} and $s_{i'j'}$ agree over $U_{ij} \times_U U_{i'j'}$ because it maps to $U_i \times_U U_{i'}$ and we have the equality there. Hence $(s_{ij}) \in H^0(\{U_{ij} \to U\}, \mathcal{F})$ gives rise to an element $s \in \mathcal{F}^+(U)$. We leave it to the reader to verify that $s|_{U_i} = s_i$.

Proof of (3). This is immediate from the definitions because the sheaf property says exactly that every map $\mathcal{F} \to H^0(\mathcal{U}, \mathcal{F})$ is bijective (for every covering $\mathcal{U}$ of U).

Statement (4) is now obvious. □

00WG **Definition 10.11.** Let $\mathcal{C}$ be a site and let $\mathcal{F}$ be a presheaf of sets on $\mathcal{C}$. The sheaf $\mathcal{F}^\# := \mathcal{F}^{++}$ together with the canonical map $\mathcal{F} \to \mathcal{F}^\#$ is called the *sheaf associated to* $\mathcal{F}$.

00WH **Proposition 10.12.** *The canonical map $\mathcal{F} \to \mathcal{F}^\#$ has the following universal property: For any map $\mathcal{F} \to \mathcal{G}$, where $\mathcal{G}$ is a sheaf of sets, there is a unique map $\mathcal{F}^\# \to \mathcal{G}$ such that $\mathcal{F} \to \mathcal{F}^\# \to \mathcal{G}$ equals the given map.*

Proof. By Lemma 10.4 we get a commutative diagram

$$\begin{array}{ccccc} \mathcal{F} & \longrightarrow & \mathcal{F}^+ & \longrightarrow & \mathcal{F}^{++} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{G} & \longrightarrow & \mathcal{G}^+ & \longrightarrow & \mathcal{G}^{++} \end{array}$$

and by Theorem 10.10 the lower horizontal maps are isomorphisms. The uniqueness follows from Lemma 10.8 which says that every section of $\mathcal{F}^\#$ locally comes from sections of $\mathcal{F}$. □

It is clear from this result that the functor $\mathcal{F} \mapsto (\mathcal{F} \to \mathcal{F}^\#)$ is unique up to unique isomorphism of functors. Actually, let us temporarily denote $i : Sh(\mathcal{C}) \to PSh(\mathcal{C})$ the functor of inclusion. The result above actually says that

$$\mathrm{Mor}_{PSh(\mathcal{C})}(\mathcal{F}, i(\mathcal{G})) = \mathrm{Mor}_{Sh(\mathcal{C})}(\mathcal{F}^\#, \mathcal{G}).$$

In other words, the functor of sheafification is the left adjoint to the inclusion functor i. We finish this section with a couple of lemmas.

00WI **Lemma 10.13.** *Let $\mathcal{F} : \mathcal{I} \to Sh(\mathcal{C})$ be a diagram. Then $\mathrm{colim}_\mathcal{I} \mathcal{F}$ exists and is the sheafification of the colimit in the category of presheaves.*

Proof. Since the sheafification functor is a left adjoint it commutes with all colimits, see Categories, Lemma 24.5. Hence, since $PSh(\mathcal{C})$ has colimits, we deduce that $Sh(\mathcal{C})$ has colimits (which are the sheafifications of the colimits in presheaves). □

00WJ **Lemma 10.14.** *The functor $PSh(\mathcal{C}) \to Sh(\mathcal{C})$, $\mathcal{F} \mapsto \mathcal{F}^\#$ is exact.*

Proof. Since it is a left adjoint it is right exact, see Categories, Lemma 24.6. On the other hand, by Lemmas 10.5 and Lemma 10.6 the colimits in the construction of $\mathcal{F}^+$ are really over the directed set $\mathrm{Ob}(\mathcal{J}_U)$ where $\mathcal{U} \geq \mathcal{U}'$ if and only if $\mathcal{U}$ is a refinement of $\mathcal{U}'$. Hence by Categories, Lemma 19.2 we see that $\mathcal{F} \to \mathcal{F}^+$ commutes with finite limits (as a functor from presheaves to presheaves). Then we conclude using Lemma 10.1. □

00WK **Lemma 10.15.** *Let $\mathcal{C}$ be a site. Let $\mathcal{F}$ be a presheaf of sets on $\mathcal{C}$. Denote $\theta^2 : \mathcal{F} \to \mathcal{F}^\#$ the canonical map of $\mathcal{F}$ into its sheafification. Let U be an object of $\mathcal{C}$. Let $s \in \mathcal{F}^\#(U)$. There exists a covering $\{U_i \to U\}$ and sections $s_i \in \mathcal{F}(U_i)$ such that*

(1) *$s|_{U_i} = \theta^2(s_i)$, and*
(2) *for every i, j there exists a covering $\{U_{ijk} \to U_i \times_U U_j\}$ of $\mathcal{C}$ such that the pullback of s_i and s_j to each U_{ijk} agree.*

Conversely, given any covering $\{U_i \to U\}$, elements $s_i \in \mathcal{F}(U_i)$ such that (2) holds, then there exists a unique section $s \in \mathcal{F}^\#(U)$ such that (1) holds.

Proof. Omitted. □

11. Injective and surjective maps of sheaves

00WL

00WM **Definition 11.1.** Let $\mathcal{C}$ be a site, and let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of sheaves of sets.

(1) We say that φ is *injective* if for every object U of $\mathcal{C}$ the map $\varphi : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective.
(2) We say that φ is *surjective* if for every object U of $\mathcal{C}$ and every section $s \in \mathcal{G}(U)$ there exists a covering $\{U_i \to U\}$ such that for all i the restriction $s|_{U_i}$ is in the image of $\varphi : \mathcal{F}(U_i) \to \mathcal{G}(U_i)$.

00WN **Lemma 11.2.** *The injective (resp. surjective) maps defined above are exactly the monomorphisms (resp. epimorphisms) of the category $Sh(\mathcal{C})$. A map of sheaves is an isomorphism if and only if it is both injective and surjective.*

Proof. Omitted. □

086K **Lemma 11.3.** *Let $\mathcal{C}$ be a site. Let $\mathcal{F} \to \mathcal{G}$ be a surjection of sheaves of sets. Then the diagram*

$$\mathcal{F} \times_{\mathcal{G}} \mathcal{F} \rightrightarrows \mathcal{F} \longrightarrow \mathcal{G}$$

represents $\mathcal{G}$ as a coequalizer.

Proof. Let $\mathcal{H}$ be a sheaf of sets and let $\varphi : \mathcal{F} \to \mathcal{H}$ be a map of sheaves equalizing the two maps $\mathcal{F} \times_{\mathcal{G}} \mathcal{F} \to \mathcal{F}$. Let $\mathcal{G}' \subset \mathcal{G}$ be the presheaf image of the map $\mathcal{F} \to \mathcal{G}$. As the product $\mathcal{F} \times_{\mathcal{G}} \mathcal{F}$ may be computed in the category of presheaves we see that it is equal to the presheaf product $\mathcal{F} \times_{\mathcal{G}'} \mathcal{F}$. Hence φ induces a unique map of presheaves $\psi' : \mathcal{G}' \to \mathcal{H}$. Since $\mathcal{G}$ is the sheafification of $\mathcal{G}'$ by Lemma 11.2 we conclude that ψ' extends uniquely to a map of sheaves $\psi : \mathcal{G} \to \mathcal{H}$. We omit the verification that φ is equal to the composition of ψ and the given map. □

12. Representable sheaves

00WO Let $\mathcal{C}$ be a category. The canonical topology is the finest topology such that all representable presheaves are sheaves (it is formally defined in Definition 47.12 but we will not need this). This topology is not always the topology associated to the structure of a site on $\mathcal{C}$. We will give a collection of coverings that generates this topology in case $\mathcal{C}$ has fibered products. First we give the following general definition.

00WP **Definition 12.1.** Let $\mathcal{C}$ be a category. We say that a family $\{U_i \to U\}_{i\in I}$ is an *effective epimorphism* if all the morphisms $U_i \to U$ are representable (see Categories, Definition 6.4), and for any $X \in \mathrm{Ob}(\mathcal{C})$ the sequence

$$\mathrm{Mor}_{\mathcal{C}}(U, X) \longrightarrow \prod_{i\in I} \mathrm{Mor}_{\mathcal{C}}(U_i, X) \rightrightarrows \prod_{(i,j)\in I^2} \mathrm{Mor}_{\mathcal{C}}(U_i \times_U U_j, X)$$

is an equalizer diagram. We say that a family $\{U_i \to U\}$ is a *universal effective epimorphism* if for any morphism $V \to U$ the base change $\{U_i \times_U V \to V\}$ is an effective epimorphism.

The class of families which are universal effective epimorphisms satisfies the axioms of Definition 6.2. If $\mathcal{C}$ has fibre products, then the associated topology is the canonical topology. (In this case, to get a site argue as in Sets, Lemma 11.1.)

Conversely, suppose that $\mathcal{C}$ is a site such that all representable presheaves are sheaves. Then clearly, all coverings are universal effective epimorphisms. Thus the following definition is the "correct" one in the setting of sites.

00WQ **Definition 12.2.** We say that the topology on a site $\mathcal{C}$ is *weaker than the canonical topology*, or that the topology is *subcanonical* if all the coverings of $\mathcal{C}$ are universal effective epimorphisms.

A representable sheaf is a representable presheaf which is also a sheaf. Since it is perhaps better to avoid this terminology when the topology is not subcanonical, we only define it formally in that case.

00WR **Definition 12.3.** Let $\mathcal{C}$ be a site whose topology is subcanonical. The Yoneda embedding h (see Categories, Section 3) presents $\mathcal{C}$ as a full subcategory of the category of sheaves of $\mathcal{C}$. In this case we call sheaves of the form h_U with $U \in \mathrm{Ob}(\mathcal{C})$ *representable sheaves* on $\mathcal{C}$. Notation: Sometimes, the representable sheaf h_U associated to U is denoted $\underline{U}$.

Note that we have in the situation of the definition

$$\mathrm{Mor}_{Sh(\mathcal{C})}(h_U, \mathcal{F}) = \mathcal{F}(U)$$

for every sheaf $\mathcal{F}$, since it holds for presheaves, see (2.1.1). In general the presheaves h_U are not sheaves and to get a sheaf you have to sheafify them. In this case we still have

090I (12.3.1) $$\mathrm{Mor}_{Sh(\mathcal{C})}(h_U^\#, \mathcal{F}) = \mathrm{Mor}_{PSh(\mathcal{C})}(h_U, \mathcal{F}) = \mathcal{F}(U)$$

for every sheaf $\mathcal{F}$. Namely, the first equality holds by the adjointness property of $\#$ and the second is (2.1.1).

00WT **Lemma 12.4.** *Let $\mathcal{C}$ be a site. If $\{U_i \to U\}_{i\in I}$ is a covering of the site $\mathcal{C}$, then the morphism of presheaves of sets*

$$\coprod_{i\in I} h_{U_i} \to h_U$$

becomes surjective after sheafification.

Proof. By Lemma 11.2 above we have to show that $\coprod_{i\in I} h_{U_i}^\# \to h_U^\#$ is an epimorphism. Let $\mathcal{F}$ be a sheaf of sets. A morphism $h_U^\# \to \mathcal{F}$ corresponds to a section $s \in \mathcal{F}(U)$. Hence the injectivity of $\mathrm{Mor}(h_U^\#, \mathcal{F}) \to \prod_i \mathrm{Mor}(h_{U_i}^\#, \mathcal{F})$ follows directly from the sheaf property of $\mathcal{F}$. □

The next lemma says, in the case the topology is weaker than the canonical topology, that every sheaf is made up out of representable sheaves in a way.

00WS **Lemma 12.5.** *Let $\mathcal{C}$ be a site. Let $E \subset \mathrm{Ob}(\mathcal{C})$ be a subset such that every object of $\mathcal{C}$ has a covering by elements of E. Let $\mathcal{F}$ be a sheaf of sets. There exists a diagram of sheaves of sets*

$$\mathcal{F}_1 \rightrightarrows \mathcal{F}_0 \longrightarrow \mathcal{F}$$

which represents $\mathcal{F}$ as a coequalizer, such that $\mathcal{F}_i$, $i = 0, 1$ are coproducts of sheaves of the form $h_U^\#$ with $U \in E$.

Proof. First we show there is an epimorphism $\mathcal{F}_0 \to \mathcal{F}$ of the desired type. Namely, just take

$$\mathcal{F}_0 = \coprod\nolimits_{U\in E, s\in\mathcal{F}(U)} (h_U)^\# \longrightarrow \mathcal{F}$$

Here the arrow restricted to the component corresponding to (U, s) maps the element $\mathrm{id}_U \in h_U^\#(U)$ to the section $s \in \mathcal{F}(U)$. This is an epimorphism according to Lemma 11.2 and our condition on E. To construct $\mathcal{F}_1$ first set $\mathcal{G} = \mathcal{F}_0 \times_{\mathcal{F}} \mathcal{F}_0$ and then construct an epimorphism $\mathcal{F}_1 \to \mathcal{G}$ as above. See Lemma 11.3. □

0GLW **Lemma 12.6.** *Let $\mathcal{C}$ be a site. Let $\mathcal{F}$ be a sheaf of sets on $\mathcal{C}$. Then there exists a diagram $\mathcal{I} \to \mathcal{C}$, $i \mapsto U_i$ such that*

$$\mathcal{F} = \mathrm{colim}_{i\in\mathcal{I}} h_{U_i}^\#$$

Moreover, if $E \subset \mathrm{Ob}(\mathcal{C})$ is a subset such that every object of $\mathcal{C}$ has a covering by elements of E, then we may assume U_i is an element of E for all $i \in \mathrm{Ob}(\mathcal{I})$.

Proof. Let $\mathcal{I}$ be the category whose objects are pairs (U, s) with $U \in \mathrm{Ob}(\mathcal{C})$ and $s \in \mathcal{F}(U)$ and whose morphisms $(U, s) \to (U', s')$ are morphisms $f : U \to U'$ in $\mathcal{C}$ with $f^*s' = s$. For each object (U, s) of $\mathcal{I}$ the element s defines by the Yoneda lemma a map $s : h_U \to \mathcal{F}$ of presheaves. Hence by the universal property of sheafification a map $h_U^\# \to \mathcal{F}$. These maps are immediately seen to be compatible with the morphisms in the category $\mathcal{I}$. Hence we obtain a map $\mathrm{colim}_{(U,s)} h_U \to \mathcal{F}$ of presheaves (where the colimit is taken in the category of presheaves) and a map $\mathrm{colim}_{(U,s)}(h_U)^\# \to \mathcal{F}$ of sheaves (where the colimit is taken in the category of sheaves). Since sheafification is the left adjoint to the inclusion functor $Sh(\mathcal{C}) \to PSh(\mathcal{C})$ (Proposition 10.12) we have $\mathrm{colim}(h_U)^\# = (\mathrm{colim}\, h_U)^\#$ by Categories, Lemma 24.5. Thus it suffices to show that $\mathrm{colim}_{(U,s)} h_U \to \mathcal{F}$ is an isomorphism of presheaves. To see this we show that for every object X of $\mathcal{C}$ the map $\mathrm{colim}_{(U,s)} h_U(X) \to \mathcal{F}(X)$ is bijective. Namely, it has an inverse sending the element $t \in \mathcal{F}(X)$ to the element of the set $\mathrm{colim}_{(U,s)} h_U(X)$ corresponding to (X, t) and $\mathrm{id}_X \in h_X(X)$.

We omit the proof of the final statement. □

13. Continuous functors

00WU

00WV **Definition 13.1.** Let $\mathcal{C}$ and $\mathcal{D}$ be sites. A functor $u : \mathcal{C} \to \mathcal{D}$ is called *continuous* if for every $\{V_i \to V\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})$ we have the following

(1) $\{u(V_i) \to u(V)\}_{i\in I}$ is in $\mathrm{Cov}(\mathcal{D})$, and
(2) for any morphism $T \to V$ in $\mathcal{C}$ the morphism $u(T \times_V V_i) \to u(T) \times_{u(V)} u(V_i)$ is an isomorphism.

Recall that given a functor u as above, and a presheaf of sets $\mathcal{F}$ on $\mathcal{D}$ we have defined $u^p\mathcal{F}$ to be simply the presheaf $\mathcal{F} \circ u$, in other words

$$u^p\mathcal{F}(V) = \mathcal{F}(u(V))$$

for every object V of $\mathcal{C}$.

00WW **Lemma 13.2.** *Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a continuous functor. If $\mathcal{F}$ is a sheaf on $\mathcal{D}$ then $u^p\mathcal{F}$ is a sheaf as well.*

Proof. Let $\{V_i \to V\}$ be a covering. By assumption $\{u(V_i) \to u(V)\}$ is a covering in $\mathcal{D}$ and $u(V_i \times_V V_j) = u(V_i) \times_{u(V)} u(V_j)$. Hence the sheaf condition for $u^p\mathcal{F}$ and the covering $\{V_i \to V\}$ is precisely the same as the sheaf condition for $\mathcal{F}$ and the covering $\{u(V_i) \to u(V)\}$. □

In order to avoid confusion we sometimes denote

$$u^s : Sh(\mathcal{D}) \longrightarrow Sh(\mathcal{C})$$

the functor u^p restricted to the subcategory of sheaves of sets. Recall that u^p has a left adjoint $u_p : PSh(\mathcal{C}) \to PSh(\mathcal{D})$, see Section 5.

00WX **Lemma 13.3.** *In the situation of Lemma 13.2. The functor $u_s : \mathcal{G} \mapsto (u_p\mathcal{G})^\#$ is a left adjoint to u^s.*

Proof. Follows directly from Lemma 5.4 and Proposition 10.12. □

Here is a technical lemma.

00WY **Lemma 13.4.** *In the situation of Lemma 13.2. For any presheaf $\mathcal{G}$ on $\mathcal{C}$ we have $(u_p\mathcal{G})^\# = (u_p(\mathcal{G}^\#))^\#$.*

Proof. For any sheaf $\mathcal{F}$ on $\mathcal{D}$ we have

$$\begin{aligned}
\mathrm{Mor}_{Sh(\mathcal{D})}(u_s(\mathcal{G}^\#), \mathcal{F}) &= \mathrm{Mor}_{Sh(\mathcal{C})}(\mathcal{G}^\#, u^s\mathcal{F}) \\
&= \mathrm{Mor}_{PSh(\mathcal{C})}(\mathcal{G}^\#, u^p\mathcal{F}) \\
&= \mathrm{Mor}_{PSh(\mathcal{C})}(\mathcal{G}, u^p\mathcal{F}) \\
&= \mathrm{Mor}_{PSh(\mathcal{D})}(u_p\mathcal{G}, \mathcal{F}) \\
&= \mathrm{Mor}_{Sh(\mathcal{D})}((u_p\mathcal{G})^\#, \mathcal{F})
\end{aligned}$$

and the result follows from the Yoneda lemma. □

04D3 **Lemma 13.5.** *Let $u : \mathcal{C} \to \mathcal{D}$ be a continuous functor between sites. For any object U of $\mathcal{C}$ we have $u_s h_U^\# = h_{u(U)}^\#$.*

Proof. Follows from Lemmas 5.6 and 13.4. □

00WZ **Remark 13.6.** (Skip on first reading.) Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let us use the definition of tautologically equivalent families of maps, see Definition 8.2 to (slightly) weaken the conditions defining continuity. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. Let us call u *quasi-continuous* if for every $\mathcal{V} = \{V_i \to V\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})$ we have the following

(1') the family of maps $\{u(V_i) \to u(V)\}_{i\in I}$ is tautologically equivalent to an element of $\mathrm{Cov}(\mathcal{D})$, and
(2) for any morphism $T \to V$ in $\mathcal{C}$ the morphism $u(T \times_V V_i) \to u(T) \times_{u(V)} u(V_i)$ is an isomorphism.

We are going to see that Lemmas 13.2 and 13.3 hold in case u is quasi-continuous as well.

We first remark that the morphisms $u(V_i) \to u(V)$ are representable, since they are isomorphic to representable morphisms (by the first condition). In particular, the family $u(\mathcal{V}) = \{u(V_i) \to u(V)\}_{i\in I}$ gives rise to a zeroth Čech cohomology group $H^0(u(\mathcal{V}), \mathcal{F})$ for any presheaf $\mathcal{F}$ on $\mathcal{D}$. Let $\mathcal{U} = \{U_j \to u(V)\}_{j\in J}$ be an element of $\mathrm{Cov}(\mathcal{D})$ tautologically equivalent to $\{u(V_i) \to u(V)\}_{i\in I}$. Note that $u(\mathcal{V})$ is a refinement of $\mathcal{U}$ and vice versa. Hence by Remark 10.7 we see that $H^0(u(\mathcal{V}), \mathcal{F}) = H^0(\mathcal{U}, \mathcal{F})$. In particular, if $\mathcal{F}$ is a sheaf, then $\mathcal{F}(u(V)) = H^0(u(\mathcal{V}), \mathcal{F})$ because of the sheaf property expressed in terms of zeroth Čech cohomology groups. We conclude that $u^p\mathcal{F}$ is a sheaf if $\mathcal{F}$ is a sheaf, since $H^0(\mathcal{V}, u^p\mathcal{F}) = H^0(u(\mathcal{V}), \mathcal{F})$ which we just observed is equal to $\mathcal{F}(u(V)) = u^p\mathcal{F}(V)$. Thus Lemma 13.2 holds. Lemma 13.3 follows immediately.

14. Morphisms of sites

00X0

00X1 **Definition 14.1.** Let $\mathcal{C}$ and $\mathcal{D}$ be sites. A *morphism of sites* $f : \mathcal{D} \to \mathcal{C}$ is given by a continuous functor $u : \mathcal{C} \to \mathcal{D}$ such that the functor u_s is exact.

Notice how the functor u goes in the direction *opposite* the morphism f. If $f \leftrightarrow u$ is a morphism of sites then we use the notation $f^{-1} = u_s$ and $f_* = u^s$. The functor f^{-1} is called the *pullback functor* and the functor f_* is called the *pushforward functor*. As in topology we have the following adjointness property

$$\mathrm{Mor}_{Sh(\mathcal{D})}(f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{Sh(\mathcal{C})}(\mathcal{G}, f_*\mathcal{F})$$

The motivation for this definition comes from the following example.

00X2 **Example 14.2.** Let $f : X \to Y$ be a continuous map of topological spaces. Recall that we have sites X_{Zar} and Y_{Zar}, see Example 6.4. Consider the functor $u : Y_{Zar} \to X_{Zar}$, $V \mapsto f^{-1}(V)$. This functor is clearly continuous because inverse images of open coverings are open coverings. (Actually, this depends on how you chose sets of coverings for X_{Zar} and Y_{Zar}. But in any case the functor is quasi-continuous, see Remark 13.6.) It is easy to verify that the functor u^s equals the usual pushforward functor f_* from topology. Hence, since u_s is an adjoint and since the usual topological pullback functor f^{-1} is an adjoint as well, we get a canonical isomorphism $f^{-1} \cong u_s$. Since f^{-1} is exact we deduce that u_s is exact. Hence u defines a morphism of sites $f : X_{Zar} \to Y_{Zar}$, which we may denote f as well since we've already seen the functors u_s, u^s agree with their usual notions anyway.

0EWI **Example 14.3.** Let $\mathcal{C}$ be a category. Let

$$\mathrm{Cov}(\mathcal{C}) \supset \mathrm{Cov}'(\mathcal{C})$$

be two sets of families of morphisms with fixed target which turn $\mathcal{C}$ into a site. Denote $\mathcal{C}_\tau$ the site corresponding to $\mathrm{Cov}(\mathcal{C})$ and $\mathcal{C}_{\tau'}$ the site corresponding to $\mathrm{Cov}'(\mathcal{C})$. We claim the identity functor on $\mathcal{C}$ defines a morphism of sites

$$\epsilon : \mathcal{C}_\tau \longrightarrow \mathcal{C}_{\tau'}$$

Namely, observe that $\mathrm{id} : \mathcal{C}_{\tau'} \to \mathcal{C}_\tau$ is continuous as every τ'-covering is a τ-covering. Thus the functor $\epsilon_* = \mathrm{id}^s$ is the identity functor on underlying presheaves. Hence the left adjoint ϵ^{-1} of ϵ_* sends a τ'-sheaf $\mathcal{F}$ to the τ-sheafification of $\mathcal{F}$ (by the universal property of sheafification). Finite limits of τ'-sheaves agree with finite limits of presheaves (Lemma 10.1) and τ-sheafification commutes with finite limits (Lemma 10.14). Thus ϵ^{-1} is left exact. Since ϵ^{-1} is a left adjoint it is also right exact (Categories, Lemma 24.6). Thus ϵ^{-1} is exact and we have checked all the conditions of Definition 14.1.

03CB **Lemma 14.4.** *Let $\mathcal{C}_i$, $i = 1, 2, 3$ be sites. Let $u : \mathcal{C}_2 \to \mathcal{C}_1$ and $v : \mathcal{C}_3 \to \mathcal{C}_2$ be continuous functors which induce morphisms of sites. Then the functor $u \circ v : \mathcal{C}_3 \to \mathcal{C}_1$ is continuous and defines a morphism of sites $\mathcal{C}_1 \to \mathcal{C}_3$.*

Proof. It is immediate from the definitions that $u \circ v$ is a continuous functor. In addition, we clearly have $(u \circ v)^p = v^p \circ u^p$, and hence $(u \circ v)^s = v^s \circ u^s$. Hence functors $(u \circ v)_s$ and $u_s \circ v_s$ are both left adjoints of $(u \circ v)^s$. Therefore $(u \circ v)_s \cong u_s \circ v_s$ and we conclude that $(u \circ v)_s$ is exact as a composition of exact functors. □

03CC **Definition 14.5.** Let $\mathcal{C}_i$, $i = 1, 2, 3$ be sites. Let $f : \mathcal{C}_1 \to \mathcal{C}_2$ and $g : \mathcal{C}_2 \to \mathcal{C}_3$ be morphisms of sites given by continuous functors $u : \mathcal{C}_2 \to \mathcal{C}_1$ and $v : \mathcal{C}_3 \to \mathcal{C}_2$. The *composition* $g \circ f$ is the morphism of sites corresponding to the functor $u \circ v$.

In this situation we have $(g \circ f)_* = g_* \circ f_*$ and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ (see proof of Lemma 14.4).

00X5 **Lemma 14.6.** *Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be continuous. Assume all the categories $(\mathcal{I}^u_V)^{opp}$ of Section 5 are filtered. Then u defines a morphism of sites $\mathcal{D} \to \mathcal{C}$, in other words u_s is exact.*

Proof. Since u_s is the left adjoint of u^s we see that u_s is right exact, see Categories, Lemma 24.6. Hence it suffices to show that u_s is left exact. In other words we have to show that u_s commutes with finite limits. Because the categories $\mathcal{I}^{opp}_Y$ are filtered we see that u_p commutes with finite limits, see Categories, Lemma 19.2 (this also uses the description of limits in *PSh*, see Section 4). And since sheafification commutes with finite limits as well (Lemma 10.14) we conclude because $u_s = \# \circ u_p$. □

00X6 **Proposition 14.7.** *Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be continuous. Assume furthermore the following:*

1. *the category $\mathcal{C}$ has a final object X and $u(X)$ is a final object of $\mathcal{D}$, and*
2. *the category $\mathcal{C}$ has fibre products and u commutes with them.*

Then u defines a morphism of sites $\mathcal{D} \to \mathcal{C}$, in other words u_s is exact.

Proof. This follows from Lemmas 5.2 and 14.6. □

00X7 **Remark 14.8.** The conditions of Proposition 14.7 above are equivalent to saying that u is left exact, i.e., commutes with finite limits. See Categories, Lemmas 18.4 and 23.2. It seems more natural to phrase it in terms of final objects and fibre products since this seems to have more geometric meaning in the examples.

Lemma 19.4 will provide another way to prove a continuous functor gives rise to a morphism of sites.

00X8 **Remark 14.9.** (Skip on first reading.) Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Analogously to Definition 14.1 we say that a *quasi-morphism of sites* $f : \mathcal{D} \to \mathcal{C}$ is given by a quasi-continuous functor $u : \mathcal{C} \to \mathcal{D}$ (see Remark 13.6) such that u_s is exact. The analogue of Proposition 14.7 in this setting is obtained by replacing the word "continuous" by the word "quasi-continuous", and replacing the word "morphism" by "quasi-morphism". The proof is literally the same.

In Definition 14.1 the condition that u_s be exact cannot be omitted. For example, the conclusion of the following lemma need not hold if one only assumes that u is continuous.

08H2 **Lemma 14.10.** *Let $f : \mathcal{D} \to \mathcal{C}$ be a morphism of sites given by the functor $u : \mathcal{C} \to \mathcal{D}$. Given any object V of $\mathcal{D}$ there exists a covering $\{V_j \to V\}$ such that for every j there exists a morphism $V_j \to u(U_j)$ for some object U_j of $\mathcal{C}$.*

Proof. Since $f^{-1} = u_s$ is exact we have $f^{-1}* = *$ where $*$ denotes the final object of the category of sheaves (Example 10.2). Since $f^{-1}* = u_s*$ is the sheafification of u_p* we see there exists a covering $\{V_j \to V\}$ such that $(u_p*)(V_j)$ is nonempty. Since $(u_p*)(V_j)$ is a colimit over the category $\mathcal{I}^u_{V_j}$ whose objects are morphisms $V_j \to u(U)$ the lemma follows. □

15. Topoi

00X9 Here is a definition of a topos which is suitable for our purposes. Namely, a topos is the category of sheaves on a site. In order to specify a topos you just specify the site. The real difference between a topos and a site lies in the definition of morphisms. Namely, it turns out that there are lots of morphisms of topoi which do not come from morphisms of the underlying sites.

00XA **Definition 15.1** (Topoi)**.** A *topos* is the category $Sh(\mathcal{C})$ of sheaves on a site $\mathcal{C}$.

(1) Let $\mathcal{C}$, $\mathcal{D}$ be sites. A *morphism of topoi* f from $Sh(\mathcal{D})$ to $Sh(\mathcal{C})$ is given by a pair of functors $f_* : Sh(\mathcal{D}) \to Sh(\mathcal{C})$ and $f^{-1} : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ such that
   (a) we have
   $$\mathrm{Mor}_{Sh(\mathcal{D})}(f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{Sh(\mathcal{C})}(\mathcal{G}, f_*\mathcal{F})$$
   bifunctorially, and
   (b) the functor f^{-1} commutes with finite limits, i.e., is left exact.
(2) Let $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$ be sites. Given morphisms of topoi $f : Sh(\mathcal{D}) \to Sh(\mathcal{C})$ and $g : Sh(\mathcal{E}) \to Sh(\mathcal{D})$ the *composition* $f \circ g$ is the morphism of topoi defined by the functors $(f \circ g)_* = f_* \circ g_*$ and $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.

Suppose that $\alpha : \mathcal{S}_1 \to \mathcal{S}_2$ is an equivalence of (possibly "big") categories. If $\mathcal{S}_1$, $\mathcal{S}_2$ are topoi, then setting $f_* = \alpha$ and f^{-1} equal to a quasi-inverse of α gives a morphism $f : \mathcal{S}_1 \to \mathcal{S}_2$ of topoi. Moreover this morphism is an equivalence in the 2-category of topoi (see Section 36). Thus it makes sense to say "$\mathcal{S}$ is a topos" if $\mathcal{S}$ is equivalent to the category of sheaves on a site (and not necessarily equal to the category of sheaves on a site). We will occasionally use this abuse of notation.

The *empty topos* is topos of sheaves on the site $\mathcal{C}$, where $\mathcal{C}$ is the empty category. We will sometimes write $\emptyset$ for this site. This is a site which has a unique sheaf

(since $\emptyset$ has no objects). Thus $Sh(\emptyset)$ is equivalent to the category having a single object and a single morphism.

The *punctual topos* is the topos of sheaves on the site $\mathcal{C}$ which has a single object pt and one morphism id_{pt} and whose only covering is the covering $\{\mathrm{id}_{pt}\}$. We will simply write pt for this site. It is clear that the category of sheaves = the category of presheaves = the category of sets. In a formula $Sh(pt) = Sets$.

Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $f : Sh(\mathcal{D}) \to Sh(\mathcal{C})$ be a morphism of topoi. Note that f_* commutes with all limits and that f^{-1} commutes with all colimits, see Categories, Lemma 24.5. In particular, the condition on f^{-1} in the definition above guarantees that f^{-1} is exact. Morphisms of topoi are often constructed using either Lemma 21.1 or the following lemma.

00XC **Lemma 15.2.** *Given a morphism of sites $f : \mathcal{D} \to \mathcal{C}$ corresponding to the functor $u : \mathcal{C} \to \mathcal{D}$ the pair of functors $(f^{-1} = u_s, f_* = u^s)$ is a morphism of topoi.*

Proof. This is obvious from Definition 14.1. □

00XD **Remark 15.3.** There are many sites that give rise to the topos $Sh(pt)$. A useful example is the following. Suppose that S is a set (of sets) which contains at least one nonempty element. Let $\mathcal{S}$ be the category whose objects are elements of S and whose morphisms are arbitrary set maps. Assume that $\mathcal{S}$ has fibre products. For example this will be the case if $S = \mathcal{P}(\text{infinite set})$ is the power set of any infinite set (exercise in set theory). Make $\mathcal{S}$ into a site by declaring surjective families of maps to be coverings (and choose a suitable sufficiently large set of covering families as in Sets, Section 11). We claim that $Sh(\mathcal{S})$ is equivalent to the category of sets.

We first prove this in case S contains $e \in S$ which is a singleton. In this case, there is an equivalence of topoi $i : Sh(pt) \to Sh(\mathcal{S})$ given by the functors

05UW (15.3.1) $$i^{-1}\mathcal{F} = \mathcal{F}(e), \quad i_*E = (U \mapsto \mathrm{Mor}_{Sets}(U, E))$$

Namely, suppose that $\mathcal{F}$ is a sheaf on $\mathcal{S}$. For any $U \in \mathrm{Ob}(\mathcal{S}) = S$ we can find a covering $\{\varphi_u : e \to U\}_{u \in U}$, where φ_u maps the unique element of e to $u \in U$. The sheaf condition implies in this case that $\mathcal{F}(U) = \prod_{u \in U} \mathcal{F}(e)$. In other words $\mathcal{F}(U) = \mathrm{Mor}_{Sets}(U, \mathcal{F}(e))$. Moreover, this rule is compatible with restriction mappings. Hence the functor

$$i_* : Sets = Sh(pt) \longrightarrow Sh(\mathcal{S}), \quad E \longmapsto (U \mapsto \mathrm{Mor}_{Sets}(U, E))$$

is an equivalence of categories, and its inverse is the functor i^{-1} given above.

If $\mathcal{S}$ does not contain a singleton, then the functor i_* as defined above still makes sense. To show that it is still an equivalence in this case, choose any nonempty $\tilde{e} \in S$ and a map $\varphi : \tilde{e} \to \tilde{e}$ whose image is a singleton. For any sheaf $\mathcal{F}$ set

$$\mathcal{F}(e) := \mathrm{Im}(\mathcal{F}(\varphi) : \mathcal{F}(\tilde{e}) \longrightarrow \mathcal{F}(\tilde{e}))$$

and show that this is a quasi-inverse to i_*. Details omitted.

00XB **Remark 15.4.** (Set theoretical issues related to morphisms of topoi. Skip on a first reading.) A morphism of topoi as defined above is not a set but a class. In other words it is given by a mathematical formula rather than a mathematical object. Although we may contemplate the collection of all morphisms between two given topoi, it is not a good idea to introduce it as a mathematical object. On the other hand, suppose $\mathcal{C}$ and $\mathcal{D}$ are given sites. Consider a functor $\Phi : \mathcal{C} \to Sh(\mathcal{D})$.

Such a thing is a set, in other words, it is a mathematical object. We may, in succession, ask the following questions on Φ.

(1) Is it true, given a sheaf $\mathcal{F}$ on $\mathcal{D}$, that the rule $U \mapsto \mathrm{Mor}_{Sh(\mathcal{D})}(\Phi(U), \mathcal{F})$ defines a sheaf on $\mathcal{C}$? If so, this defines a functor $\Phi_* : Sh(\mathcal{D}) \to Sh(\mathcal{C})$.
(2) Is it true that Φ_* has a left adjoint? If so, write Φ^{-1} for this left adjoint.
(3) Is it true that Φ^{-1} is exact?

If the last question still has the answer "yes", then we obtain a morphism of topoi (Φ_*, Φ^{-1}). Moreover, given any morphism of topoi (f_*, f^{-1}) we may set $\Phi(U) = f^{-1}(h_U^\#)$ and obtain a functor Φ as above with $f_* \cong \Phi_*$ and $f^{-1} \cong \Phi^{-1}$ (compatible with adjoint property). The upshot is that by working with the collection of Φ instead of morphisms of topoi, we (a) replaced the notion of a morphism of topoi by a mathematical object, and (b) the collection of Φ forms a class (and not a collection of classes). Of course, more can be said, for example one can work out more precisely the significance of conditions (2) and (3) above; we do this in the case of points of topoi in Section 32.

00XE **Remark 15.5.** (Skip on first reading.) Let $\mathcal{C}$ and $\mathcal{D}$ be sites. A quasi-morphism of sites $f : \mathcal{D} \to \mathcal{C}$ (see Remark 14.9) gives rise to a morphism of topoi f from $Sh(\mathcal{D})$ to $Sh(\mathcal{C})$ exactly as in Lemma 15.2.

16. G-sets and morphisms

04D4 Let $\varphi : G \to H$ be a homomorphism of groups. Choose (suitable) sites $\mathcal{T}_G$ and $\mathcal{T}_H$ as in Example 6.5 and Section 9. Let $u : \mathcal{T}_H \to \mathcal{T}_G$ be the functor which assigns to a H-set U the G-set U_φ which has the same underlying set but G action defined by $g \cdot \xi = \varphi(g)\xi$ for $g \in G$ and $\xi \in U$. It is clear that u commutes with finite limits and is continuous[4]. Applying Proposition 14.7 and Lemma 15.2 we obtain a morphism of topoi

$$f : Sh(\mathcal{T}_G) \longrightarrow Sh(\mathcal{T}_H)$$

associated with φ. Using Proposition 9.1 we see that we get a pair of adjoint functors

$$f_* : G\text{-}Sets \longrightarrow H\text{-}Sets, \quad f^{-1} : H\text{-}Sets \longrightarrow G\text{-}Sets.$$

Let's work out what are these functors in this case.

We first work out a formula for f_*. Recall that given a G-set S the corresponding sheaf $\mathcal{F}_S$ on $\mathcal{T}_G$ is given by the rule $\mathcal{F}_S(U) = \mathrm{Mor}_G(U, S)$. And on the other hand, given a sheaf $\mathcal{G}$ on $\mathcal{T}_H$ the corresponding H-set is given by the rule $\mathcal{G}({}_H H)$. Hence we see that

$$f_*S = \mathrm{Mor}_{G\text{-}Sets}(({}_H H)_\varphi, S).$$

If we work this out a little bit more then we get

$$f_*S = \{a : H \to S \mid a(gh) = ga(h)\}$$

with left H-action given by $(h \cdot a)(h') = a(h'h)$ for any element $a \in f_*S$.

Next, we explicitly compute f^{-1}. Note that since the topology on $\mathcal{T}_G$ and $\mathcal{T}_H$ is subcanonical, all representable presheaves are sheaves. Moreover, given an object V of $\mathcal{T}_H$ we see that $f^{-1}h_V$ is equal to $h_{u(V)}$ (see Lemma 13.5). Hence we see that

[4]Set theoretical remark: First choose $\mathcal{T}_H$. Then choose $\mathcal{T}_G$ to contain $u(\mathcal{T}_H)$ and such that every covering in $\mathcal{T}_H$ corresponds to a covering in $\mathcal{T}_G$. This is possible by Sets, Lemmas 10.1, 10.2 and 11.1.

$f^{-1}S = S_\varphi$ for representable sheaves. Since every sheaf on $\mathcal{T}_H$ is a coproduct of representable sheaves we conclude that this is true in general. Hence we see that for any H-set T we have

$$f^{-1}T = T_\varphi.$$

The adjunction between f^{-1} and f_* is evidenced by the formula

$$\mathrm{Mor}_{G\text{-}Sets}(T_\varphi, S) = \mathrm{Mor}_{H\text{-}Sets}(T, f_*S)$$

with f_*S as above. This can be proved directly. Moreover, it is then clear that (f^{-1}, f_*) form an adjoint pair and that f^{-1} is exact. So alternatively to the above the morphism of topoi $f : G\text{-}Sets \to H\text{-}Sets$ can be defined directly in this manner.

17. Quasi-compact objects and colimits

090G To be able to use the same language as in the case of topological spaces we introduce the following terminology.

090H **Definition 17.1.** Let $\mathcal{C}$ be a site. An object U of $\mathcal{C}$ is *quasi-compact* if given a covering $\mathcal{U} = \{U_i \to U\}_{i\in I}$ in $\mathcal{C}$ there exists another covering $\mathcal{V} = \{V_j \to U\}_{j\in J}$ and a morphism $\mathcal{V} \to \mathcal{U}$ of families of maps with fixed target given by $\mathrm{id} : U \to U$, $\alpha : J \to I$, and $V_j \to U_{\alpha(j)}$ (see Definition 8.1) such that the image of α is a finite subset of I.

Of course the usual notion is sufficient to conclude that U is quasi-compact.

0D05 **Lemma 17.2.** *Let $\mathcal{C}$ be a site. Let U be an object of $\mathcal{C}$. Consider the following conditions*

1. *U is quasi-compact,*
2. *for every covering $\{U_i \to U\}_{i\in I}$ in $\mathcal{C}$ there exists a finite covering $\{V_j \to U\}_{j=1,\ldots,m}$ of $\mathcal{C}$ refining $\mathcal{U}$, and*
3. *for every covering $\{U_i \to U\}_{i\in I}$ in $\mathcal{C}$ there exists a finite subset $I' \subset I$ such that $\{U_i \to U\}_{i\in I'}$ is a covering in $\mathcal{C}$.*

Then we always have (3) $\Rightarrow$ (2) $\Rightarrow$ (1) but the reverse implications do not hold in general.

Proof. The implications are immediate from the definitions. Let $X = [0, 1] \subset \mathbf{R}$ as a topological space (with the usual ϵ-δ topology). Let $\mathcal{C}$ be the category of open subspaces of X with inclusions as morphisms and usual open coverings (compare with Example 6.4). However, then we change the notion of covering in $\mathcal{C}$ to exclude all finite coverings, except for the coverings of the form $\{U \to U\}$. It is easy to see that this will be a site as in Definition 6.2. In this site the object $X = U = [0, 1]$ is quasi-compact in the sense of Definition 17.1 but U does not satisfy (2). We leave it to the reader to make an example where (2) holds but not (3). □

Here is the topos theoretic meaning of a quasi-compact object.

0D06 **Lemma 17.3.** *Let $\mathcal{C}$ be a site. Let U be an object of $\mathcal{C}$. The following are equivalent*

1. *U is quasi-compact, and*
2. *for every surjection of sheaves $\coprod_{i\in I} \mathcal{F}_i \to h_U^\#$ there is a finite subset $J \subset I$ such that $\coprod_{i\in J} \mathcal{F}_i \to h_U^\#$ is surjective.*

Proof. Assume (1) and let $\coprod_{i\in I}\mathcal{F}_i \to h_U^\#$ be a surjection. Then id_U is a section of $h_U^\#$ over U. Hence there exists a covering $\{U_a \to U\}_{a\in A}$ and for each $a \in A$ a section s_a of $\coprod_{i\in I}\mathcal{F}_i$ over U_a mapping to id_U. By the construction of coproducts as sheafification of coproducts of presheaves (Lemma 10.13), for each a there exists a covering $\{U_{ab} \to U_a\}_{b\in B_a}$ and for all $b \in B_a$ an $\iota(b) \in I$ and a section s_b of $\mathcal{F}_{\iota(b)}$ over U_{ab} mapping to $\mathrm{id}_U|_{U_{ab}}$. Thus after replacing the covering $\{U_a \to U\}_{a\in A}$ by $\{U_{ab} \to U\}_{a\in A, b\in B_a}$ we may assume we have a map $\iota : A \to I$ and for each $a \in A$ a section s_a of $\mathcal{F}_{\iota(a)}$ over U_a mapping to id_U. Since U is quasi-compact, there is a covering $\{V_c \to U\}_{c\in C}$, a map $\alpha : C \to A$ with finite image, and $V_c \to U_{\alpha(c)}$ over U. Then we see that $J = \mathrm{Im}(\iota \circ \alpha) \subset I$ works because $\coprod_{c\in C} h_{V_c}^\# \to h_U^\#$ is surjective (Lemma 12.4) and factors through $\coprod_{i\in J}\mathcal{F}_i \to h_U^\#$. (Here we use that the composition $h_{V_c}^\# \to h_{U_{\alpha(c)}} \xrightarrow{s_{\alpha(c)}} \mathcal{F}_{\iota(\alpha(c))} \to h_U^\#$ is the map $h_{V_c}^\# \to h_U^\#$ coming from the morphism $V_c \to U$ because $s_{\alpha(c)}$ maps to $\mathrm{id}_U|_{U_{\alpha(c)}}$.)

Assume (2). Let $\{U_i \to U\}_{i\in I}$ be a covering. By Lemma 12.4 we see that $\coprod_{i\in I} h_{U_i}^\# \to h_U^\#$ is surjective. Thus we find a finite subset $J \subset I$ such that $\coprod_{j\in J} h_{U_j}^\# \to h_U^\#$ is surjective. Then arguing as above we find a covering $\{V_c \to U\}_{c\in C}$ of U in $\mathcal{C}$ and a map $\iota : C \to J$ such that id_U lifts to a section of s_c of $h_{U_{\iota(c)}}^\#$ over V_c. Refining the covering even further we may assume $s_c \in h_{U_{\iota(c)}}(V_c)$ mapping to id_U. Then $s_c : V_c \to U_{\iota(c)}$ is a morphism over U and we conclude. □

The lemma above motivates the following definition.

0D07 **Definition 17.4.** An object $\mathcal{F}$ of a topos $Sh(\mathcal{C})$ is *quasi-compact* if for any surjective map $\coprod_{i\in I}\mathcal{F}_i \to \mathcal{F}$ of $Sh(\mathcal{C})$ there exists a finite subset $J \subset I$ such that $\coprod_{i\in J}\mathcal{F}_i \to \mathcal{F}$ is surjective. A topos $Sh(\mathcal{C})$ is said to be *quasi-compact* if its final object $*$ is a quasi-compact object.

By Lemma 17.3 if the site $\mathcal{C}$ has a final object X, then $Sh(\mathcal{C})$ is quasi-compact if and only if X is quasi-compact.

0GMP **Lemma 17.5.** *Let $\mathcal{C}$ be a site.*

(1) *If $U \to V$ is a morphism of $\mathcal{C}$ such that $h_U^\# \to h_V^\#$ is surjective and U is quasi-compact, then V is quasi-compact.*
(2) *If $\mathcal{F} \to \mathcal{G}$ is a surjection of sheaves of sets and $\mathcal{F}$ is quasi-compact, then $\mathcal{G}$ is quasi-compact.*

Proof. Omitted. □

0GMQ **Lemma 17.6.** *Let $\mathcal{C}$ be a site. If $n \geq 1$ and $\mathcal{F}_1, \ldots, \mathcal{F}_n$ are quasi-compact sheaves on $\mathcal{C}$, then $\coprod_{i=1,\ldots,n}\mathcal{F}_i$ is quasi-compact.*

Proof. Omitted. □

The following two lemmas form the analogue of Sheaves, Lemma 29.1 for sites.

0738 **Lemma 17.7.** *Let $\mathcal{C}$ be a site. Let $\mathcal{I} \to Sh(\mathcal{C})$, $i \mapsto \mathcal{F}_i$ be a filtered diagram of sheaves of sets. Let $U \in \mathrm{Ob}(\mathcal{C})$. Consider the canonical map*

$$\Psi : \mathrm{colim}_i \mathcal{F}_i(U) \longrightarrow (\mathrm{colim}_i \mathcal{F}_i)(U)$$

With the terminology introduced above:

(1) *If all the transition maps are injective then Ψ is injective for any U.*

(2) *If U is quasi-compact, then Ψ is injective.*
(3) *If U is quasi-compact and all the transition maps are injective then Ψ is an isomorphism.*
(4) *If U has a cofinal system of coverings $\{U_j \to U\}_{j\in J}$ with J finite and $U_j \times_U U_{j'}$ quasi-compact for all $j, j' \in J$, then Ψ is bijective.*

Proof. Assume all the transition maps are injective. In this case the presheaf $\mathcal{F}' : V \mapsto \operatorname{colim}_i \mathcal{F}_i(V)$ is separated (see Definition 10.9). By Lemma 10.13 we have $(\mathcal{F}')^\# = \operatorname{colim}_i \mathcal{F}_i$. By Theorem 10.10 we see that $\mathcal{F}' \to (\mathcal{F}')^\#$ is injective. This proves (1).

Assume U is quasi-compact. Suppose that $s \in \mathcal{F}_i(U)$ and $s' \in \mathcal{F}_{i'}(U)$ give rise to elements on the left hand side which have the same image under Ψ. This means we can choose a covering $\{U_a \to U\}_{a\in A}$ and for each $a \in A$ an index $i_a \in I$, $i_a \geq i$, $i_a \geq i'$ such that $\varphi_{ii_a}(s) = \varphi_{i'i_a}(s')$. Because U is quasi-compact we can choose a covering $\{V_b \to U\}_{b\in B}$, a map $\alpha : B \to A$ with finite image, and morphisms $V_b \to U_{\alpha(b)}$ over U. Pick $i'' \in I$ to be $\geq$ than all of the $i_{\alpha(b)}$ which is possible because the image of α is finite. We conclude that $\varphi_{ii''}(s)$ and $\varphi_{i'i''}(s)$ agree on V_b for all $b \in B$ and hence that $\varphi_{ii''}(s) = \varphi_{i'i''}(s)$. This proves (2).

Assume U is quasi-compact and all transition maps injective. Let s be an element of the target of Ψ. There exists a covering $\{U_a \to U\}_{a\in A}$ and for each $a \in A$ an index $i_a \in I$ and a section $s_a \in \mathcal{F}_{i_a}(U_a)$ such that $s|_{U_a}$ comes from s_a for all $a \in A$. Because U is quasi-compact we can choose a covering $\{V_b \to U\}_{b\in B}$, a map $\alpha : B \to A$ with finite image, and morphisms $V_b \to U_{\alpha(b)}$ over U. Pick $i \in I$ to be $\geq$ than all of the $i_{\alpha(b)}$ which is possible because the image of α is finite. By (1) the sections $s_b = \varphi_{i_{\alpha(b)}i}(s_{\alpha(b)})|_{V_b}$ agree over $V_b \times_U V_{b'}$. Hence they glue to a section $s' \in \mathcal{F}_i(U)$ which maps to s under Ψ. This proves (3).

Assume the hypothesis of (4). By Lemma 17.2 the object U is quasi-compact, hence Ψ is injective by (2). To prove surjectivity, let s be an element of the target of Ψ. By assumption there exists a finite covering $\{U_j \to U\}_{j=1,\ldots,m}$, with $U_j \times_U U_{j'}$ quasi-compact for all $1 \leq j, j' \leq m$ and for each j an index $i_j \in I$ and $s_j \in \mathcal{F}_{i_j}(U_j)$ such that $s|_{U_j}$ is the image of s_j for all j. Since $U_j \times_U U_{j'}$ is quasi-compact we can apply (2) and we see that there exists an $i_{jj'} \in I$, $i_{jj'} \geq i_j$, $i_{jj'} \geq i_{j'}$ such that $\varphi_{i_j i_{jj'}}(s_j)$ and $\varphi_{i_{j'} i_{jj'}}(s_{j'})$ agree over $U_j \times_U U_{j'}$. Choose an index $i \in I$ wich is bigger or equal than all the $i_{jj'}$. Then we see that the sections $\varphi_{i_j i}(s_j)$ of $\mathcal{F}_i$ glue to a section of $\mathcal{F}_i$ over U. This section is mapped to the element s as desired. □

0GMR **Lemma 17.8.** *Let $\mathcal{C}$ be a site. Let $\mathcal{I} \to Sh(\mathcal{C})$, $i \mapsto \mathcal{F}_i$ be a filtered diagram of sheaves of sets. Consider the canonical map*

$$\Psi : \operatorname{colim}_i \Gamma(\mathcal{C}, \mathcal{F}_i) \longrightarrow \Gamma(\mathcal{C}, \operatorname{colim}_i \mathcal{F}_i)$$

We have the following:

(1) *If all the transition maps are injective then Ψ is injective.*
(2) *If $Sh(\mathcal{C})$ is quasi-compact, then Ψ is injective.*
(3) *If $Sh(\mathcal{C})$ is quasi-compact and all the transition maps are injective then Ψ is an isomorphism.*
(4) *Assume there exists a set $S \subset \operatorname{Ob}(Sh(\mathcal{C}))$ with the following properties:*
   (a) *for every surjection $\mathcal{F} \to *$ there exists a $\mathcal{K} \in S$ and a map $\mathcal{K} \to \mathcal{F}$ such that $\mathcal{K} \to *$ is surjective,*

(b) *for $\mathcal{K} \in S$ the product $\mathcal{K} \times \mathcal{K}$ is quasi-compact.*
Then Ψ is bijective.

Proof. Proof of (1). Assume all the transition maps are injective. In this case the presheaf $\mathcal{F}' : V \mapsto \operatorname{colim}_i \mathcal{F}_i(V)$ is separated (see Definition 10.9). By Lemma 10.13 we have $(\mathcal{F}')^\# = \operatorname{colim}_i \mathcal{F}_i$. By Theorem 10.10 we see that $\mathcal{F}' \to (\mathcal{F}')^\#$ is injective. This proves (1).

Proof of (2). Assume $Sh(\mathcal{C})$ is quasi-compact. Recall that $\Gamma(\mathcal{C}, \mathcal{F}) = \operatorname{Mor}(*, \mathcal{F})$ for all $\mathcal{F}$ in $Sh(\mathcal{C})$. Let $a_i, b_i : * \to \mathcal{F}_i$ and for $i' \geq i$ denote $a_{i'}, b_{i'} : * \to \mathcal{F}_{i'}$ the composition with the transition maps of the system. Set $a = \operatorname{colim}_{i' \geq i} a_{i'}$ and similary for b. For $i' \geq i$ denote

$$E_{i'} = \operatorname{Equalizer}(a_{i'}, b_{i'}) \subset * \quad \text{and} \quad E = \operatorname{Equalizer}(a, b) \subset *$$

By Categories, Lemma 19.2 we have $E = \operatorname{colim}_{i' \geq i} E_{i'}$. It follows that $\coprod_{i' \geq i} E_{i'} \to E$ is a surjective map of sheaves. Hence, if $E = *$, i.e., if $a = b$, then because $*$ is quasi-compact, we see that $E_{i'} = *$ for some $i' \geq i$, and we conclude $a_{i'} = b_{i'}$ for some $i' \geq i$. This proves (2).

Proof of (3). Assume $Sh(\mathcal{C})$ is quasi-compact and all transition maps are injective. Let $a : * \to \operatorname{colim} \mathcal{F}_i$ be a map. Then $E_i = a^{-1}(\mathcal{F}_i) \subset *$ is a subsheaf and we have $\operatorname{colim} E_i = *$ (by the reference above). Hence for some i we have $E_i = *$ and we see that the image of a is contained in $\mathcal{F}_i$ as desired.

Proof of (4). Let $S \subset \operatorname{Ob}(Sh(\mathcal{C}))$ satisfy (4)(a), (b). Applying (4)(a) to $\operatorname{id} : * \to *$ we find there exists a $\mathcal{K} \in S$ such that $\mathcal{K} \to *$ is surjective. The maps $\mathcal{K} \times \mathcal{K} \to \mathcal{K} \to *$ are surjective. By (4)(b) and Lemma 17.5 we conclude that $\mathcal{K}$ and $Sh(\mathcal{C})$ are quasi-compact. Thus Ψ is injective by (2). Set $\mathcal{F} = \operatorname{colim} \mathcal{F}_i$. Let $s : * \to \mathcal{F}$ be a global section of the colimit. Since $\coprod \mathcal{F}_i \to \mathcal{F}$ is surjective, we see that the projection

$$\coprod\nolimits_{i \in I} * \times_{s, \mathcal{F}} \mathcal{F}_i \to *$$

is surjective. By (4)(a) we obtain $\mathcal{K} \in S$ and a map $\mathcal{K} \to \coprod_{i \in I} * \times_{s, \mathcal{F}} \mathcal{F}_i$ with $\mathcal{K} \to *$ surjective. Since $\mathcal{K}$ is quasi-compact we obtain a factorization $\mathcal{K} \to \coprod_{i' \in I'} * \times_{s, \mathcal{F}} \mathcal{F}_{i'}$ for some finite subset $I' \subset I$. Let $i \in I$ be an upper bound for the finite subset I'. The transition maps define a map $\coprod_{i' \in I'} \mathcal{F}_{i'} \to \mathcal{F}_i$. This in turn produces a map $\mathcal{K} \to * \times_{s, \mathcal{F}} \mathcal{F}_i$. In other words, we obtain $\mathcal{K} \in S$ with $\mathcal{K} \to *$ surjective and a commutative diagram

$$\begin{array}{ccccc} \mathcal{K} \times \mathcal{K} & \rightrightarrows & \mathcal{K} & \longrightarrow & * \\ & & \downarrow & & \downarrow{\scriptstyle s} \\ & & \mathcal{F}_i & \longrightarrow & \mathcal{F} \end{array} = \operatorname{colim} \mathcal{F}_i$$

Observe that the top row of this diagram is a coequalizer. Hence it suffices to show that after increasing i the two induced maps $a_i, b_i : \mathcal{K} \times \mathcal{K} \to \mathcal{F}_i$ are equal. This is done shown in the next paragraph using the exact same argument as in the proof of (2) and we urge the reader to skip the rest of the proof.

For $i' \geq i$ denote $a_{i'}, b_{i'} : \mathcal{K} \times \mathcal{K} \to \mathcal{F}_{i'}$ the composition of a_i, b_i with the transition maps of the system. Set $a = \operatorname{colim}_{i' \geq i} a_{i'} : \mathcal{K} \times \mathcal{K} \to \mathcal{F}$ and similary for b. We have $a = b$ by the commutativity of the diagram above. For $i' \geq i$ denote

$$E_{i'} = \operatorname{Equalizer}(a_{i'}, b_{i'}) \subset \mathcal{K} \times \mathcal{K} \quad \text{and} \quad E = \operatorname{Equalizer}(a, b) \subset \mathcal{K} \times \mathcal{K}$$

By Categories, Lemma 19.2 we have $E = \operatorname{colim}_{i' \geq i} E_{i'}$. It follows that $\coprod_{i' \geq i} E_{i'} \to E$ is a surjective map of sheaves. Since $a = b$ we have $E = \mathcal{K} \times \mathcal{K}$. As $\mathcal{K} \times \mathcal{K}$ is quasi-compact by (4)(b), we see that $E_{i'} = \mathcal{K} \times \mathcal{K}$ for some $i' \geq i$, and we conclude $a_{i'} = b_{i'}$ for some $i' \geq i$. □

0GMS **Remark 17.9.** Let $\mathcal{C}$ be a site. There are several ways to ensure that the hypotheses of part (4) of Lemma 17.8 are satisfied. Here are a few.

(1) Assume there exists a set $\mathcal{B} \subset \mathrm{Ob}(\mathcal{C})$ with the following properties:
   (a) for every surjection $\mathcal{F} \to *$ there exist $m \geq 0$ and $U_1, \ldots, U_m \in \mathcal{B}$ with $\mathcal{F}(U_j)$ nonempty and $\coprod h^\#_{U_j} \to *$ surjective,
   (b) for $U, U' \in \mathcal{B}$ the sheaf $h^\#_U \times h^\#_{U'}$ is quasi-compact.

(2) Assume there exists a set $\mathcal{B} \subset \mathrm{Ob}(\mathcal{C})$ with the following properties:
   (a) there exist $m \geq 0$ and $U_1, \ldots, U_m \in \mathcal{B}$ with $\coprod h^\#_{U_j} \to *$ surjective,
   (b) for $U \in \mathcal{B}$ any covering of U can be refined by a finite covering $\{U_j \to U\}_{j=1,\ldots,m}$ with $U_j \in \mathcal{B}$, and
   (c) for $U, U' \in \mathcal{B}$ there exist $m \geq 0$, $U_1, \ldots, U_m \in \mathcal{B}$, and morphisms $U_j \to U$ and $U_j \to U'$ such that $\coprod h^\#_{U_j} \to h^\#_U \times h^\#_{U'}$ is surjective.

(3) Suppose that
   (a) $Sh(\mathcal{C})$ is quasi-compact,
   (b) every object of $\mathcal{C}$ has a covering whose members are quasi-compact objects,
   (c) if U and U' are quasi-compact, then the sheaf $h^\#_U \times h^\#_{U'}$ is quasi-compact.

In cases (1) and (2) we set $S \subset \mathrm{Ob}(Sh(\mathcal{C}))$ equal to the set of finite coproducts of the sheaves $h^\#_U$ for $U \in \mathcal{B}$. In case (3) we set $S \subset \mathrm{Ob}(Sh(\mathcal{C}))$ equal to the set of finite coproducts of the sheaves $h^\#_U$ for $U \in \mathrm{Ob}(\mathcal{C})$ quasi-compact.

Later we will need a bound on what can happen with colimits as follows.

0GS0 **Lemma 17.10.** *Let $\mathcal{C}$ be a site. Let β be an ordinal. Let $\beta \to Sh(\mathcal{C})$, $\alpha \mapsto \mathcal{F}_\alpha$ be a system of sheaves over β. For $U \in \mathrm{Ob}(\mathcal{C})$ consider the canonical map*

$$\operatorname{colim}_{\alpha<\beta} \mathcal{F}_\alpha(U) \longrightarrow (\operatorname{colim}_{\alpha<\beta} \mathcal{F}_\alpha)(U)$$

If the cofinality of β is large enough, then this map is bijective for all U.

Proof. The left hand side is the value on U of the colimit $\mathcal{F}_{\mathrm{colim}}$ taken in the category of presheaves, see Section 4. Recall that $\operatorname{colim}_{\alpha<\beta} \mathcal{F}_\alpha$ is the sheafification $\mathcal{F}^\#_{\mathrm{colim}}$ of $\mathcal{F}_{\mathrm{colim}}$, see Lemma 10.13. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be an element of the set $\mathrm{Cov}(\mathcal{C})$ of coverings of $\mathcal{C}$. If the cofinality of β is larger than the cardinality of I, then we claim

$$H^0(\mathcal{U}, \mathcal{F}_{\mathrm{colim}}) = \operatorname{colim} H^0(\mathcal{U}, \mathcal{F}_\alpha) = \operatorname{colim} \mathcal{F}_\alpha(U) = \mathcal{F}_{\mathrm{colim}}(U)$$

The second and third equality signs are clear. For the first, say $s = (s_i) \in H^0(\mathcal{U}, \mathcal{F}_{\mathrm{colim}})$. Then for each i the element s_i comes from an element $s_{i,\alpha_i} \in \mathcal{F}_{\alpha_i}(U_i)$ for some $\alpha_i < \beta$. By the assumption on cofinality, we can choose $\alpha_i = \alpha$ independent of i. Then s_i and s_j map to the same element of $\mathcal{F}_{\alpha_{i,j}}(U_i \times_U U_j)$ for some $\alpha_{i,j} < \beta$. Since the cardinality if $I \times I$ is also less than the cofinality of β, we see that we may after increasing α assume $\alpha_{i,j} = \alpha$ for all i, j. This proves that the natural map $\operatorname{colim} H^0(\mathcal{U}, \mathcal{F}_\alpha) \to H^0(\mathcal{U}, \mathcal{F}_{\mathrm{colim}})$ is surjective. A very similar argument shows that it is injective. In particular, we see that $\mathcal{F}_{\mathrm{colim}}$ satisfies the

sheaf condition for $\mathcal{U}$. Thus if the cofinality of β is larger than the supremum of the cardinalities of the set of index sets I of coverings, then we conclude. □

18. Colimits of sites

0EXI We need an analogue of Lemma 17.7 in the case that the site is the limit of an inverse system of sites. For simplicity we only explain the construction in case the index sets of coverings are finite.

0A34 **Situation 18.1.** Here we are given

(1) a cofiltered index category $\mathcal{I}$,
(2) for $i \in \mathrm{Ob}(\mathcal{I})$ a site $\mathcal{C}_i$ such that every covering in $\mathcal{C}_i$ has a finite index set,
(3) for a morphism $a : i \to j$ in $\mathcal{I}$ a morphism of sites $f_a : \mathcal{C}_i \to \mathcal{C}_j$ given by a continuous functor $u_a : \mathcal{C}_j \to \mathcal{C}_i$,

such that $f_a \circ f_b = f_c$ whenever $c = a \circ b$ in $\mathcal{I}$.

09YL **Lemma 18.2.** *In Situation 18.1 we can construct a site $(\mathcal{C}, Cov(\mathcal{C}))$ as follows*

(1) *as a category $\mathcal{C} = \operatorname{colim} \mathcal{C}_i$, and*
(2) *$Cov(\mathcal{C})$ is the union of the images of $Cov(\mathcal{C}_i)$ by $u_i : \mathcal{C}_i \to \mathcal{C}$.*

Proof. Our definition of composition of morphisms of sites implies that $u_b \circ u_a = u_c$ whenever $c = a \circ b$ in $\mathcal{I}$. The formula $\mathcal{C} = \operatorname{colim} \mathcal{C}_i$ means that $\mathrm{Ob}(\mathcal{C}) = \operatorname{colim} \mathrm{Ob}(\mathcal{C}_i)$ and $\mathrm{Arrows}(\mathcal{C}) = \operatorname{colim} \mathrm{Arrows}(\mathcal{C}_i)$. Then source, target, and composition are inherited from the source, target, and composition on $\mathrm{Arrows}(\mathcal{C}_i)$. In this way we obtain a category. Denote $u_i : \mathcal{C}_i \to \mathcal{C}$ the obvious functor. Remark that given any finite diagram in $\mathcal{C}$ there exists an i such that this diagram is the image of a diagram in $\mathcal{C}_i$.

Let $\{U^t \to U\}$ be a covering of $\mathcal{C}$. We first prove that if $V \to U$ is a morphism of $\mathcal{C}$, then $U^t \times_U V$ exists. By our remark above and our definition of coverings, we can find an i, a covering $\{U_i^t \to U_i\}$ of $\mathcal{C}_i$ and a morphism $V_i \to U_i$ whose image by u_i is the given data. We claim that $U^t \times_U V$ is the image of $U_i^t \times_{U_i} V_i$ by u_i. Namely, for every $a : j \to i$ in $\mathcal{I}$ the functor u_a is continuous, hence $u_a(U_i^t \times_{U_i} V_i) = u_a(U_i^t) \times_{u_a(U_i)} u_a(V_i)$. In particular we can replace i by j, if we so desire. Thus, if W is another object of $\mathcal{C}$, then we may assume $W = u_i(W_i)$ and we see that

$$\begin{aligned}
&\mathrm{Mor}_{\mathcal{C}}(W, u_i(U_i^t \times_{U_i} V_i)) \\
&= \operatorname{colim}_{a:j\to i} \mathrm{Mor}_{\mathcal{C}_j}(u_a(W_i), u_a(U_i^t \times_{U_i} V_i)) \\
&= \operatorname{colim}_{a:j\to i} \mathrm{Mor}_{\mathcal{C}_j}(u_a(W_i), u_a(U_i^t)) \times_{\mathrm{Mor}_{\mathcal{C}_j}(u_a(W_i), u_a(U_i))} \mathrm{Mor}_{\mathcal{C}_j}(u_a(W_i), u_a(V_i)) \\
&= \mathrm{Mor}_{\mathcal{C}}(W, U^t) \times_{\mathrm{Mor}_{\mathcal{C}}(W,U)} \mathrm{Mor}_{\mathcal{C}}(W, V)
\end{aligned}$$

as filtered colimits commute with finite limits (Categories, Lemma 19.2). It also follows that $\{U^t \times_U V \to V\}$ is a covering in $\mathcal{C}$. In this way we see that axiom (3) of Definition 6.2 holds.

To verify axiom (2) of Definition 6.2 let $\{U^t \to U\}_{t\in T}$ be a covering of $\mathcal{C}$ and for each t let $\{U^{ts} \to U^t\}$ be a covering of $\mathcal{C}$. Then we can find an i and a covering $\{U_i^t \to U_i\}_{t\in T}$ of $\mathcal{C}_i$ whose image by u_i is $\{U^t \to U\}$. Since T is **finite** we may choose an $a : j \to i$ in $\mathcal{I}$ and coverings $\{U_j^{ts} \to u_a(U_i^t)\}$ of $\mathcal{C}_j$ whose image by u_j gives $\{U^{ts} \to U^t\}$. Then we conclude that $\{U^{ts} \to U\}$ is a covering of $\mathcal{C}$ by an application of axiom (2) to the site $\mathcal{C}_j$.

We omit the proof of axiom (1) of Definition 6.2. □

0A35 **Lemma 18.3.** *In Situation 18.1 let $u_i : \mathcal{C}_i \to \mathcal{C}$ be as constructed in Lemma 18.2. Then u_i defines a morphism of sites $f_i : \mathcal{C} \to \mathcal{C}_i$. For $U_i \in \mathrm{Ob}(\mathcal{C}_i)$ and sheaf $\mathcal{F}$ on $\mathcal{C}_i$ we have*

09YM (18.3.1) $$f_i^{-1}\mathcal{F}(u_i(U_i)) = \mathrm{colim}_{a:j\to i} f_a^{-1}\mathcal{F}(u_a(U_i))$$

Proof. It is immediate from the arguments in the proof of Lemma 18.2 that the functors u_i are continuous. To finish the proof we have to show that $f_i^{-1} := u_{i,s}$ is an exact functor $Sh(\mathcal{C}_i) \to Sh(\mathcal{C})$. In fact it suffices to show that f_i^{-1} is left exact, because it is right exact as a left adjoint (Categories, Lemma 24.6). We first prove (18.3.1) and then we deduce exactness.

For an arbitrary object V of $\mathcal{C}$ we can pick a $a : j \to i$ and an object $V_j \in \mathrm{Ob}(\mathcal{C})$ with $V = u_j(V_j)$. Then we can set

$$\mathcal{G}(V) = \mathrm{colim}_{b:k\to j} f_{a\circ b}^{-1}\mathcal{F}(u_b(V_j))$$

The value $\mathcal{G}(V)$ of the colimit is independent of the choice of $b : j \to i$ and of the object V_j with $u_j(V_j) = V$; we omit the verification. Moreover, if $\alpha : V \to V'$ is a morphism of $\mathcal{C}$, then we can choose $b : j \to i$ and a morphism $\alpha_j : V_j \to V_j'$ with $u_j(\alpha_j) = \alpha$. This induces a map $\mathcal{G}(V') \to \mathcal{G}(V)$ by using the restrictions along the morphisms $u_b(\alpha_j) : u_b(V_j) \to u_b(V_j')$. A check shows that $\mathcal{G}$ is a presheaf (omitted). In fact, $\mathcal{G}$ satisfies the sheaf condition. Namely, any covering $\mathcal{U} = \{U^t \to U\}$ in $\mathcal{C}$ comes from a finite level. Say $\mathcal{U}_j = \{U_j^t \to U_j\}$ is mapped to $\mathcal{U}$ by u_j for some $a : j \to i$ in $\mathcal{I}$. Then we have

$$H^0(\mathcal{U}, \mathcal{G}) = \mathrm{colim}_{b:k\to j} H^0(u_b(\mathcal{U}_j), f_{b\circ a}^{-1}\mathcal{F}) = \mathrm{colim}_{b:k\to j} f_{b\circ a}^{-1}\mathcal{F}(u_b(U_j)) = \mathcal{G}(U)$$

as desired. The first equality holds because filtered colimits commute with finite limits (Categories, Lemma 19.2). By construction $\mathcal{G}(U)$ is given by the right hand side of (18.3.1). Hence (18.3.1) is true if we can show that $\mathcal{G}$ is equal to $f_i^{-1}\mathcal{F}$.

In this paragraph we check that $\mathcal{G}$ is canonically isomorphic to $f_i^{-1}\mathcal{F}$. We strongly encourage the reader to skip this paragraph. To check this we have to show there is a bijection $\mathrm{Mor}_{Sh(\mathcal{C})}(\mathcal{G}, \mathcal{H}) = \mathrm{Mor}_{Sh(\mathcal{C}_i)}(\mathcal{F}, f_{i,*}\mathcal{H})$ functorial in the sheaf $\mathcal{H}$ on $\mathcal{C}$ where $f_{i,*} = u_i^p$. A map $\mathcal{G} \to \mathcal{H}$ is the same thing as a compatible system of maps

$$\varphi_{a,b,V_j} : f_{a\circ b}^{-1}\mathcal{F}(u_b(V_j)) \longrightarrow \mathcal{H}(u_j(V_j))$$

for all $a : j \to i$, $b : k \to j$ and $V_j \in \mathrm{Ob}(\mathcal{C}_j)$. The compatibilities force the maps φ_{a,b,V_j} to be equal to $\varphi_{a\circ b,\mathrm{id},u_b(V_j)}$. Given $a : j \to i$, the family of maps $\varphi_{a,\mathrm{id},V_j}$ corresponds to a map of sheaves $\varphi_a : f_a^{-1}\mathcal{F} \to f_{j,*}\mathcal{H}$. The compatibilities between the $\varphi_{a,\mathrm{id},u_a(V_i)}$ and the $\varphi_{\mathrm{id},\mathrm{id},V_i}$ implies that φ_a is the adjoint of the map φ_{id} via

$$\mathrm{Mor}_{Sh(\mathcal{C}_j)}(f_a^{-1}\mathcal{F}, f_{j,*}\mathcal{H}) = \mathrm{Mor}_{Sh(\mathcal{C}_i)}(\mathcal{F}, f_{a,*}f_{j,*}\mathcal{H}) = \mathrm{Mor}_{Sh(\mathcal{C}_i)}(\mathcal{F}, f_{i,*}\mathcal{H})$$

Thus finally we see that the whole system of maps φ_{a,b,V_j} is determined by the map $\varphi_{id} : \mathcal{F} \to f_{i,*}\mathcal{H}$. Conversely, given such a map $\psi : \mathcal{F} \to f_{i,*}\mathcal{H}$ we can read the argument just given backwards to construct the family of maps φ_{a,b,V_j}. This finishes the proof that $\mathcal{G} = f_i^{-1}\mathcal{F}$.

Assume (18.3.1) holds. Then the functor $\mathcal{F} \mapsto f_i^{-1}\mathcal{F}(U)$ commutes with finite limits because finite limits of sheaves are computed in the category of presheaves (Lemma 10.1), the functors f_a^{-1} commutes with finite limits, and filtered colimits

commute with finite limits. To see that $\mathcal{F} \mapsto f_i^{-1}\mathcal{F}(V)$ commutes with finite limits for a general object V of $\mathcal{C}$, we can use the same argument using the formula for $f_i^{-1}\mathcal{F}(V) = \mathcal{G}(V)$ given above. Thus f_i^{-1} is left exact and the proof of the lemma is complete. □

09YN **Lemma 18.4.** *In Situation 18.1 assume given*

(1) *a sheaf $\mathcal{F}_i$ on $\mathcal{C}_i$ for all $i \in \mathrm{Ob}(\mathcal{I})$,*
(2) *for $a : j \to i$ a map $\varphi_a : f_a^{-1}\mathcal{F}_i \to \mathcal{F}_j$ of sheaves on $\mathcal{C}_j$*

such that $\varphi_c = \varphi_b \circ f_b^{-1}\varphi_a$ whenever $c = a \circ b$. Set $\mathcal{F} = \operatorname{colim} f_i^{-1}\mathcal{F}_i$ on the site $\mathcal{C}$ of Lemma 18.2. Let $i \in \mathrm{Ob}(\mathcal{I})$ and $X_i \in Ob(\mathcal{C}_i)$. Then

$$\operatorname{colim}_{a:j\to i} \mathcal{F}_j(u_a(X_i)) = \mathcal{F}(u_i(X_i))$$

Proof. A formal argument shows that

$$\operatorname{colim}_{a:j\to i} \mathcal{F}_i(u_a(X_i)) = \operatorname{colim}_{a:j\to i} \operatorname{colim}_{b:k\to j} f_b^{-1}\mathcal{F}_j(u_{a\circ b}(X_i))$$

By (18.3.1) we see that the inner colimit is equal to $f_j^{-1}\mathcal{F}_j(u_i(X_i))$ hence we conclude by Lemma 17.7. □

0EXJ **Lemma 18.5.** *In Situation 18.1 assume we have a sheaf $\mathcal{F}$ on $\mathcal{C}$. Then*

$$\mathcal{F} = \operatorname{colim} f_i^{-1} f_{i,*}\mathcal{F}$$

where the transition maps are $f_j^{-1}\varphi_a$ for $a : j \to i$ where $\varphi_a : f_a^{-1} f_{i,*}\mathcal{F} \to f_{j,*}\mathcal{F}$ is a canonical map satisfying a cocycle condition as in Lemma 18.4.

Proof. For the morphism

$$\varphi_a : f_a^{-1} f_{i,*}\mathcal{F} \to f_{j,*}\mathcal{F}$$

we choose the adjoint to the identity map

$$f_{i,*}\mathcal{F} \to f_{a,*} f_{j,*}\mathcal{F}$$

Hence φ_a is the counit for the adjunction given by $(f_a^{-1}, f_{a,*})$. We must prove that for all $a : j \to i$ and $b : k \to i$ with composition $c = a \circ b$ we have $\varphi_c = \varphi_b \circ f_b^{-1}\varphi_a$. This follows from Categories, Lemma 24.9. Lastly, we must prove that the map given by adjunction

$$\operatorname{colim}_{i\in I} f_i^{-1} f_{i,*}\mathcal{F} \longrightarrow \mathcal{F}$$

is an isomorphism. For an object U of $\mathcal{C}$ we need to show the map

$$(\operatorname{colim}_{i\in I} f_i^{-1}\mathcal{F}_i)(U) \to \mathcal{F}(U)$$

is bijective. Choose an i and an object U_i of $\mathcal{C}_i$ with $u_i(U_i) = U$. Then the left hand side is equal to

$$(\operatorname{colim}_{i\in I} f_i^{-1}\mathcal{F}_i)(U) = \operatorname{colim}_{a:j\to i} f_{j,*}\mathcal{F}(u_a(U_i))$$

by Lemma 18.4. Since $u_j(u_a(U_i)) = U$ we have $f_{j,*}\mathcal{F}(u_a(U_i)) = \mathcal{F}(U)$ for all $a : j \to i$ by definition. Hence the value of the colimit is $\mathcal{F}(U)$ and the proof is complete. □

19. More functoriality of presheaves

00XF In this section we revisit the material of Section 5. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor between categories. Recall that

$$u^p : PSh(\mathcal{D}) \longrightarrow PSh(\mathcal{C})$$

is the functor that associates to $\mathcal{G}$ on $\mathcal{D}$ the presheaf $u^p\mathcal{G} = \mathcal{G} \circ u$. It turns out that this functor not only has a left adjoint (namely u_p) but also a right adjoint.

Namely, for any $V \in \mathrm{Ob}(\mathcal{D})$ we define a category ${}_V\mathcal{I} = {}^u_V\mathcal{I}$. Its objects are pairs $(U, \psi : u(U) \to V)$. Note that the arrow is in the opposite direction from the arrow we used in defining the category $\mathcal{I}^u_V$ in Section 5. A morphism $(U, \psi) \to (U', \psi')$ is given by a morphism $\alpha : U \to U'$ such that $\psi = \psi' \circ u(\alpha)$. In addition, given any presheaf of sets $\mathcal{F}$ on $\mathcal{C}$ we introduce the functor ${}_V\mathcal{F} : {}_V\mathcal{I}^{opp} \to Sets$, which is defined by the rule ${}_V\mathcal{F}(U, \psi) = \mathcal{F}(U)$. We define

$${}_pu(\mathcal{F})(V) := \lim_{{}_V\mathcal{I}^{opp}} {}_V\mathcal{F}$$

As a limit there are projection maps $c(\psi) : {}_pu(\mathcal{F})(V) \to \mathcal{F}(U)$ for every object (U, ψ) of ${}_V\mathcal{I}$. In fact,

$${}_pu(\mathcal{F})(V) = \left\{ \begin{array}{c} \text{collections } s_{(U,\psi)} \in \mathcal{F}(U) \\ \forall \beta : (U_1, \psi_1) \to (U_2, \psi_2) \text{ in } {}_V\mathcal{I} \\ \text{we have } \beta^* s_{(U_2,\psi_2)} = s_{(U_1,\psi_1)} \end{array} \right\}$$

where the correspondence is given by $s \mapsto s_{(U,\psi)} = c(\psi)(s)$. We leave it to the reader to define the restriction mappings ${}_pu(\mathcal{F})(V) \to {}_pu(\mathcal{F})(V')$ associated to any morphism $V' \to V$ of $\mathcal{D}$. The resulting presheaf will be denoted ${}_pu\mathcal{F}$.

00XG **Lemma 19.1.** *There is a canonical map ${}_pu\mathcal{F}(u(U)) \to \mathcal{F}(U)$, which is compatible with restriction maps.*

Proof. This is just the projection map $c(\mathrm{id}_{u(U)})$ above. □

Note that any map of presheaves $\mathcal{F} \to \mathcal{F}'$ gives rise to compatible systems of maps between functors ${}_V\mathcal{F} \to {}_V\mathcal{F}'$, and hence to a map of presheaves ${}_pu\mathcal{F} \to {}_pu\mathcal{F}'$. In other words, we have defined a functor

$${}_pu : PSh(\mathcal{C}) \longrightarrow PSh(\mathcal{D})$$

00XH **Lemma 19.2.** *The functor ${}_pu$ is a right adjoint to the functor u^p. In other words the formula*

$$\mathrm{Mor}_{PSh(\mathcal{C})}(u^p\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{PSh(\mathcal{D})}(\mathcal{G}, {}_pu\mathcal{F})$$

holds bifunctorially in $\mathcal{F}$ and $\mathcal{G}$.

Proof. This is proved in exactly the same way as the proof of Lemma 5.4. We note that the map $u^p{}_pu\mathcal{F} \to \mathcal{F}$ from Lemma 19.1 is the map that is used to go from the right to the left.

Alternately, think of a presheaf of sets $\mathcal{F}$ on $\mathcal{C}$ as a presheaf $\mathcal{F}'$ on $\mathcal{C}^{opp}$ with values in $Sets^{opp}$, and similarly on $\mathcal{D}$. Check that $({}_pu\mathcal{F})' = u_p(\mathcal{F}')$, and that $(u^p\mathcal{G})' = u^p(\mathcal{G}')$. By Remark 5.5 we have the adjointness of u_p and u^p for presheaves with values in $Sets^{opp}$. The result then follows formally from this. □

Thus given a functor $u : \mathcal{C} \to \mathcal{D}$ of categories we obtain a sequence of functors

$$u_p, u^p, {}_pu$$

between categories of presheaves where in each consecutive pair the first is left adjoint to the second.

09VQ **Lemma 19.3.** *Let $u : \mathcal{C} \to \mathcal{D}$ and $v : \mathcal{D} \to \mathcal{C}$ be functors of categories. Assume that v is right adjoint to u. Then we have*

1. *$u^p h_V = h_{v(V)}$ for any V in $\mathcal{D}$,*
2. *the category $\mathcal{I}^v_U$ has an initial object,*
3. *the category ${}^u_V\mathcal{I}$ has a final object,*
4. *${}_pu = v^p$, and*
5. *$u^p = v_p$.*

Proof. Proof of (1). Let V be an object of $\mathcal{D}$. We have $u^p h_V = h_{v(V)}$ because $u^p h_V(U) = \mathrm{Mor}_{\mathcal{D}}(u(U), V) = \mathrm{Mor}_{\mathcal{C}}(U, v(V))$ by assumption.

Proof of (2). Let U be an object of $\mathcal{C}$. Let $\eta : U \to v(u(U))$ be the map adjoint to the map $\mathrm{id} : u(U) \to u(U)$. Then we claim $(u(U), \eta)$ is an initial object of $\mathcal{I}^v_U$. Namely, given an object $(V, \phi : U \to v(V))$ of $\mathcal{I}^v_U$ the morphism ϕ is adjoint to a map $\psi : u(U) \to V$ which then defines a morphism $(u(U), \eta) \to (V, \phi)$.

Proof of (3). Let V be an object of $\mathcal{D}$. Let $\xi : u(v(V)) \to V$ be the map adjoint to the map $\mathrm{id} : v(V) \to v(V)$. Then we claim $(v(V), \xi)$ is a final object of ${}^u_V\mathcal{I}$. Namely, given an object $(U, \psi : u(U) \to V)$ of ${}^u_V\mathcal{I}$ the morphism ψ is adjoint to a map $\phi : U \to v(V)$ which then defines a morphism $(U, \psi) \to (v(V), \xi)$.

Hence for any presheaf $\mathcal{F}$ on $\mathcal{C}$ we have

$$\begin{aligned} v^p\mathcal{F}(V) &= \mathcal{F}(v(V)) \\ &= \mathrm{Mor}_{PSh(\mathcal{C})}(h_{v(V)}, \mathcal{F}) \\ &= \mathrm{Mor}_{PSh(\mathcal{C})}(u^p h_V, \mathcal{F}) \\ &= \mathrm{Mor}_{PSh(\mathcal{D})}(h_V, {}_pu\mathcal{F}) \\ &= {}_pu\mathcal{F}(V) \end{aligned}$$

which proves part (4). Part (5) follows by the uniqueness of adjoint functors. □

09VR **Lemma 19.4.** *A continuous functor of sites which has a continuous left adjoint defines a morphism of sites.*

Proof. Let $u : \mathcal{C} \to \mathcal{D}$ be a continuous functor of sites. Let $w : \mathcal{D} \to \mathcal{C}$ be a continuous left adjoint. Then $u_p = w^p$ by Lemma 19.3. Hence $u_s = w^s$ has a left adjoint, namely w_s (Lemma 13.3). Thus u_s has both a right and a left adjoint, whence is exact (Categories, Lemma 24.6). □

20. Cocontinuous functors

00XI There is another way to construct morphisms of topoi. This involves using cocontinuous functors between sites defined as follows.

00XJ **Definition 20.1.** Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. The functor u is called *cocontinuous* if for every $U \in \mathrm{Ob}(\mathcal{C})$ and every covering $\{V_j \to u(U)\}_{j \in J}$ of $\mathcal{D}$ there exists a covering $\{U_i \to U\}_{i \in I}$ of $\mathcal{C}$ such that the family of maps $\{u(U_i) \to u(U)\}_{i \in I}$ refines the covering $\{V_j \to u(U)\}_{j \in J}$.

Note that $\{u(U_i) \to u(U)\}_{i \in I}$ is in general *not* a covering of the site $\mathcal{D}$.

00XK **Lemma 20.2.** *Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be cocontinuous. Let $\mathcal{F}$ be a sheaf on $\mathcal{C}$. Then ${}_pu\mathcal{F}$ is a sheaf on $\mathcal{D}$, which we will denote ${}_su\mathcal{F}$.*

Proof. Let $\{V_j \to V\}_{j \in J}$ be a covering of the site $\mathcal{D}$. We have to show that

$$ {}_pu\mathcal{F}(V) \longrightarrow \prod {}_pu\mathcal{F}(V_j) \rightrightarrows \prod {}_pu\mathcal{F}(V_j \times_V V_{j'}) $$

is an equalizer diagram. Since ${}_pu$ is right adjoint to u^p we have

$$ {}_pu\mathcal{F}(V) = \mathrm{Mor}_{PSh(\mathcal{D})}(h_V, {}_pu\mathcal{F}) = \mathrm{Mor}_{PSh(\mathcal{C})}(u^p h_V, \mathcal{F}) = \mathrm{Mor}_{Sh(\mathcal{C})}((u^p h_V)^\#, \mathcal{F}) $$

Hence it suffices to show that

07GF (20.2.1)
$$ \coprod u^p h_{V_j \times_V V_{j'}} \rightrightarrows \coprod u^p h_{V_j} \longrightarrow u^p h_V $$

becomes a coequalizer diagram after sheafification. (Recall that a coproduct in the category of sheaves is the sheafification of the coproduct in the category of presheaves, see Lemma 10.13.)

We first show that the second arrow of (20.2.1) becomes surjective after sheafification. To do this we use Lemma 11.2. Thus it suffices to show a section s of $u^p h_V$ over U lifts to a section of $\coprod u^p h_{V_j}$ on the members of a covering of U. Note that s is a morphism $s : u(U) \to V$. Then $\{V_j \times_{V,s} u(U) \to u(U)\}$ is a covering of $\mathcal{D}$. Hence, as u is cocontinuous, there is a covering $\{U_i \to U\}$ such that $\{u(U_i) \to u(U)\}$ refines $\{V_j \times_{V,s} u(U) \to u(U)\}$. This means that each restriction $s|_{U_i} : u(U_i) \to V$ factors through a morphism $s_i : u(U_i) \to V_j$ for some j, i.e., $s|_{U_i}$ is in the image of $u^p h_{V_j}(U_i) \to u^p h_V(U_i)$ as desired.

Let $s, s' \in (\coprod u^p h_{V_j})^\#(U)$ map to the same element of $(u^p h_V)^\#(U)$. To finish the proof of the lemma we show that after replacing U by the members of a covering that s, s' are the image of the same section of $\coprod u^p h_{V_j \times_V V_{j'}}$ by the two maps of (20.2.1). We may first replace U by the members of a covering and assume that $s \in u^p h_{V_j}(U)$ and $s' \in u^p h_{V_{j'}}(U)$. A second such replacement guarantees that s and s' have the same image in $u^p h_V(U)$ instead of in the sheafification. Hence $s : u(U) \to V_j$ and $s' : u(U) \to V_{j'}$ are morphisms of $\mathcal{D}$ such that

$$ \begin{array}{ccc} u(U) & \xrightarrow[s']{} & V_{j'} \\ {\scriptstyle s}\downarrow & & \downarrow \\ V_j & \longrightarrow & V \end{array} $$

is commutative. Thus we obtain $t = (s, s') : u(U) \to V_j \times_V V_{j'}$, i.e., a section $t \in u^p h_{V_j \times_V V_{j'}}(U)$ which maps to s, s' as desired. □

00XL **Lemma 20.3.** *Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be cocontinuous. The functor $Sh(\mathcal{D}) \to Sh(\mathcal{C})$, $\mathcal{G} \mapsto (u^p \mathcal{G})^\#$ is a left adjoint to the functor ${}_su$ introduced in Lemma 20.2 above. Moreover, it is exact.*

Proof. Let us prove the adjointness property as follows

$$ \begin{aligned} \mathrm{Mor}_{Sh(\mathcal{C})}((u^p \mathcal{G})^\#, \mathcal{F}) &= \mathrm{Mor}_{PSh(\mathcal{C})}(u^p \mathcal{G}, \mathcal{F}) \\ &= \mathrm{Mor}_{PSh(\mathcal{D})}(\mathcal{G}, {}_pu\mathcal{F}) \\ &= \mathrm{Mor}_{Sh(\mathcal{D})}(\mathcal{G}, {}_su\mathcal{F}). \end{aligned} $$

Thus it is a left adjoint and hence right exact, see Categories, Lemma 24.6. We have seen that sheafification is left exact, see Lemma 10.14. Moreover, the inclusion $i : Sh(\mathcal{D}) \to PSh(\mathcal{D})$ is left exact by Lemma 10.1. Finally, the functor u^p is left exact because it is a right adjoint (namely to u_p). Thus the functor is the composition $^\# \circ u^p \circ i$ of left exact functors, hence left exact. □

We finish this section with a technical lemma.

00XM **Lemma 20.4.** *In the situation of Lemma 20.3. For any presheaf $\mathcal{G}$ on $\mathcal{D}$ we have $(u^p\mathcal{G})^\# = (u^p(\mathcal{G}^\#))^\#$.*

Proof. For any sheaf $\mathcal{F}$ on $\mathcal{C}$ we have

$$\begin{aligned} \mathrm{Mor}_{Sh(\mathcal{C})}((u^p(\mathcal{G}^\#))^\#, \mathcal{F}) &= \mathrm{Mor}_{Sh(\mathcal{D})}(\mathcal{G}^\#, {}_su\mathcal{F}) \\ &= \mathrm{Mor}_{Sh(\mathcal{D})}(\mathcal{G}^\#, {}_pu\mathcal{F}) \\ &= \mathrm{Mor}_{PSh(\mathcal{D})}(\mathcal{G}, {}_pu\mathcal{F}) \\ &= \mathrm{Mor}_{PSh(\mathcal{C})}(u^p\mathcal{G}, \mathcal{F}) \\ &= \mathrm{Mor}_{Sh(\mathcal{C})}((u^p\mathcal{G})^\#, \mathcal{F}) \end{aligned}$$

and the result follows from the Yoneda lemma. □

09W7 **Remark 20.5.** Let $u : \mathcal{C} \to \mathcal{D}$ be a functor between categories. Given morphisms $g : u(U) \to V$ and $f : W \to V$ in $\mathcal{D}$ we can consider the functor

$$\mathcal{C}^{opp} \longrightarrow Sets, \quad T \longmapsto \mathrm{Mor}_{\mathcal{C}}(T, U) \times_{\mathrm{Mor}_{\mathcal{D}}(u(T), V)} \mathrm{Mor}_{\mathcal{D}}(u(T), W)$$

If this functor is representable, denote $U \times_{g,V,f} W$ the corresponding object of $\mathcal{C}$. Assume that $\mathcal{C}$ and $\mathcal{D}$ are sites. Consider the property P: for every covering $\{f_j : V_j \to V\}$ of $\mathcal{D}$ and any morphism $g : u(U) \to V$ we have

(1) $U \times_{g,V,f_i} V_i$ exists for all i, and
(2) $\{U \times_{g,V,f_i} V_i \to U\}$ is a covering of $\mathcal{C}$.

Please note the similarity with the definition of continuous functors. If u has P then u is cocontinuous (details omitted). Many of the cocontinuous functors we will encounter satisfy P.

21. Cocontinuous functors and morphisms of topoi

00XN It is clear from the above that a cocontinuous functor u gives a morphism of topoi in the same direction as u. Thus this is in the opposite direction from the morphism of topoi associated (under certain conditions) to a continuous u as in Definition 14.1, Proposition 14.7, and Lemma 15.2.

00XO **Lemma 21.1.** *Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be cocontinuous. The functors $g_* = {}_su$ and $g^{-1} = (u^p\ )^\#$ define a morphism of topoi g from $Sh(\mathcal{C})$ to $Sh(\mathcal{D})$.*

Proof. This is exactly the content of Lemma 20.3. □

03L5 **Lemma 21.2.** *Let $u : \mathcal{C} \to \mathcal{D}$, and $v : \mathcal{D} \to \mathcal{E}$ be cocontinuous functors. Then $v \circ u$ is cocontinuous and we have $h = g \circ f$ where $f : Sh(\mathcal{C}) \to Sh(\mathcal{D})$, resp. $g : Sh(\mathcal{D}) \to Sh(\mathcal{E})$, resp. $h : Sh(\mathcal{C}) \to Sh(\mathcal{E})$ is the morphism of topoi associated to u, resp. v, resp. $v \circ u$.*

Proof. Let $U \in \mathrm{Ob}(\mathcal{C})$. Let $\{E_i \to v(u(U))\}$ be a covering of U in $\mathcal{E}$. By assumption there exists a covering $\{D_j \to u(U)\}$ in $\mathcal{D}$ such that $\{v(D_j) \to v(u(U))\}$ refines $\{E_i \to v(u(U))\}$. Also by assumption there exists a covering $\{C_l \to U\}$ in $\mathcal{C}$ such that $\{u(C_l) \to u(U)\}$ refines $\{D_j \to u(U)\}$. Then it is true that $\{v(u(C_l)) \to v(u(U))\}$ refines the covering $\{E_i \to v(u(U))\}$. This proves that $v \circ u$ is cocontinuous. To prove the last assertion it suffices to show that ${}_sv \circ {}_su = {}_s(v \circ u)$. It suffices to prove that ${}_pv \circ {}_pu = {}_p(v \circ u)$, see Lemma 20.2. Since ${}_pu$, resp. ${}_pv$, resp. ${}_p(v \circ u)$ is right adjoint to u^p, resp. v^p, resp. $(v \circ u)^p$ it suffices to prove that $u^p \circ v^p = (v \circ u)^p$. And this is direct from the definitions. □

00XP **Example 21.3.** Let X be a topological space. Let $j : U \to X$ be the inclusion of an open subspace. Recall that we have sites X_{Zar} and U_{Zar}, see Example 6.4. Recall that we have the functor $u : X_{Zar} \to U_{Zar}$ associated to j which is continuous and gives rise to a morphism of sites $U_{Zar} \to X_{Zar}$, see Example 14.2. This also gives a morphism of topoi (j_*, j^{-1}). Next, consider the functor $v : U_{Zar} \to X_{Zar}$, $V \mapsto v(V) = V$ (just the same open but now thought of as an object of X_{Zar}). This functor is cocontinuous. Namely, if $v(V) = \bigcup_{j \in J} W_j$ is an open covering in X, then each W_j must be a subset of U and hence is of the form $v(V_j)$, and trivially $V = \bigcup_{j \in J} V_j$ is an open covering in U. We conclude by Lemma 21.1 above that there is a morphism of topoi associated to v

$$Sh(U) \longrightarrow Sh(X)$$

given by ${}_sv$ and $(v^p)^\#$. We claim that actually $(v^p)^\# = j^{-1}$ and that ${}_sv = j_*$, in other words, that this is the same morphism of topoi as the one given above. Perhaps the easiest way to see this is to realize that for any sheaf $\mathcal{G}$ on X we have $v^p\mathcal{G}(V) = \mathcal{G}(V)$ which according to Sheaves, Lemma 31.1 is a description of $j^{-1}\mathcal{G}$ (and hence sheafification is superfluous in this case). The equality of ${}_sv$ and j_* follows by uniqueness of adjoint functors (but may also be computed directly).

00XQ **Example 21.4.** This example is a slight generalization of Example 21.3. Let $f : X \to Y$ be a continuous map of topological spaces. Assume that f is open. Recall that we have sites X_{Zar} and Y_{Zar}, see Example 6.4. Recall that we have the functor $u : Y_{Zar} \to X_{Zar}$ associated to f which is continuous and gives rise to a morphism of sites $X_{Zar} \to Y_{Zar}$, see Example 14.2. This also gives a morphism of topoi (f_*, f^{-1}). Next, consider the functor $v : X_{Zar} \to Y_{Zar}$, $U \mapsto v(U) = f(U)$. This functor is cocontinuous. Namely, if $f(U) = \bigcup_{j \in J} V_j$ is an open covering in Y, then setting $U_j = f^{-1}(V_j) \cap U$ we get an open covering $U = \bigcup U_j$ such that $f(U) = \bigcup f(U_j)$ is a refinement of $f(U) = \bigcup V_j$. We conclude by Lemma 21.1 above that there is a morphism of topoi associated to v

$$Sh(X) \longrightarrow Sh(Y)$$

given by ${}_sv$ and $(v^p)^\#$. We claim that actually $(v^p)^\# = f^{-1}$ and that ${}_sv = f_*$, in other words, that this is the same morphism of topoi as the one given above. For any sheaf $\mathcal{G}$ on Y we have $v^p\mathcal{G}(U) = \mathcal{G}(f(U))$. On the other hand, we may compute $u_p\mathcal{G}(U) = \mathrm{colim}_{f(U) \subset V} \mathcal{G}(V) = \mathcal{G}(f(U))$ because clearly $(f(U), U \subset f^{-1}(f(U)))$ is an initial object of the category $\mathcal{I}^u_U$ of Section 5. Hence $u_p = v^p$ and we conclude $f^{-1} = u_s = (v^p)^\#$. The equality of ${}_sv$ and f_* follows by uniqueness of adjoint functors (but may also be computed directly).

In the first Example 21.3 the functor v is also continuous. But in the second Example 21.4 it is generally not continuous because condition (2) of Definition 13.1

may fail. Hence the following lemma applies to the first example, but not to the second.

00XR **Lemma 21.5.** *Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. Assume that*

(a) *u is cocontinuous, and*
(b) *u is continuous.*

Let $g : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be the associated morphism of topoi. Then

(1) *sheafification in the formula $g^{-1} = (u^p\ )^\#$ is unnecessary, in other words $g^{-1}(\mathcal{G})(U) = \mathcal{G}(u(U))$,*
(2) *g^{-1} has a left adjoint $g_! = (u_p\ )^\#$, and*
(3) *g^{-1} commutes with arbitrary limits and colimits.*

Proof. By Lemma 13.2 for any sheaf $\mathcal{G}$ on $\mathcal{D}$ the presheaf $u^p\mathcal{G}$ is a sheaf on $\mathcal{C}$. And then we see the adjointness by the following string of equalities

$$\begin{aligned} \mathrm{Mor}_{Sh(\mathcal{C})}(\mathcal{F}, g^{-1}\mathcal{G}) &= \mathrm{Mor}_{PSh(\mathcal{C})}(\mathcal{F}, u^p\mathcal{G}) \\ &= \mathrm{Mor}_{PSh(\mathcal{D})}(u_p\mathcal{F}, \mathcal{G}) \\ &= \mathrm{Mor}_{Sh(\mathcal{D})}(g_!\mathcal{F}, \mathcal{G}) \end{aligned}$$

The statement on limits and colimits follows from the discussion in Categories, Section 24. □

In the situation of Lemma 21.5 above we see that we have a sequence of adjoint functors

$$g_!,\ g^{-1},\ g_*.$$

The functor $g_!$ is *not* exact in general, because it does not transform a final object of $Sh(\mathcal{C})$ into a final object of $Sh(\mathcal{D})$ in general. See Sheaves, Remark 31.13. On the other hand, in the topological setting of Example 21.3 the functor $j_!$ is exact on abelian sheaves, see Modules, Lemma 3.4. The following lemma gives the generalization to the case of sites.

00XS **Lemma 21.6.** *Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. Assume that*

(a) *u is cocontinuous,*
(b) *u is continuous, and*
(c) *fibre products and equalizers exist in $\mathcal{C}$ and u commutes with them.*

In this case the functor $g_!$ above commutes with fibre products and equalizers (and more generally with finite connected limits).

Proof. Assume (a), (b), and (c). We have $g_! = (u_p\ )^\#$. Recall (Lemma 10.1) that limits of sheaves are equal to the corresponding limits as presheaves. And sheafification commutes with finite limits (Lemma 10.14). Thus it suffices to show that u_p commutes with fibre products and equalizers. To do this it suffices that colimits over the categories $(\mathcal{I}^u_V)^{opp}$ of Section 5 commute with fibre products and equalizers. This follows from Lemma 5.1 and Categories, Lemma 19.9. □

The following lemma deals with a case that is even more like the morphism associated to an open immersion of topological spaces.

00XT **Lemma 21.7.** *Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. Assume that*

(a) *u is cocontinuous,*
(b) *u is continuous, and*
(c) *u is fully faithful.*

For $g_!, g^{-1}, g_*$ as above the canonical maps $\mathcal{F} \to g^{-1}g_!\mathcal{F}$ and $g^{-1}g_*\mathcal{F} \to \mathcal{F}$ are isomorphisms for all sheaves $\mathcal{F}$ on $\mathcal{C}$.

Proof. Let X be an object of $\mathcal{C}$. In Lemmas 20.2 and 21.5 we have seen that sheafification is not necessary for the functors $g^{-1} = (u^p\ )^\#$ and $g_* = ({}_pu\ )^\#$. We may compute $(g^{-1}g_*\mathcal{F})(X) = g_*\mathcal{F}(u(X)) = \lim \mathcal{F}(Y)$. Here the limit is over the category of pairs $(Y, u(Y) \to u(X))$ where the morphisms $u(Y) \to u(X)$ are not required to be of the form $u(\alpha)$ with α a morphism of $\mathcal{C}$. By assumption (c) we see that they automatically come from morphisms of $\mathcal{C}$ and we deduce that the limit is the value on $(X, u(\mathrm{id}_X))$, i.e., $\mathcal{F}(X)$. This proves that $g^{-1}g_*\mathcal{F} = \mathcal{F}$.

On the other hand, $(g^{-1}g_!\mathcal{F})(X) = g_!\mathcal{F}(u(X)) = (u_p\mathcal{F})^\#(u(X))$, and $u_p\mathcal{F}(u(X)) = \operatorname{colim} \mathcal{F}(Y)$. Here the colimit is over the category of pairs $(Y, u(X) \to u(Y))$ where the morphisms $u(X) \to u(Y)$ are not required to be of the form $u(\alpha)$ with α a morphism of $\mathcal{C}$. By assumption (c) we see that they automatically come from morphisms of $\mathcal{C}$ and we deduce that the colimit is the value on $(X, u(\mathrm{id}_X))$, i.e., $\mathcal{F}(X)$. Thus for every $X \in \mathrm{Ob}(\mathcal{C})$ we have $u_p\mathcal{F}(u(X)) = \mathcal{F}(X)$. Since u is cocontinuous and continuous any covering of $u(X)$ in $\mathcal{D}$ can be refined by a covering (!) $\{u(X_i) \to u(X)\}$ of $\mathcal{D}$ where $\{X_i \to X\}$ is a covering in $\mathcal{C}$. This implies that $(u_p\mathcal{F})^+(u(X)) = \mathcal{F}(X)$ also, since in the colimit defining the value of $(u_p\mathcal{F})^+$ on $u(X)$ we may restrict to the cofinal system of coverings $\{u(X_i) \to u(X)\}$ as above. Hence we see that $(u_p\mathcal{F})^+(u(X)) = \mathcal{F}(X)$ for all objects X of $\mathcal{C}$ as well. Repeating this argument one more time gives the equality $(u_p\mathcal{F})^\#(u(X)) = \mathcal{F}(X)$ for all objects X of $\mathcal{C}$. This produces the desired equality $g^{-1}g_!\mathcal{F} = \mathcal{F}$. □

Finally, here is a case that does not have any corresponding topological example. We will use this lemma to see what happens when we enlarge a "partial universe" of schemes keeping the same topology. In the situation of the lemma, the morphism of topoi $g : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ identifies $Sh(\mathcal{C})$ as a subtopos of $Sh(\mathcal{D})$ (Section 43) and moreover, the given embedding has a retraction.

00XU **Lemma 21.8.** *Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. Assume that*

(a) *u is cocontinuous,*
(b) *u is continuous,*
(c) *u is fully faithful,*
(d) *fibre products exist in $\mathcal{C}$ and u commutes with them, and*
(e) *there exist final objects $e_\mathcal{C} \in \mathrm{Ob}(\mathcal{C})$, $e_\mathcal{D} \in \mathrm{Ob}(\mathcal{D})$ such that $u(e_\mathcal{C}) = e_\mathcal{D}$.*

Let $g_!, g^{-1}, g_*$ be as above. Then, u defines a morphism of sites $f : \mathcal{D} \to \mathcal{C}$ with $f_* = g^{-1}$, $f^{-1} = g_!$. The composition

$$Sh(\mathcal{C}) \xrightarrow{g} Sh(\mathcal{D}) \xrightarrow{f} Sh(\mathcal{C})$$

is isomorphic to the identity morphism of the topos $Sh(\mathcal{C})$. Moreover, the functor f^{-1} is fully faithful.

Proof. By assumption the functor u satisfies the hypotheses of Proposition 14.7. Hence u defines a morphism of sites and hence a morphism of topoi f as in Lemma 15.2. The formulas $f_* = g^{-1}$ and $f^{-1} = g_!$ are clear from the lemma cited and Lemma 21.5. We have $f_* \circ g_* = g^{-1} \circ g_* \cong \mathrm{id}$, and $g^{-1} \circ f^{-1} = g^{-1} \circ g_! \cong \mathrm{id}$ by Lemma 21.7.

We still have to show that f^{-1} is fully faithful. Let $\mathcal{F}, \mathcal{G} \in \mathrm{Ob}(Sh(\mathcal{C}))$. We have to show that the map

$$\mathrm{Mor}_{Sh(\mathcal{C})}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathrm{Mor}_{Sh(\mathcal{D})}(f^{-1}\mathcal{F}, f^{-1}\mathcal{G})$$

is bijective. But the right hand side is equal to

$$\begin{aligned} \mathrm{Mor}_{Sh(\mathcal{D})}(f^{-1}\mathcal{F}, f^{-1}\mathcal{G}) &= \mathrm{Mor}_{Sh(\mathcal{C})}(\mathcal{F}, f_*f^{-1}\mathcal{G}) \\ &= \mathrm{Mor}_{Sh(\mathcal{C})}(\mathcal{F}, g^{-1}f^{-1}\mathcal{G}) \\ &= \mathrm{Mor}_{Sh(\mathcal{C})}(\mathcal{F}, \mathcal{G}) \end{aligned}$$

(the first equality by adjunction) which proves what we want. □

00XV **Example 21.9.** Let X be a topological space. Let $i : Z \to X$ be the inclusion of a subset (with induced topology). Consider the functor $u : X_{Zar} \to Z_{Zar}$, $U \mapsto u(U) = Z \cap U$. At first glance it may appear that this functor is cocontinuous as well. After all, since Z has the induced topology, shouldn't any covering of $U \cap Z$ it come from a covering of U in X? Not so! Namely, what if $U \cap Z = \emptyset$? In that case, the empty covering is a covering of $U \cap Z$, and the empty covering can only be refined by the empty covering. Thus we conclude that u cocontinuous $\Rightarrow$ every nonempty open U of X has nonempty intersection with Z. But this is not sufficient. For example, if $X = \mathbf{R}$ the real number line with the usual topology, and $Z = \mathbf{R} \setminus \{0\}$, then there is an open covering of Z, namely $Z = \{x < 0\} \cup \bigcup_n \{1/n < x\}$ which cannot be refined by the restriction of any open covering of X.

22. Cocontinuous functors which have a right adjoint

00XW It may happen that a cocontinuous functor u has a right adjoint v. In this case it is often the case that v is continuous, and if so, then it defines a morphism of topoi (which is the same as the one defined by u).

00XX **Lemma 22.1.** *Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$, and $v : \mathcal{D} \to \mathcal{C}$ be functors. Assume that u is cocontinuous, and that v is a right adjoint to u. Let $g : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be the morphism of topoi associated to u, see Lemma 21.1. Then $g_*\mathcal{F}$ is equal to the presheaf $v^p\mathcal{F}$, in other words, $(g_*\mathcal{F})(V) = \mathcal{F}(v(V))$.*

Proof. We have $u^p h_V = h_{v(V)}$ by Lemma 19.3. By Lemma 20.4 this implies that $g^{-1}(h_V^\#) = (u^p h_V^\#)^\# = (u^p h_V)^\# = h_{v(V)}^\#$. Hence for any sheaf $\mathcal{F}$ on $\mathcal{C}$ we have

$$\begin{aligned} (g_*\mathcal{F})(V) &= \mathrm{Mor}_{Sh(\mathcal{D})}(h_V^\#, g_*\mathcal{F}) \\ &= \mathrm{Mor}_{Sh(\mathcal{C})}(g^{-1}(h_V^\#), \mathcal{F}) \\ &= \mathrm{Mor}_{Sh(\mathcal{C})}(h_{v(V)}^\#, \mathcal{F}) \\ &= \mathcal{F}(v(V)) \end{aligned}$$

which proves the lemma. □

In the situation of Lemma 22.1 we see that v^p transforms sheaves into sheaves. Hence we can define $v^s = v^p$ restricted to sheaves. Just as in Lemma 13.3 we see that $v_s : \mathcal{G} \mapsto (v_p\mathcal{G})^\#$ is a left adjoint to v^s. On the other hand, we have $v^s = g_*$ and g^{-1} is a left adjoint of g_* as well. We conclude that $g^{-1} = v_s$ is exact.

00XY **Lemma 22.2.** *In the situation of Lemma 22.1. We have $g_* = v^s = v^p$ and $g^{-1} = v_s = (v_p\ )^\#$. If v is continuous then v defines a morphism of sites f from $\mathcal{C}$ to $\mathcal{D}$ whose associated morphism of topoi is equal to the morphism g associated to the cocontinuous functor u. In other words, a continuous functor which has a cocontinuous left adjoint defines a morphism of sites.*

Proof. Clear from the discussion above the lemma and Definitions 14.1 and Lemma 15.2. □

0EWJ **Example 22.3.** This example continues the discussion of Example 14.3 from which we borrow the notation $\mathcal{C}, \tau, \tau', \epsilon$. Observe that the identity functor $v : \mathcal{C}_{\tau'} \to \mathcal{C}_\tau$ is a continuous functor and the identity functor $u : \mathcal{C}_\tau \to \mathcal{C}_{\tau'}$ is a cocontinuous functor. Moreover u is left adjoint to v. Hence the results of Lemmas 22.1 and 22.2 apply and we conclude v defines a morphism of sites, namely

$$\epsilon : \mathcal{C}_\tau \longrightarrow \mathcal{C}_{\tau'}$$

whose corresponding morphism of topoi is the same as the morphism of topoi associated to the cocontinuous functor u.

23. Cocontinuous functors which have a left adjoint

08NG It may happen that a cocontinuous functor u has a left adjoint w.

08NH **Lemma 23.1.** *Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $g : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be the morphism of topoi associated to a continuous and cocontinuous functor $u : \mathcal{C} \to \mathcal{D}$, see Lemmas 21.1 and 21.5.*

1. *If $w : \mathcal{D} \to \mathcal{C}$ is a left adjoint to u, then*
   (a) *$g_!\mathcal{F}$ is the sheaf associated to the presheaf $w^p\mathcal{F}$, and*
   (b) *$g_!$ is exact.*
2. *if w is a continuous left adjoint, then $g_!$ has a left adjoint.*
3. *If w is a cocontinuous left adjoint, then $g_! = h^{-1}$ and $g^{-1} = h_*$ where $h : Sh(\mathcal{D}) \to Sh(\mathcal{C})$ is the morphism of topoi associated to w.*

Proof. Recall that $g_!\mathcal{F}$ is the sheafification of $u_p\mathcal{F}$. Hence (1)(a) follows from the fact that $u_p = w^p$ by Lemma 19.3.

To see (1)(b) note that $g_!$ commutes with all colimits as $g_!$ is a left adjoint (Categories, Lemma 24.5). Let $i \mapsto \mathcal{F}_i$ be a finite diagram in $Sh(\mathcal{C})$. Then $\lim \mathcal{F}_i$ is computed in the category of presheaves (Lemma 10.1). Since w^p is a right adjoint (Lemma 5.4) we see that $w^p \lim \mathcal{F}_i = \lim w^p\mathcal{F}_i$. Since sheafification is exact (Lemma 10.14) we conclude by (1)(a).

Assume w is continuous. Then $g_! = (w^p\ )^\# = w^s$ but sheafification isn't necessary and one has the left adjoint w_s, see Lemmas 13.2 and 13.3.

Assume w is cocontinuous. The equality $g_! = h^{-1}$ follows from (1)(a) and the definitions. The equality $g^{-1} = h_*$ follows from the equality $g_! = h^{-1}$ and uniqueness of adjoint functor. Alternatively one can deduce it from Lemma 22.1. □

24. Existence of lower shriek

09YW In this section we discuss some cases of morphisms of topoi f for which f^{-1} has a left adjoint $f_!$.

09YX **Lemma 24.1.** *Let $\mathcal{C}$, $\mathcal{D}$ be two sites. Let $f : Sh(\mathcal{D}) \to Sh(\mathcal{C})$ be a morphism of topoi. Let $E \subset \mathrm{Ob}(\mathcal{D})$ be a subset such that*

(1) *for $V \in E$ there exists a sheaf $\mathcal{G}$ on $\mathcal{C}$ such that $f^{-1}\mathcal{F}(V) = \mathrm{Mor}_{Sh(\mathcal{C})}(\mathcal{G}, \mathcal{F})$ functorially for $\mathcal{F}$ in $Sh(\mathcal{C})$,*
(2) *every object of $\mathcal{D}$ has a covering by objects of E.*

Then f^{-1} has a left adjoint $f_!$.

Proof. By the Yoneda lemma (Categories, Lemma 3.5) the sheaf $\mathcal{G}_V$ corresponding to $V \in E$ is defined up to unique isomorphism by the formula $f^{-1}\mathcal{F}(V) = \mathrm{Mor}_{Sh(\mathcal{C})}(\mathcal{G}_V, \mathcal{F})$. Recall that $f^{-1}\mathcal{F}(V) = \mathrm{Mor}_{Sh(\mathcal{D})}(h_V^\#, f^{-1}\mathcal{F})$. Denote $i_V : h_V^\# \to f^{-1}\mathcal{G}_V$ the map corresponding to id in $\mathrm{Mor}(\mathcal{G}_V, \mathcal{G}_V)$. Functoriality in (1) implies that the bijection is given by

$$\mathrm{Mor}_{Sh(\mathcal{C})}(\mathcal{G}_V, \mathcal{F}) \to \mathrm{Mor}_{Sh(\mathcal{D})}(h_V^\#, f^{-1}\mathcal{F}), \quad \varphi \mapsto f^{-1}\varphi \circ i_V$$

For any $V_1, V_2 \in E$ there is a canonical map

$$\mathrm{Mor}_{Sh(\mathcal{D})}(h_{V_2}^\#, h_{V_1}^\#) \to \mathrm{Hom}_{Sh(\mathcal{C})}(\mathcal{G}_{V_2}, \mathcal{G}_{V_1}), \quad \varphi \mapsto f_!(\varphi)$$

which is characterized by $f^{-1}(f_!(\varphi)) \circ i_{V_2} = i_{V_1} \circ \varphi$. Note that $\varphi \mapsto f_!(\varphi)$ is compatible with composition; this can be seen directly from the characterization. Hence $h_V^\# \mapsto \mathcal{G}_V$ and $\varphi \mapsto f_!\varphi$ is a functor from the full subcategory of $Sh(\mathcal{D})$ whose objects are the $h_V^\#$.

Let J be a set and let $J \to E$, $j \mapsto V_j$ be a map. Then we have a functorial bijection

$$\mathrm{Mor}_{Sh(\mathcal{C})}(\coprod \mathcal{G}_{V_j}, \mathcal{F}) \longrightarrow \mathrm{Mor}_{Sh(\mathcal{D})}(\coprod h_{V_j}^\#, f^{-1}\mathcal{F})$$

using the product of the bijections above. Hence we can extend the functor $f_!$ to the full subcategory of $Sh(\mathcal{D})$ whose objects are coproducts of $h_V^\#$ with $V \in E$.

Given an arbitrary sheaf $\mathcal{H}$ on $\mathcal{D}$ we choose a coequalizer diagram

$$\mathcal{H}_1 \rightrightarrows \mathcal{H}_0 \longrightarrow \mathcal{H}$$

where $\mathcal{H}_i = \coprod h_{V_{i,j}}^\#$ is a coproduct with $V_{i,j} \in E$. This is possible by assumption (2), see Lemma 12.5 (for those worried about set theoretical issues, note that the construction given in Lemma 12.5 is canonical). Define $f_!(\mathcal{H})$ to be the sheaf on $\mathcal{C}$ which makes

$$f_!\mathcal{H}_1 \rightrightarrows f_!\mathcal{H}_0 \longrightarrow f_!\mathcal{H}$$

a coequalizer diagram. Then

$$\begin{aligned}
\mathrm{Mor}(f_!\mathcal{H}, \mathcal{F}) &= \mathrm{Equalizer}(\ \mathrm{Mor}(f_!\mathcal{H}_0, \mathcal{F}) \rightrightarrows \mathrm{Mor}(f_!\mathcal{H}_1, \mathcal{F})\ ) \\
&= \mathrm{Equalizer}(\ \mathrm{Mor}(\mathcal{H}_0, f^{-1}\mathcal{F}) \rightrightarrows \mathrm{Mor}(\mathcal{H}_1, f^{-1}\mathcal{F})\ ) \\
&= \mathrm{Hom}(\mathcal{H}, f^{-1}\mathcal{F})
\end{aligned}$$

Hence we see that we can extend $f_!$ to the whole category of sheaves on $\mathcal{D}$. □

25. Localization

00XZ Let $\mathcal{C}$ be a site. Let $U \in \mathrm{Ob}(\mathcal{C})$. See Categories, Example 2.13 for the definition of the category $\mathcal{C}/U$ of objects over U. We turn $\mathcal{C}/U$ into a site by declaring a family of morphisms $\{V_j \to V\}$ of objects over U to be a covering of $\mathcal{C}/U$ if and only if it is a covering in $\mathcal{C}$. Consider the forgetful functor

$$j_U : \mathcal{C}/U \longrightarrow \mathcal{C}.$$

This is clearly cocontinuous and continuous. Hence by the results of the previous sections we obtain a morphism of topoi

$$j_U : Sh(\mathcal{C}/U) \longrightarrow Sh(\mathcal{C})$$

given by j_U^{-1} and j_{U*}, as well as a functor $j_{U!}$.

00Y0 **Definition 25.1.** Let $\mathcal{C}$ be a site. Let $U \in \mathrm{Ob}(\mathcal{C})$.

(1) The site $\mathcal{C}/U$ is called the *localization of the site $\mathcal{C}$ at the object U*.
(2) The morphism of topoi $j_U : Sh(\mathcal{C}/U) \to Sh(\mathcal{C})$ is called the *localization morphism*.
(3) The functor j_{U*} is called the *direct image functor*.
(4) For a sheaf $\mathcal{F}$ on $\mathcal{C}$ the sheaf $j_U^{-1}\mathcal{F}$ is called the *restriction of $\mathcal{F}$ to $\mathcal{C}/U$*.
(5) For a sheaf $\mathcal{G}$ on $\mathcal{C}/U$ the sheaf $j_{U!}\mathcal{G}$ is called the *extension of $\mathcal{G}$ by the empty set*.

The restriction $j_U^{-1}\mathcal{F}$ is the sheaf defined by the rule $j_U^{-1}\mathcal{F}(X/U) = \mathcal{F}(X)$ as expected. The extension by the empty set also has a very easy description in this case; here it is.

03CD **Lemma 25.2.** *Let $\mathcal{C}$ be a site. Let $U \in \mathrm{Ob}(\mathcal{C})$. Let $\mathcal{G}$ be a presheaf on $\mathcal{C}/U$. Then $j_{U!}(\mathcal{G}^\#)$ is the sheaf associated to the presheaf*

$$V \longmapsto \coprod\nolimits_{\varphi \in \mathrm{Mor}_{\mathcal{C}}(V,U)} \mathcal{G}(V \xrightarrow{\varphi} U)$$

with obvious restriction mappings.

Proof. By Lemma 21.5 we have $j_{U!}(\mathcal{G}^\#) = ((j_U)_p\mathcal{G}^\#)^\#$. By Lemma 13.4 this is equal to $((j_U)_p\mathcal{G})^\#$. Hence it suffices to prove that $(j_U)_p$ is given by the formula above for any presheaf $\mathcal{G}$ on $\mathcal{C}/U$. OK, and by the definition in Section 5 we have

$$(j_U)_p\mathcal{G}(V) = \mathrm{colim}_{(W/U, V \to W)} \mathcal{G}(W)$$

Now it is clear that the category of pairs $(W/U, V \to W)$ has an object $O_\varphi = (\varphi : V \to U, \mathrm{id} : V \to V)$ for every $\varphi : V \to U$, and moreover for any object there is a unique morphism from one of the O_φ into it. The result follows. □

03HU **Lemma 25.3.** *Let $\mathcal{C}$ be a site. Let $U \in \mathrm{Ob}(\mathcal{C})$. Let X/U be an object of $\mathcal{C}/U$. Then we have $j_{U!}(h^\#_{X/U}) = h^\#_X$.*

Proof. Denote $p : X \to U$ the structure morphism of X. By Lemma 25.2 we see $j_{U!}(h^\#_{X/U})$ is the sheaf associated to the presheaf

$$V \longmapsto \coprod\nolimits_{\varphi \in \mathrm{Mor}_{\mathcal{C}}(V,U)} \{\psi : V \to X \mid p \circ \psi = \varphi\}$$

This is clearly the same thing as $\mathrm{Mor}_{\mathcal{C}}(V, X)$. Hence the lemma follows. □

We have $j_{U!}(*) = h_U^\#$ by either of the two lemmas above. Hence for every sheaf $\mathcal{G}$ over $\mathcal{C}/U$ there is a canonical map of sheaves $j_{U!}\mathcal{G} \to h_U^\#$. This characterizes sheaves in the essential image of $j_{U!}$.

00Y1 **Lemma 25.4.** *Let $\mathcal{C}$ be a site. Let $U \in \mathrm{Ob}(\mathcal{C})$. The functor $j_{U!}$ gives an equivalence of categories*

$$Sh(\mathcal{C}/U) \longrightarrow Sh(\mathcal{C})/h_U^\#$$

Proof. Let us denote objects of $\mathcal{C}/U$ as pairs (X, a) where X is an object of $\mathcal{C}$ and $a : X \to U$ is a morphism of $\mathcal{C}$. Similarly, objects of $Sh(\mathcal{C})/h_U^\#$ are pairs $(\mathcal{F}, \varphi)$. The functor $Sh(\mathcal{C}/U) \to Sh(\mathcal{C})/h_U^\#$ sends $\mathcal{G}$ to the pair $(j_{U!}\mathcal{G}, \gamma)$ where γ is the composition of $j_{U!}\mathcal{G} \to j_{U!}*$ with the identification $j_{U!}* = h_U^\#$.

Let us construct a functor from $Sh(\mathcal{C})/h_U^\#$ to $Sh(\mathcal{C}/U)$. Suppose that $(\mathcal{F}, \varphi)$ is given. For an object (X, a) of $\mathcal{C}/U$ we consider the set $\mathcal{F}_\varphi(X, a)$ of elements $s \in \mathcal{F}(X)$ which under φ map to the image of $a \in \mathrm{Mor}_\mathcal{C}(X, U) = h_U(X)$ in $h_U^\#(X)$. It is easy to see that $(X, a) \mapsto \mathcal{F}_\varphi(X, a)$ is a sheaf on $\mathcal{C}/U$. Clearly, the rule $(\mathcal{F}, \varphi) \mapsto \mathcal{F}_\varphi$ defines a functor $Sh(\mathcal{C})/h_U^\# \to Sh(\mathcal{C}/U)$.

Consider also the functor $PSh(\mathcal{C})/h_U \to PSh(\mathcal{C}/U)$, $(\mathcal{F}, \varphi) \mapsto \mathcal{F}_\varphi$ where $\mathcal{F}_\varphi(X, a)$ is defined as the set of elements of $\mathcal{F}(X)$ mapping to $a \in h_U(X)$. We claim that the diagram

$$\begin{array}{ccc} PSh(\mathcal{C})/h_U & \longrightarrow & PSh(\mathcal{C}/U) \\ \downarrow & & \downarrow \\ Sh(\mathcal{C})/h_U^\# & \longrightarrow & Sh(\mathcal{C}/U) \end{array}$$

commutes, where the vertical arrows are given by sheafification. To see this[5], it suffices to prove that the construction commutes with the functor $\mathcal{F} \mapsto \mathcal{F}^+$ of Lemmas 10.3 and 10.4 and Theorem 10.10. Commutation with $\mathcal{F} \mapsto \mathcal{F}^+$ follows from the fact that given (X, a) the categories of coverings of (X, a) in $\mathcal{C}/U$ and coverings of X in $\mathcal{C}$ are canonically identified.

Next, let $PSh(\mathcal{C}/U) \to PSh(\mathcal{C})/h_U$ send $\mathcal{G}$ to the pair $(j_{U!}^{PSh}\mathcal{G}, \gamma)$ where $j_{U!}^{PSh}\mathcal{G}$ the presheaf defined by the formula in Lemma 25.2 and γ is the composition of $j_{U!}^{PSh}\mathcal{G} \to j_{U!}*$ with the identification $j_{U!}^{PSh}* = h_U$ (obvious from the formula). Then it is immediately clear that the diagram

$$\begin{array}{ccc} PSh(\mathcal{C}/U) & \longrightarrow & PSh(\mathcal{C})/h_U \\ \downarrow & & \downarrow \\ Sh(\mathcal{C}/U) & \longrightarrow & Sh(\mathcal{C})/h_U^\# \end{array}$$

[5]An alternative is to describe $\mathcal{F}_\varphi$ by the cartesian diagram

$$\begin{array}{ccc} \mathcal{F}_\varphi & \longrightarrow & * \\ \downarrow & & \downarrow \\ \mathcal{F}|_{\mathcal{C}/U} & \longrightarrow & h_U|_{\mathcal{C}/U} \end{array} \quad \text{for presheaves and} \quad \begin{array}{ccc} \mathcal{F}_\varphi & \longrightarrow & * \\ \downarrow & & \downarrow \\ \mathcal{F}|_{\mathcal{C}/U} & \longrightarrow & h_U^\#|_{\mathcal{C}/U} \end{array}$$

for sheaves and use that restriction to $\mathcal{C}/U$ commutes with sheafification.

commutes, where the vertical arrows are sheafification. Putting everything together it suffices to show there are functorial isomorphisms $(j_{U!}^{PSh}\mathcal{G})_\gamma = \mathcal{G}$ for $\mathcal{G}$ in $PSh(\mathcal{C}/U)$ and $j_{U!}^{PSh}\mathcal{F}_\varphi = \mathcal{F}$ for $(\mathcal{F}, \varphi)$ in $PSh(\mathcal{C})/h_U$. The value of the presheaf $(j_{U!}^{PSh}\mathcal{G})_\gamma$ on (X, a) is the fibre of the map

$$\coprod\nolimits_{a':X\to U} \mathcal{G}(X, a') \to \mathrm{Mor}_{\mathcal{C}}(X, U)$$

over a which is $\mathcal{G}(X, a)$. This proves the first equality. The value of the presheaf $j_{U!}^{PSh}\mathcal{F}_\varphi$ is on X is

$$\coprod\nolimits_{a:X\to U} \mathcal{F}_\varphi(X, a) = \mathcal{F}(X)$$

because given a set map $S \to S'$ the set S is the disjoint union of its fibres. □

Lemma 25.4 says the functor $j_{U!}$ is the composition

$$Sh(\mathcal{C}/U) \to Sh(\mathcal{C})/h_U^\# \to Sh(\mathcal{C})$$

where the first arrow is an equivalence.

04BB **Lemma 25.5.** *Let $\mathcal{C}$ be a site. Let $U \in \mathrm{Ob}(\mathcal{C})$. The functor $j_{U!}$ commutes with fibre products and equalizers (and more generally finite connected limits). In particular, if $\mathcal{F} \subset \mathcal{F}'$ in $Sh(\mathcal{C}/U)$, then $j_{U!}\mathcal{F} \subset j_{U!}\mathcal{F}'$.*

Proof. Via Lemma 25.4 and the fact that an equivalence of categories commutes with all limits, this reduces to the fact that the functor $Sh(\mathcal{C})/h_U^\# \to Sh(\mathcal{C})$ commutes with fibre products and equalizers. Alternatively, one can prove this directly using the description of $j_{U!}$ in Lemma 25.2 using that sheafification is exact. (Also, in case $\mathcal{C}$ has fibre products and equalizers, the result follows from Lemma 21.6.) □

0E8E **Lemma 25.6.** *Let $\mathcal{C}$ be a site. Let $U \in \mathrm{Ob}(\mathcal{C})$. The functor $j_{U!}$ reflects injections and surjections.*

Proof. We have to show $j_{U!}$ reflects monomorphisms and epimorphisms, see Lemma 11.2. Via Lemma 25.4 this reduces to the fact that the functor $Sh(\mathcal{C})/h_U^\# \to Sh(\mathcal{C})$ reflects monomorphisms and epimorphisms. □

03EE **Lemma 25.7.** *Let $\mathcal{C}$ be a site. Let $U \in \mathrm{Ob}(\mathcal{C})$. For any sheaf $\mathcal{F}$ on $\mathcal{C}$ we have $j_{U!}j_U^{-1}\mathcal{F} = \mathcal{F} \times h_U^\#$.*

Proof. This is clear from the description of $j_{U!}$ in Lemma 25.2. □

03EH **Lemma 25.8.** *Let $\mathcal{C}$ be a site. Let $f : V \to U$ be a morphism of $\mathcal{C}$. Then there exists a commutative diagram*

$$\begin{array}{ccc} \mathcal{C}/V & \xrightarrow{\quad j \quad} & \mathcal{C}/U \\ & {\scriptstyle j_V}\searrow \quad \swarrow{\scriptstyle j_U} & \\ & \mathcal{C} & \end{array}$$

of continuous and cocontinuous functors. The functor $j : \mathcal{C}/V \to \mathcal{C}/U$, $(a : W \to V) \mapsto (f \circ a : W \to U)$ is identified with the functor $j_{V/U} : (\mathcal{C}/U)/(V/U) \to \mathcal{C}/U$ via the identification $(\mathcal{C}/U)/(V/U) = \mathcal{C}/V$. Moreover we have $j_{V!} = j_{U!} \circ j_!$, $j_V^{-1} = j^{-1} \circ j_U^{-1}$, and $j_{V*} = j_{U*} \circ j_*$.

Proof. The commutativity of the diagram is immediate. The agreement of j with $j_{V/U}$ follows from the definitions. By Lemma 21.2 we see that the following diagram of morphisms of topoi

04IK (25.8.1)
$$\begin{array}{ccc} Sh(\mathcal{C}/V) & \xrightarrow{\quad j \quad} & Sh(\mathcal{C}/U) \\ & {\scriptstyle j_V}\searrow \quad \swarrow{\scriptstyle j_U} & \\ & Sh(\mathcal{C}) & \end{array}$$

is commutative. This proves that $j_V^{-1} = j^{-1} \circ j_U^{-1}$ and $j_{V*} = j_{U*} \circ j_*$. The equality $j_{V!} = j_{U!} \circ j_!$ follows formally from adjointness properties. □

04IL **Lemma 25.9.** *Notation $\mathcal{C}$, $f : V \to U$, j_U, j_V, and j as in Lemma 25.8. Via the identifications $Sh(\mathcal{C}/V) = Sh(\mathcal{C})/h_V^\#$ and $Sh(\mathcal{C}/U) = Sh(\mathcal{C})/h_U^\#$ of Lemma 25.4 we have*

(1) *the functor j^{-1} has the following description*
$$j^{-1}(\mathcal{H} \xrightarrow{\varphi} h_U^\#) = (\mathcal{H} \times_{\varphi, h_U^\#, f} h_V^\# \to h_V^\#).$$

(2) *the functor $j_!$ has the following description*
$$j_!(\mathcal{H} \xrightarrow{\varphi} h_V^\#) = (\mathcal{H} \xrightarrow{h_f \circ \varphi} h_U^\#)$$

Proof. Proof of (2). Recall that the identification $Sh(\mathcal{C}/V) \to Sh(\mathcal{C})/h_V^\#$ sends $\mathcal{G}$ to $j_{V!}\mathcal{G} \to j_{V!}(*) = h_V^\#$ and similarly for $Sh(\mathcal{C}/U) \to Sh(\mathcal{C})/h_U^\#$. Thus $j_!\mathcal{G}$ is mapped to $j_{U!}(j_!\mathcal{G}) \to j_{U!}(*) = h_U^\#$ and (2) follows because $j_{U!}j_! = j_{V!}$ by Lemma 25.8.

The reader can now prove (1) by using that j^{-1} is the right adjoint to $j_!$ and using that the rule in (1) is the right adjoint to the rule in (2). Here is a direct proof. Suppose that $\varphi : \mathcal{H} \to h_U^\#$ is an object of $Sh(\mathcal{C})/h_U^\#$. By the proof of Lemma 25.4 this corresponds to the sheaf $\mathcal{H}_\varphi$ on $\mathcal{C}/U$ defined by the rule
$$(a : W \to U) \longmapsto \{s \in \mathcal{H}(W) \mid \varphi(s) = a\}$$
on $\mathcal{C}/U$. The pullback $j^{-1}\mathcal{H}_\varphi$ to $\mathcal{C}/V$ is given by the rule
$$(a : W \to V) \longmapsto \{s \in \mathcal{H}(W) \mid \varphi(s) = f \circ a\}$$
by the description of $j^{-1} = j_{U/V}^{-1}$ as the restriction of $\mathcal{H}_\varphi$ to $\mathcal{C}/V$. On the other hand, applying the rule to the object
$$\mathcal{H}' = \mathcal{H} \times_{\varphi, h_U^\#, f} h_V^\# \xrightarrow{\quad \varphi' \quad} h_V^\#$$
of $Sh(\mathcal{C})/h_V^\#$ we get $\mathcal{H}'_{\varphi'}$ given by
$$\begin{aligned}(a : W \to V) \longmapsto & \{s' \in \mathcal{H}'(W) \mid \varphi'(s') = a\} \\ = & \{(s, a') \in \mathcal{H}(W) \times h_V^\#(W) \mid a' = a \text{ and } \varphi(s) = f \circ a'\}\end{aligned}$$
which is exactly the same rule as the one describing $j^{-1}\mathcal{H}_\varphi$ above. □

0494 **Remark 25.10.** Localization and presheaves. Let $\mathcal{C}$ be a category. Let U be an object of $\mathcal{C}$. Strictly speaking the functors j_U^{-1}, j_{U*} and $j_{U!}$ have not been defined for presheaves. But of course, we can think of a presheaf as a sheaf for the chaotic topology on $\mathcal{C}$ (see Example 6.6). Hence we also obtain a functor

$$j_U^{-1} : PSh(\mathcal{C}) \longrightarrow PSh(\mathcal{C}/U)$$

and functors

$$j_{U*}, j_{U!} : PSh(\mathcal{C}/U) \longrightarrow PSh(\mathcal{C})$$

which are right, left adjoint to j_U^{-1}. By Lemma 25.2 we see that $j_{U!}\mathcal{G}$ is the presheaf

$$V \longmapsto \coprod\nolimits_{\varphi \in \mathrm{Mor}_{\mathcal{C}}(V,U)} \mathcal{G}(V \xrightarrow{\varphi} U)$$

In addition the functor $j_{U!}$ commutes with fibre products and equalizers.

09W8 **Remark 25.11.** Let $\mathcal{C}$ be a site. Let $U \to V$ be a morphism of $\mathcal{C}$. The cocontinuous functors $\mathcal{C}/U \to \mathcal{C}$ and $j : \mathcal{C}/U \to \mathcal{C}/V$ (Lemma 25.8) satisfy property P of Remark 20.5. For example, if we have objects (X/U), (W/V), a morphism $g : j(X/U) \to (W/V)$, and a covering $\{f_i : (W_i/V) \to (W/V)\}$ then $(X \times_W W_i/U)$ is an avatar of $(X/U) \times_{g,(W/V),f_i} (W_i/V)$ and the family $\{(X \times_W W_i/U) \to (X/U)\}$ is a covering of $\mathcal{C}/U$.

26. Glueing sheaves

04TP This section is the analogue of Sheaves, Section 33.

04TQ **Lemma 26.1.** *Let $\mathcal{C}$ be a site. Let $\{U_i \to U\}$ be a covering of $\mathcal{C}$. Let $\mathcal{F}$, $\mathcal{G}$ be sheaves on $\mathcal{C}$. Given a collection*

$$\varphi_i : \mathcal{F}|_{\mathcal{C}/U_i} \longrightarrow \mathcal{G}|_{\mathcal{C}/U_i}$$

of maps of sheaves such that for all $i, j \in I$ the maps φ_i, φ_j restrict to the same map $\varphi_{ij} : \mathcal{F}|_{\mathcal{C}/U_i \times_U U_j} \to \mathcal{G}|_{\mathcal{C}/U_i \times_U U_j}$ then there exists a unique map of sheaves

$$\varphi : \mathcal{F}|_{\mathcal{C}/U} \longrightarrow \mathcal{G}|_{\mathcal{C}/U}$$

whose restriction to each $\mathcal{C}/U_i$ agrees with φ_i.

Proof. The restrictions used in the lemma are those of Lemma 25.8. Let V/U be an object of $\mathcal{C}/U$. Set $V_i = U_i \times_U V$ and denote $\mathcal{V} = \{V_i \to V\}$. Observe that $(U_i \times_U U_j) \times_U V = V_i \times_V V_j$. Then we have $\mathcal{F}|_{\mathcal{C}/U_i}(V_i/U_i) = \mathcal{F}(V_i)$ and $\mathcal{F}|_{\mathcal{C}/U_i \times_U U_j}(V_i \times_V V_j/U_i \times_U U_j) = \mathcal{F}(V_i \times_V V_j)$ and similarly for $\mathcal{G}$. Thus we can define φ on sections over V as the dotted arrows in the diagram

$$\begin{array}{ccccccc}
\mathcal{F}(V) & = & H^0(\mathcal{V}, \mathcal{F}) & \longrightarrow & \prod \mathcal{F}(V_i) & \rightrightarrows & \prod \mathcal{F}(V_i \times_V V_j) \\
 & & \vdots & & \prod \varphi_i \downarrow & & \prod \varphi_{ij} \downarrow \\
\mathcal{G}(V) & = & H^0(\mathcal{V}, \mathcal{G}) & \longrightarrow & \prod \mathcal{G}(V_i) & \rightrightarrows & \prod \mathcal{G}(V_i \times_V V_j)
\end{array}$$

The equality signs come from the sheaf condition; see Section 10 for the notation $H^0(\mathcal{V}, -)$. We omit the verification that these maps are compatible with the restriction maps. □

The previous lemma implies that given two sheaves $\mathcal{F}$, $\mathcal{G}$ on a site $\mathcal{C}$ the rule

$$U \longmapsto \mathrm{Mor}_{Sh(\mathcal{C}/U)}(\mathcal{F}|_{\mathcal{C}/U}, \mathcal{G}|_{\mathcal{C}/U})$$

defines a sheaf $\mathcal{H}om(\mathcal{F}, \mathcal{G})$. This is a kind of *internal hom sheaf*. It is seldom used in the setting of sheaves of sets, and more usually in the setting of sheaves of modules, see Modules on Sites, Section 27.

0BWQ **Lemma 26.2.** *Let $\mathcal{C}$ be a site. Let $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ be sheaves on $\mathcal{C}$. There is a canonical bijection*

$$\mathrm{Mor}_{Sh(\mathcal{C})}(\mathcal{F} \times \mathcal{G}, \mathcal{H}) = \mathrm{Mor}_{Sh(\mathcal{C})}(\mathcal{F}, \mathcal{H}om(\mathcal{G}, \mathcal{H}))$$

which is functorial in all three entries.

Proof. The lemma says that the functors $- \times \mathcal{G}$ and $\mathcal{H}om(\mathcal{G}, -)$ are adjoint to each other. To show this, we use the notion of unit and counit, see Categories, Section 24. The unit

$$\eta_{\mathcal{F}} : \mathcal{F} \longrightarrow \mathcal{H}om(\mathcal{G}, \mathcal{F} \times \mathcal{G})$$

sends $s \in \mathcal{F}(U)$ to the map $\mathcal{G}|_{\mathcal{C}/U} \to \mathcal{F}|_{\mathcal{C}/U} \times \mathcal{G}|_{\mathcal{C}/U}$ which over V/U is given by

$$\mathcal{G}(V) \longrightarrow \mathcal{F}(V) \times \mathcal{G}(V), \quad t \longmapsto (s|_V, t).$$

The counit

$$\epsilon_{\mathcal{H}} : \mathcal{H}om(\mathcal{G}, \mathcal{H}) \times \mathcal{G} \longrightarrow \mathcal{H}$$

is the evaluation map. It is given by the rule

$$\mathrm{Mor}_{Sh(\mathcal{C}/U)}(\mathcal{G}|_{\mathcal{C}/U}, \mathcal{H}|_{\mathcal{C}/U}) \times \mathcal{G}(U) \longrightarrow \mathcal{H}(U), \quad (\varphi, s) \longmapsto \varphi(s).$$

Then for each $\varphi : \mathcal{F} \times \mathcal{G} \to \mathcal{H}$, the corresponding morphism $\mathcal{F} \to \mathcal{H}om(\mathcal{G}, \mathcal{H})$ is given by mapping each section $s \in \mathcal{F}(U)$ to the morphism of sheaves on $\mathcal{C}/U$ which on sections over V/U is given by

$$\mathcal{G}(V) \longrightarrow \mathcal{H}(V), \quad t \longmapsto \varphi(s|_V, t).$$

Conversely, for each $\psi : \mathcal{F} \to \mathcal{H}om(\mathcal{G}, \mathcal{H})$, the corresponding morphism $\mathcal{F} \times \mathcal{G} \to \mathcal{H}$ is given by

$$\mathcal{F}(U) \times \mathcal{G}(U) \longrightarrow \mathcal{H}(U), \quad (s, t) \longmapsto \psi(s)(t)$$

on sections over an object U. We omit the details of the proof showing that these constructions are mutually inverse. □

0D7X **Lemma 26.3.** *Let $\mathcal{C}$ be a site and $U \in \mathrm{Ob}(\mathcal{C})$. Then $\mathcal{H}om(h_U^\#, \mathcal{F}) = j_*(\mathcal{F}|_{\mathcal{C}/U})$ for $\mathcal{F}$ in $Sh(\mathcal{C})$.*

Proof. This can be shown by directly constructing an isomorphism of sheaves. Instead we argue as follows. Let $\mathcal{G}$ be a sheaf on $\mathcal{C}$. Then

$$\begin{aligned} \mathrm{Mor}(\mathcal{G}, j_*(\mathcal{F}|_{\mathcal{C}/U})) &= \mathrm{Mor}(\mathcal{G}|_{\mathcal{C}/U}, \mathcal{F}|_{\mathcal{C}/U}) \\ &= \mathrm{Mor}(j_!(\mathcal{G}|_{\mathcal{C}/U}), \mathcal{F}) \\ &= \mathrm{Mor}(\mathcal{G} \times h_U^\#, \mathcal{F}) \\ &= \mathrm{Mor}(\mathcal{G}, \mathcal{H}om(h_U^\#, \mathcal{F})) \end{aligned}$$

and we conclude by the Yoneda lemma. Here we used Lemmas 26.2 and 25.7. □

Let $\mathcal{C}$ be a site. Let $\{U_i \to U\}_{i\in I}$ be a covering of $\mathcal{C}$. For each $i \in I$ let $\mathcal{F}_i$ be a sheaf of sets on $\mathcal{C}/U_i$. For each pair $i, j \in I$, let

$$\varphi_{ij} : \mathcal{F}_i|_{\mathcal{C}/U_i\times_U U_j} \longrightarrow \mathcal{F}_j|_{\mathcal{C}/U_i\times_U U_j}$$

be an isomorphism of sheaves of sets. Assume in addition that for every triple of indices $i, j, k \in I$ the following diagram is commutative

$$\begin{array}{ccc} \mathcal{F}_i|_{\mathcal{C}/U_i\times_U U_j\times_U U_k} & \xrightarrow{\varphi_{ik}} & \mathcal{F}_k|_{\mathcal{C}/U_i\times_U U_j\times_U U_k} \\ & {\scriptstyle\varphi_{ij}}\searrow \quad \nearrow{\scriptstyle\varphi_{jk}} & \\ & \mathcal{F}_j|_{\mathcal{C}/U_i\times_U U_j\times_U U_k} & \end{array}$$

We will call such a collection of data $(\mathcal{F}_i, \varphi_{ij})$ a *glueing data for sheaves of sets with respect to the covering* $\{U_i \to U\}_{i\in I}$.

04TR **Lemma 26.4.** *Let $\mathcal{C}$ be a site. Let $\{U_i \to U\}_{i\in I}$ be a covering of $\mathcal{C}$. Given any glueing data $(\mathcal{F}_i, \varphi_{ij})$ for sheaves of sets with respect to the covering $\{U_i \to U\}_{i\in I}$ there exists a sheaf of sets $\mathcal{F}$ on $\mathcal{C}/U$ together with isomorphisms*

$$\varphi_i : \mathcal{F}|_{\mathcal{C}/U_i} \to \mathcal{F}_i$$

such that the diagrams

$$\begin{array}{ccc} \mathcal{F}|_{\mathcal{C}/U_i\times_U U_j} & \xrightarrow[\varphi_i]{} & \mathcal{F}_i|_{\mathcal{C}/U_i\times_U U_j} \\ {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle\varphi_{ij}} \\ \mathcal{F}|_{\mathcal{C}/U_i\times_U U_j} & \xrightarrow{\varphi_j} & \mathcal{F}_j|_{\mathcal{C}/U_i\times_U U_j} \end{array}$$

are commutative.

Proof. Let us describe how to construct the sheaf $\mathcal{F}$ on $\mathcal{C}/U$. Let $a : V \to U$ be an object of $\mathcal{C}/U$. Then

$$\mathcal{F}(V/U) = \{(s_i)_{i\in I} \in \prod_{i\in I} \mathcal{F}_i(U_i \times_U V/U_i) \mid \varphi_{ij}(s_i|_{U_i\times_U U_j\times_U V}) = s_j|_{U_i\times_U U_j\times_U V}\}$$

We omit the construction of the restriction mappings. We omit the verification that this is a sheaf. We omit the construction of the isomorphisms φ_i, and we omit proving the commutativity of the diagrams of the lemma. □

Let $\mathcal{C}$ be a site. Let $\{U_i \to U\}_{i\in I}$ be a covering of $\mathcal{C}$. Let $\mathcal{F}$ be a sheaf on $\mathcal{C}/U$. Associated to $\mathcal{F}$ we have its *canonical glueing data* given by the restrictions $\mathcal{F}|_{\mathcal{C}/U_i}$ and the canonical isomorphisms

$$\left(\mathcal{F}|_{\mathcal{C}/U_i}\right)|_{\mathcal{C}/U_i\times_U U_j} = \left(\mathcal{F}|_{\mathcal{C}/U_j}\right)|_{\mathcal{C}/U_i\times_U U_j}$$

coming from the fact that the composition of the functors $\mathcal{C}/U_i \times_U U_j \to \mathcal{C}/U_i \to \mathcal{C}/U$ and $\mathcal{C}/U_i \times_U U_j \to \mathcal{C}/U_j \to \mathcal{C}/U$ are equal.

04TS **Lemma 26.5.** *Let $\mathcal{C}$ be a site. Let $\{U_i \to U\}_{i\in I}$ be a covering of $\mathcal{C}$. The category $Sh(\mathcal{C}/U)$ is equivalent to the category of glueing data via the functor that associates to $\mathcal{F}$ on $\mathcal{C}/U$ the canonical glueing data.*

Proof. In Lemma 26.1 we saw that the functor is fully faithful, and in Lemma 26.4 we proved that it is essentially surjective (by explicitly constructing a quasi-inverse functor). □

Let $\mathcal{C}$ be a site. We are going to discuss a version of glueing sheaves on the entire site $\mathcal{C}$. For each object U in $\mathcal{C}$, let $\mathcal{F}_U$ be a sheaf on $\mathcal{C}/U$. Recall that there is a functor $j_f : \mathcal{C}/V \to \mathcal{C}/U$ associated to each morphism $f : V \to U$ in $\mathcal{C}$, given by $(a : W \to V) \mapsto (f \circ a : W \to U)$. For each such f, let

$$c_f : j_f^{-1}\mathcal{F}_U \to \mathcal{F}_V$$

be an isomorphism of sheaves. Assume that given any two arrows $f : V \to U$ and $g : W \to V$ in $\mathcal{C}$, the composition $c_g \circ j_g^{-1} c_f$ is equal to $c_{f \circ g}$. We will call such a collection of data $(\mathcal{F}_U, c_f)$ an *absolute glueing data for sheaves of sets on* $\mathcal{C}$. A morphism of absolute glueing data $(\mathcal{F}_U, c_f) \to (\mathcal{G}_U, c'_f)$ is given by a collection (φ_U) of morphisms of sheaves $\varphi_U : \mathcal{F}_U \to \mathcal{G}_U$, such that

$$\begin{array}{ccc} j_f^{-1}\mathcal{F}_U & \xrightarrow{c_f} & \mathcal{F}_V \\ {\scriptstyle j_f^{-1}\varphi_U}\downarrow & & \downarrow{\scriptstyle \varphi_V} \\ j_f^{-1}\mathcal{G}_U & \xrightarrow{c'_f} & \mathcal{G}_V \end{array}$$

commutes for every morphism $f : V \to U$ in $\mathcal{C}$.

Associated to any sheaf $\mathcal{F}$ on $\mathcal{C}$ is its *canonical absolute glueing data* $(\mathcal{F}|_{\mathcal{C}/U}, c_f)$, where the canonical isomorphisms $c_f : j_f^{-1}\mathcal{F}|_{\mathcal{C}/U} \to \mathcal{F}|_{\mathcal{C}/V}$ for $f : V \to U$ come from the relation $j_V = j_U \circ j_f$ as in Lemma 25.8. Any morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of sheaves of $\mathcal{C}$ induces a morphism $(\varphi|_{\mathcal{C}/U})$ of canonical absolute glueing data.

0GWK **Lemma 26.6.** *Let $\mathcal{C}$ be a site. The category $Sh(\mathcal{C})$ is equivalent to the category of absolute glueing data via the functor that associates to $\mathcal{F}$ on $\mathcal{C}$ the canonical absolute glueing data.*

Proof. Given an absolute glueing data $(\mathcal{F}_U, c_f)$, we construct a sheaf $\mathcal{F}$ on $\mathcal{C}$ by setting $\mathcal{F}(U) = \mathcal{F}_U(U)$, where restriction along $f : V \to U$ given by the commutative diagram

$$\begin{array}{ccccc} \mathcal{F}_U(U) & \longrightarrow & \mathcal{F}_U(V) & \xrightarrow{c_f} & \mathcal{F}_V(V) \\ \| & & & & \| \\ \mathcal{F}(U) & & \longrightarrow & & \mathcal{F}(V) \end{array}$$

The compatibility condition $c_g \circ j_g^{-1} c_f = c_{f \circ g}$ ensures that $\mathcal{F}$ is a presheaf, and also ensures that the maps $c_f : \mathcal{F}_U(V) \to \mathcal{F}(V)$ define an isomorphism $\mathcal{F}_U \to \mathcal{F}|_{\mathcal{C}/U}$ for each U. Since each $\mathcal{F}_U$ is a sheaf, this implies that $\mathcal{F}$ is a sheaf as well. The functor $(\mathcal{F}_U, c_f) \mapsto \mathcal{F}$ just constructed is quasi-inverse to the functor which takes a sheaf on $\mathcal{C}$ to its canonical glueing data. Further details omitted. □

0GWL **Remark 26.7.** There is a variant of Lemma 26.6 which comes up in algebraic geometry. Namely, suppose that $\mathcal{C}$ is a site with all fibre products and for each $U \in \mathrm{Ob}(\mathcal{C})$ we are given a full subcategory $U_\tau \subset \mathcal{C}/U$ with the following properties

(1) U/U is in U_τ,
(2) for V/U in U_τ and covering $\{V_j \to V\}$ of $\mathcal{C}$ we have V_j/U in U_τ and
(3) for a morphism $U' \to U$ of $\mathcal{C}$ and V/U in U_τ the base change $V' = V \times_U U'$ is in U'_τ.

In this setting U_τ is a site for all U in $\mathcal{C}$ and the base change functor $U_\tau \to U'_\tau$ defines a morphism $f_\tau : U_\tau \to U'_\tau$ of sites for all morphisms $f : U' \to U$ of $\mathcal{C}$. The glueing statement we obtain then reads as follows: A sheaf $\mathcal{F}$ on $\mathcal{C}$ is given by the following data:

(1) for every $U \in \mathrm{Ob}(\mathcal{C})$ a sheaf $\mathcal{F}_U$ on U_τ,
(2) for every $f : U' \to U$ in $\mathcal{C}$ a map $c_f : f_\tau^{-1}\mathcal{F}_U \to \mathcal{F}_{U'}$.

These data are subject to the following conditions:

(a) given $f : U' \to U$ and $g : U'' \to U'$ in $\mathcal{C}$ the composition $c_g \circ g_\tau^{-1} c_f$ is equal to $c_{f\circ g}$, and
(b) if $f : U' \to U$ is in U_τ then c_f is an isomorphism.

If we ever need this we will precisely state and prove this here. (Note that this result is slightly different from the statements above as we are not requiring all the maps c_f to be isomorphisms!)

27. More localization

04IM In this section we prove a few lemmas on localization where we impose some additional hypotheses on the site on or the object we are localizing at.

03HT **Lemma 27.1.** *Let $\mathcal{C}$ be a site. Let $U \in \mathrm{Ob}(\mathcal{C})$. If the topology on $\mathcal{C}$ is subcanonical, see Definition 12.2, and if $\mathcal{G}$ is a sheaf on $\mathcal{C}/U$, then*

$$j_{U!}(\mathcal{G})(V) = \coprod\nolimits_{\varphi \in \mathrm{Mor}_{\mathcal{C}}(V,U)} \mathcal{G}(V \xrightarrow{\varphi} U),$$

in other words sheafification is not necessary in Lemma 25.2.

Proof. Let $\mathcal{V} = \{V_i \to V\}_{i\in I}$ be a covering of V in the site $\mathcal{C}$. We are going to check the sheaf condition for the presheaf $\mathcal{H}$ of Lemma 25.2 directly. Let $(s_i, \varphi_i)_{i\in I} \in \prod_i \mathcal{H}(V_i)$, This means $\varphi_i : V_i \to U$ is a morphism in $\mathcal{C}$, and $s_i \in \mathcal{G}(V_i \xrightarrow{\varphi_i} U)$. The restriction of the pair (s_i, φ_i) to $V_i \times_V V_j$ is the pair $(s_i|_{V_i\times_V V_j/U}, \mathrm{pr}_1 \circ \varphi_i)$, and likewise the restriction of the pair (s_j, φ_j) to $V_i \times_V V_j$ is the pair $(s_j|_{V_i\times_V V_j/U}, \mathrm{pr}_2 \circ \varphi_j)$. Hence, if the family (s_i, φ_i) lies in $\check{H}^0(\mathcal{V}, \mathcal{H})$, then we see that $\mathrm{pr}_1 \circ \varphi_i = \mathrm{pr}_2 \circ \varphi_j$. The condition that the topology on $\mathcal{C}$ is weaker than the canonical topology then implies that there exists a unique morphism $\varphi : V \to U$ such that φ_i is the composition of $V_i \to V$ with φ. At this point the sheaf condition for $\mathcal{G}$ guarantees that the sections s_i glue to a unique section $s \in \mathcal{G}(V \xrightarrow{\varphi} U)$. Hence $(s, \varphi) \in \mathcal{H}(V)$ as desired. □

03CE **Lemma 27.2.** *Let $\mathcal{C}$ be a site. Let $U \in \mathrm{Ob}(\mathcal{C})$. Assume $\mathcal{C}$ has products of pairs of objects. Then*

(1) *the functor j_U has a continuous right adjoint, namely the functor $v(X) = X \times U/U$,*
(2) *the functor v defines a morphism of sites $\mathcal{C}/U \to \mathcal{C}$ whose associated morphism of topoi equals $j_U : Sh(\mathcal{C}/U) \to Sh(\mathcal{C})$, and*
(3) *we have $j_{U*}\mathcal{F}(X) = \mathcal{F}(X \times U/U)$.*

Proof. The functor v being right adjoint to j_U means that given Y/U and X we have

$$\mathrm{Mor}_{\mathcal{C}}(Y, X) = \mathrm{Mor}_{\mathcal{C}/U}(Y/U, X \times U/U)$$

which is clear. To check that v is continuous let $\{X_i \to X\}$ be a covering of $\mathcal{C}$. By the third axiom of a site (Definition 6.2) we see that

$$\{X_i \times_X (X \times U) \to X \times_X (X \times U)\} = \{X_i \times U \to X \times U\}$$

is a covering of $\mathcal{C}$ also. Hence v is continuous. The other statements of the lemma follow from Lemmas 22.1 and 22.2. □

09W9 **Lemma 27.3.** *Let $\mathcal{C}$ be a site. Let $U \to V$ be a morphism of $\mathcal{C}$. Assume $\mathcal{C}$ has fibre products. Let j be as in Lemma 25.8. Then*

1. *the functor $j : \mathcal{C}/U \to \mathcal{C}/V$ has a continuous right adjoint, namely the functor $v : (X/V) \mapsto (X \times_V U/U)$,*
2. *the functor v defines a morphism of sites $\mathcal{C}/U \to \mathcal{C}/V$ whose associated morphism of topoi equals j, and*
3. *we have $j_*\mathcal{F}(X/V) = \mathcal{F}(X \times_V U/U)$.*

Proof. Follows from Lemma 27.2 since j may be viewed as a localization functor by Lemma 25.8. □

A fundamental property of an open immersion is that the restriction of the push-forward and the restriction of the extension by the empty set produces back the original sheaf. This is not always true for the functors associated to j_U above. It is true when U is a "subobject of the final object".

00Y2 **Lemma 27.4.** *Let $\mathcal{C}$ be a site. Let $U \in \mathrm{Ob}(\mathcal{C})$. Assume that every X in $\mathcal{C}$ has at most one morphism to U. Let $\mathcal{F}$ be a sheaf on $\mathcal{C}/U$. The canonical maps $\mathcal{F} \to j_U^{-1} j_{U!}\mathcal{F}$ and $j_U^{-1} j_{U*}\mathcal{F} \to \mathcal{F}$ are isomorphisms.*

Proof. This is a special case of Lemma 21.7 because the assumption on U is equivalent to the fully faithfulness of the localization functor $\mathcal{C}/U \to \mathcal{C}$. □

0EYV **Lemma 27.5.** *Let $\mathcal{C}$ be a site. Let*

$$\begin{array}{ccc} U' & \longrightarrow & U \\ \downarrow & & \downarrow \\ V' & \longrightarrow & V \end{array}$$

be a commutative diagram of $\mathcal{C}$. The morphisms of Lemma 25.8 produce commutative diagrams

$$\begin{array}{ccc} \mathcal{C}/U' & \xrightarrow[j_{U'/U}]{} & \mathcal{C}/U \\ {\scriptstyle j_{U'/V'}}\downarrow & & \downarrow{\scriptstyle j_{U/V}} \\ \mathcal{C}/V' & \xrightarrow[j_{V'/V}]{} & \mathcal{C}/V \end{array} \quad \text{and} \quad \begin{array}{ccc} Sh(\mathcal{C}/U') & \xrightarrow[j_{U'/U}]{} & Sh(\mathcal{C}/U) \\ {\scriptstyle j_{U'/V'}}\downarrow & & \downarrow{\scriptstyle j_{U/V}} \\ Sh(\mathcal{C}/V') & \xrightarrow[j_{V'/V}]{} & Sh(\mathcal{C}/V) \end{array}$$

of continuous and cocontinuous functors and of topoi. Moreover, if the initial diagram of $\mathcal{C}$ is cartesian, then we have $j_{V'/V}^{-1} \circ j_{U/V,*} = j_{U'/V',*} \circ j_{U'/U}^{-1}$.

Proof. The commutativity of the left square in the first statement of the lemma is immediate from the definitions. It implies the commutativity of the diagram of topoi by Lemma 21.2. Assume the diagram is cartesian. By the uniqueness of adjoint functors, to show $j_{V'/V}^{-1} \circ j_{U/V,*} = j_{U'/V',*} \circ j_{U'/U}^{-1}$ is equivalent to showing

$j_{U/V}^{-1} \circ j_{V'/V!} = j_{U'/U!} \circ j_{U'/V'}^{-1}$. Via the identifications of Lemma 25.4 we may think of our diagram of topoi as

$$\begin{array}{ccc} Sh(\mathcal{C})/h_{U'}^\# & \longrightarrow & Sh(\mathcal{C})/h_U^\# \\ \downarrow & & \downarrow \\ Sh(\mathcal{C})/h_{V'}^\# & \longrightarrow & Sh(\mathcal{C})/h_V^\# \end{array}$$

and we know how to interpret the functors j^{-1} and $j_!$ by Lemma 25.9. Thus we have to show given $\mathcal{F} \to h_{V'}^\#$ that

$$\mathcal{F} \times_{h_{V'}^\#} h_{U'}^\# = \mathcal{F} \times_{h_V^\#} h_U^\#$$

as sheaves with map to $h_U^\#$. This is true because $h_{U'} = h_{V'} \times_{h_V} h_U$ and hence also

$$h_{U'}^\# = h_{V'}^\# \times_{h_V^\#} h_U^\#$$

as sheafification is exact. □

28. Localization and morphisms

04I8 The following lemma is important in order to understand relation between localization and morphisms of sites and topoi.

03CF **Lemma 28.1.** *Let $f : \mathcal{C} \to \mathcal{D}$ be a morphism of sites corresponding to the continuous functor $u : \mathcal{D} \to \mathcal{C}$. Let $V \in \mathrm{Ob}(\mathcal{D})$ and set $U = u(V)$. Then the functor $u' : \mathcal{D}/V \to \mathcal{C}/U$, $V'/V \mapsto u(V')/U$ determines a morphism of sites $f' : \mathcal{C}/U \to \mathcal{D}/V$. The morphism f' fits into a commutative diagram of topoi*

$$\begin{array}{ccc} Sh(\mathcal{C}/U) & \xrightarrow[j_U]{} & Sh(\mathcal{C}) \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Sh(\mathcal{D}/V) & \xrightarrow{j_V} & Sh(\mathcal{D}). \end{array}$$

Using the identifications $Sh(\mathcal{C}/U) = Sh(\mathcal{C})/h_U^\#$ and $Sh(\mathcal{D}/V) = Sh(\mathcal{D})/h_V^\#$ of Lemma 25.4 the functor $(f')^{-1}$ is described by the rule

$$(f')^{-1}(\mathcal{H} \xrightarrow{\varphi} h_V^\#) = (f^{-1}\mathcal{H} \xrightarrow{f^{-1}\varphi} h_U^\#).$$

Finally, we have $f'_* j_U^{-1} = j_V^{-1} f_*$.

Proof. It is clear that u' is continuous, and hence we get functors $f'_* = (u')^s = (u')^p$ (see Sections 5 and 13) and an adjoint $(f')^{-1} = (u')_s = ((u')_p\ )^\#$. The assertion $f'_* j_U^{-1} = j_V^{-1} f_*$ follows as

$$(j_V^{-1} f_* \mathcal{F})(V'/V) = f_* \mathcal{F}(V') = \mathcal{F}(u(V')) = (j_U^{-1}\mathcal{F})(u(V')/U) = (f'_* j_U^{-1}\mathcal{F})(V'/V)$$

which holds even for presheaves. What isn't clear a priori is that $(f')^{-1}$ is exact, that the diagram commutes, and that the description of $(f')^{-1}$ holds.

Let $\mathcal{H}$ be a sheaf on $\mathcal{D}/V$. Let us compute $j_{U!}(f')^{-1}\mathcal{H}$. We have

$$\begin{aligned} j_{U!}(f')^{-1}\mathcal{H} &= ((j_U)_p(u'_p\mathcal{H})^\#)^\# \\ &= ((j_U)_p u'_p\mathcal{H})^\# \\ &= (u_p(j_V)_p\mathcal{H})^\# \\ &= f^{-1}j_{V!}\mathcal{H} \end{aligned}$$

The first equality by unwinding the definitions. The second equality by Lemma 13.4. The third equality because $u \circ j_V = j_U \circ u'$. The fourth equality by Lemma 13.4 again. All of the equalities above are isomorphisms of functors, and hence we may interpret this as saying that the following diagram of categories and functors is commutative

$$\begin{array}{ccccc} Sh(\mathcal{C}/U) & \xrightarrow[j_{U!}]{} & Sh(\mathcal{C})/h_U^\# & \longrightarrow & Sh(\mathcal{C}) \\ \uparrow{\scriptstyle (f')^{-1}} & & \uparrow{\scriptstyle f^{-1}} & & \uparrow{\scriptstyle f^{-1}} \\ Sh(\mathcal{D}/V) & \xrightarrow{j_{V!}} & Sh(\mathcal{D})/h_V^\# & \longrightarrow & Sh(\mathcal{D}) \end{array}$$

The middle arrow makes sense as $f^{-1}h_V^\# = (h_{u(V)})^\# = h_U^\#$, see Lemma 13.5. In particular this proves the description of $(f')^{-1}$ given in the statement of the lemma. Since by Lemma 25.4 the left horizontal arrows are equivalences and since f^{-1} is exact by assumption we conclude that $(f')^{-1} = u'_s$ is exact. Namely, because it is a left adjoint it is already right exact (Categories, Lemma 24.5). Hence we only need to show that it transforms a final object into a final object and commutes with fibre products (Categories, Lemma 23.2). Both are clear for the induced functor $f^{-1} : Sh(\mathcal{D})/h_V^\# \to Sh(\mathcal{C})/h_U^\#$. This proves that f' is a morphism of sites.

We still have to verify that $(f')^{-1}j_V^{-1} = j_U^{-1}f^{-1}$. To see this use the formula above and the description in Lemma 25.7. Namely, combined these give, for any sheaf $\mathcal{G}$ on $\mathcal{D}$, that

$$j_{U!}(f')^{-1}j_V^{-1}\mathcal{G} = f^{-1}j_{V!}j_V^{-1}\mathcal{G} = f^{-1}(\mathcal{G} \times h_V^\#) = f^{-1}\mathcal{G} \times h_U^\# = j_{U!}j_U^{-1}f^{-1}\mathcal{G}.$$

Since the functor $j_{U!}$ induces an equivalence $Sh(\mathcal{C}/U) \to Sh(\mathcal{C})/h_U^\#$ we conclude. □

The following lemma is a special case of the more general Lemma 28.1 above.

03EF **Lemma 28.2.** *Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $u : \mathcal{D} \to \mathcal{C}$ be a functor. Let $V \in \mathrm{Ob}(\mathcal{D})$. Set $U = u(V)$. Assume that*

(1) *$\mathcal{C}$ and $\mathcal{D}$ have all finite limits,*
(2) *u is continuous, and*
(3) *u commutes with finite limits.*

There exists a commutative diagram of morphisms of sites

$$\begin{array}{ccc} \mathcal{C}/U & \xrightarrow[j_U]{} & \mathcal{C} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathcal{D}/V & \xrightarrow{j_V} & \mathcal{D} \end{array}$$

where the right vertical arrow corresponds to u, the left vertical arrow corresponds to the functor $u' : \mathcal{D}/V \to \mathcal{C}/U$, $V'/V \mapsto u(V')/u(V)$ and the horizontal arrows correspond to the functors $\mathcal{C} \to \mathcal{C}/U$, $X \mapsto X \times U$ and $\mathcal{D} \to \mathcal{D}/V$, $Y \mapsto Y \times V$ as

in Lemma 27.2. Moreover, the associated diagram of morphisms of topoi is equal to the diagram of Lemma 28.1. In particular we have $f'_* j_U^{-1} = j_V^{-1} f_*$.

Proof. Note that u satisfies the assumptions of Proposition 14.7 and hence induces a morphism of sites $f : \mathcal{C} \to \mathcal{D}$ by that proposition. It is clear that u induces a functor u' as indicated. It is clear that this functor also satisfies the assumptions of Proposition 14.7. Hence we get a morphism of sites $f' : \mathcal{C}/U \to \mathcal{D}/V$. The diagram commutes by our definition of composition of morphisms of sites (see Definition 14.5) and because

$$u(Y \times V) = u(Y) \times u(V) = u(Y) \times U$$

which shows that the diagram of categories and functors opposite to the diagram of the lemma commutes. □

At this point we can localize a site, we know how to relocalize, and we can localize a morphism of sites at an object of the site downstairs. If we combine these then we get the following kind of diagram.

04IN **Lemma 28.3.** *Let $f : \mathcal{C} \to \mathcal{D}$ be a morphism of sites corresponding to the continuous functor $u : \mathcal{D} \to \mathcal{C}$. Let $V \in \mathrm{Ob}(\mathcal{D})$, $U \in \mathrm{Ob}(\mathcal{C})$ and $c : U \to u(V)$ a morphism of $\mathcal{C}$. There exists a commutative diagram of topoi*

$$\begin{array}{ccc} Sh(\mathcal{C}/U) & \xrightarrow{j_U} & Sh(\mathcal{C}) \\ {\scriptstyle f_c}\downarrow & & \downarrow{\scriptstyle f} \\ Sh(\mathcal{D}/V) & \xrightarrow{j_V} & Sh(\mathcal{D}). \end{array}$$

We have $f_c = f' \circ j_{U/u(V)}$ where $f' : Sh(\mathcal{C}/u(V)) \to Sh(\mathcal{D}/V)$ is as in Lemma 28.1 and $j_{U/u(V)} : Sh(\mathcal{C}/U) \to Sh(\mathcal{C}/u(V))$ is as in Lemma 25.8. Using the identifications $Sh(\mathcal{C}/U) = Sh(\mathcal{C})/h_U^\#$ and $Sh(\mathcal{D}/V) = Sh(\mathcal{D})/h_V^\#$ of Lemma 25.4 the functor $(f_c)^{-1}$ is described by the rule

$$(f_c)^{-1}(\mathcal{H} \xrightarrow{\varphi} h_V^\#) = (f^{-1}\mathcal{H} \times_{f^{-1}\varphi, h^\#_{u(V)}, c} h_U^\# \to h_U^\#).$$

Finally, given any morphisms $b : V' \to V$, $a : U' \to U$ and $c' : U' \to u(V')$ such that

$$\begin{array}{ccc} U' & \xrightarrow{c'} & u(V') \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle u(b)} \\ U & \xrightarrow{c} & u(V) \end{array}$$

commutes, then the diagram

$$\begin{array}{ccc} Sh(\mathcal{C}/U') & \xrightarrow{j_{U'/U}} & Sh(\mathcal{C}/U) \\ {\scriptstyle f_{c'}}\downarrow & & \downarrow{\scriptstyle f_c} \\ Sh(\mathcal{D}/V') & \xrightarrow{j_{V'/V}} & Sh(\mathcal{D}/V). \end{array}$$

commutes.

Proof. This lemma proves itself, and is more a collection of things we know at this stage of the development of theory. For example the commutativity of the first square follows from the commutativity of Diagram (25.8.1) and the commutativity

of the diagram in Lemma 28.1. The description of f_c^{-1} follows on combining Lemma 25.9 with Lemma 28.1. The commutativity of the last square then follows from the equality

$$f^{-1}\mathcal{H} \times_{h^\#_{u(V)},c} h^\#_U \times_{h^\#_U} h^\#_{U'} = f^{-1}(\mathcal{H} \times_{h^\#_V} h^\#_{V'}) \times_{h^\#_{u(V'),c'}} h^\#_{U'}$$

which is formal using that $f^{-1}h^\#_V = h^\#_{u(V)}$ and $f^{-1}h^\#_{V'} = h^\#_{u(V')}$, see Lemma 13.5. □

In the following lemma we find another kind of functoriality of localization, in case the morphism of topoi comes from a cocontinuous functor. This is a kind of diagram which is different from the diagram in Lemma 28.1, and in particular, in general the equality $f'_* j_U^{-1} = j_V^{-1} f_*$ seen in Lemma 28.1 does not hold in the situation of the following lemma.

03EG **Lemma 28.4.** *Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a cocontinuous functor. Let U be an object of $\mathcal{C}$, and set $V = u(U)$. We have a commutative diagram*

$$\begin{array}{ccc} \mathcal{C}/U & \xrightarrow{j_U} & \mathcal{C} \\ {\scriptstyle u'}\downarrow & & \downarrow{\scriptstyle u} \\ \mathcal{D}/V & \xrightarrow{j_V} & \mathcal{D} \end{array}$$

where the left vertical arrow is $u' : \mathcal{C}/U \to \mathcal{D}/V$, $U'/U \mapsto V'/V$. Then u' is cocontinuous also and we get a commutative diagram of topoi

$$\begin{array}{ccc} Sh(\mathcal{C}/U) & \xrightarrow{j_U} & Sh(\mathcal{C}) \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Sh(\mathcal{D}/V) & \xrightarrow{j_V} & Sh(\mathcal{D}) \end{array}$$

where f (resp. f') corresponds to u (resp. u').

Proof. The commutativity of the first diagram is clear. It implies the commutativity of the second diagram provided we show that u' is cocontinuous.

Let U'/U be an object of $\mathcal{C}/U$. Let $\{V_j/V \to u(U')/V\}_{j\in J}$ be a covering of $u(U')/V$ in $\mathcal{D}/V$. Since u is cocontinuous there exists a covering $\{U'_i \to U'\}_{i\in I}$ such that the family $\{u(U'_i) \to u(U')\}$ refines the covering $\{V_j \to u(U')\}$ in $\mathcal{D}$. In other words, there exists a map of index sets $\alpha : I \to J$ and morphisms $\phi_i : u(U'_i) \to V_{\alpha(i)}$ over U'. Think of U'_i as an object over U via the composition $U'_i \to U' \to U$. Then $\{U'_i/U \to U'/U\}$ is a covering of $\mathcal{C}/U$ such that $\{u(U'_i)/V \to u(U')/V\}$ refines $\{V_j/V \to u(U')/V\}$ (use the same α and the same maps ϕ_i). Hence $u' : \mathcal{C}/U \to \mathcal{D}/V$ is cocontinuous. □

0D5R **Lemma 28.5.** *Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a cocontinuous functor. Let V be an object of $\mathcal{D}$. Let ${}^u_V\mathcal{I}$ be the category introduced in Section 19. We have a commutative diagram*

$$\begin{array}{ccc} {}^u_V\mathcal{I} & \xrightarrow{j} & \mathcal{C} \\ {\scriptstyle u'}\downarrow & & \downarrow{\scriptstyle u} \\ \mathcal{D}/V & \xrightarrow{j_V} & \mathcal{D} \end{array} \quad \text{where} \quad \begin{array}{c} j : (U, \psi) \mapsto U \\ u' : (U, \psi) \mapsto (\psi : u(U) \to V) \end{array}$$

Declare a family of morphisms $\{(U_i, \psi_i) \to (U, \psi)\}$ of ${}^u_V\mathcal{I}$ to be a covering if and only if $\{U_i \to U\}$ is a covering in $\mathcal{C}$. Then

(1) *${}^u_V\mathcal{I}$ is a site,*
(2) *j is continuous and cocontinuous,*
(3) *u' is cocontinuous,*
(4) *we get a commutative diagram of topoi*

$$\begin{array}{ccc} Sh({}^u_V\mathcal{I}) & \xrightarrow{j} & Sh(\mathcal{C}) \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Sh(\mathcal{D}/V) & \xrightarrow{j_V} & Sh(\mathcal{D}) \end{array}$$

where f (resp. f') corresponds to u (resp. u'), and
(5) *we have $f'_* j^{-1} = j_V^{-1} f_*$.*

Proof. Parts (1), (2), (3), and (4) are straightforward consequences of the definitions and the fact that the functor j commutes with fibre products. We omit the details. To see (5) recall that f_* is given by ${}_s u = {}_p u$. Hence the value of $j_V^{-1} f_* \mathcal{F}$ on V'/V is the value of ${}_p u\mathcal{F}$ on V' which is the limit of the values of $\mathcal{F}$ on the category ${}^u_{V'}\mathcal{I}$. Clearly, there is an equivalence of categories

$${}^u_{V'}\mathcal{I} \to {}^{u'}_{V'/V}\mathcal{I}$$

Since the value of $f'_* j^{-1}\mathcal{F}$ on V'/V is given by the limit of the values of $j^{-1}\mathcal{F}$ on the category ${}^{u'}_{V'/V}\mathcal{I}$ and since the values of $j^{-1}\mathcal{F}$ on objects of ${}^u_V\mathcal{I}$ are just the values of $\mathcal{F}$ (by Lemma 21.5 as j is continuous and cocontinuous) we see that (5) is true. □

The following two results are of a slightly different nature.

0FN1 **Lemma 28.6.** *Assume given sites $\mathcal{C}', \mathcal{C}, \mathcal{D}', \mathcal{D}$ and functors*

$$\begin{array}{ccc} \mathcal{C}' & \xrightarrow{v'} & \mathcal{C} \\ {\scriptstyle u'}\downarrow & & \downarrow{\scriptstyle u} \\ \mathcal{D}' & \xrightarrow{v} & \mathcal{D} \end{array}$$

Assume

(1) *u, u', v, and v' are cocontinuous giving rise to morphisms of topoi f, f', g, and g' by Lemma 21.1,*
(2) *$v \circ u' = u \circ v'$,*
(3) *v and v' are continuous as well as cocontinuous, and*
(4) *for any object V' of $\mathcal{D}'$ the functor ${}^{u'}_{V'}\mathcal{I} \to {}^u_{v(V')}\mathcal{I}$ given by v is cofinal.*

Then $f'_* \circ (g')^{-1} = g^{-1} \circ f_*$ and $g'_! \circ (f')^{-1} = f^{-1} \circ g_!$.

Proof. The categories ${}^{u'}_{V'}\mathcal{I}$ and ${}^u_{v(V')}\mathcal{I}$ are defined in Section 19. The functor in condition (4) sends the object $\psi : u'(U') \to V'$ of ${}^{u'}_{V'}\mathcal{I}$ to the object $v(\psi) : uv'(U') = vu'(U') \to v(V')$ of ${}^u_{v(V')}\mathcal{I}$. Recall that g^{-1} is given by v^p (Lemma 21.5) and f_* is given by ${}_s u = {}_p u$. Hence the value of $g^{-1} f_* \mathcal{F}$ on V' is the value of ${}_p u\mathcal{F}$ on $v(V')$ which is the limit

$$\lim_{u(U) \to v(V') \in \mathrm{Ob}({}^u_{v(V')}\mathcal{I}^{opp})} \mathcal{F}(U)$$

By the same reasoning, the value of $f'_*(g')^{-1}\mathcal{F}$ on V' is given by the limit

$$\lim_{u'(U')\to V'\in \mathrm{Ob}(^{u'}_{V'}\mathcal{I}^{opp})} \mathcal{F}(v'(U'))$$

Thus assumption (4) and Categories, Lemma 17.4 show that these agree and the first equality of the lemma is proved. The second equality follows from the first by uniqueness of adjoints. □

0FN2 **Lemma 28.7.** *Assume given sites $\mathcal{C}', \mathcal{C}, \mathcal{D}', \mathcal{D}$ and functors*

$$\begin{array}{ccc} \mathcal{C}' & \xrightarrow{v'} & \mathcal{C} \\ {\scriptstyle u'}\uparrow & & \uparrow{\scriptstyle u} \\ \mathcal{D}' & \xrightarrow{v} & \mathcal{D} \end{array}$$

With notation as in Sections 14 and 21 assume

(1) *u and u' are continuous giving rise to morphisms of sites f and f',*
(2) *v and v' are cocontinuous giving rise to morphisms of topoi g and g',*
(3) *$u \circ v = v' \circ u'$, and*
(4) *v and v' are continuous as well as cocontinuous.*

Then[6] *$f'_* \circ (g')^{-1} = g^{-1} \circ f_*$ and $g'_! \circ (f')^{-1} = f^{-1} \circ g_!$.*

Proof. Namely, we have

$$f'_*(g')^{-1}\mathcal{F} = (u')^p((v')^p\mathcal{F})^\# = (u')^p(v')^p\mathcal{F}$$

The first equality by definition and the second by Lemma 21.5. We have

$$g^{-1}f_*\mathcal{F} = (v^p u^p \mathcal{F})^\# = ((u')^p(v')^p\mathcal{F})^\# = (u')^p(v')^p\mathcal{F}$$

The first equality by definition, the second because $u \circ v = v' \circ u'$, the third because we already saw that $(u')^p(v')^p\mathcal{F}$ is a sheaf. This proves $f'_* \circ (g')^{-1} = g^{-1} \circ f_*$ and the equality $g'_! \circ (f')^{-1} = f^{-1} \circ g_!$ follows by uniqueness of left adjoints. □

29. Morphisms of topoi

039Z In this section we show that any morphism of topoi is equivalent to a morphism of topoi which comes from a morphism of sites. Please compare with [AGV71, Exposé IV, Proposition 4.9.4].

03A0 **Lemma 29.1.** *Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. Assume that*

(1) *u is cocontinuous,*
(2) *u is continuous,*
(3) *given $a, b : U' \to U$ in $\mathcal{C}$ such that $u(a) = u(b)$, then there exists a covering $\{f_i : U'_i \to U'\}$ in $\mathcal{C}$ such that $a \circ f_i = b \circ f_i$,*
(4) *given $U', U \in \mathrm{Ob}(\mathcal{C})$ and a morphism $c : u(U') \to u(U)$ in $\mathcal{D}$ there exists a covering $\{f_i : U'_i \to U'\}$ in $\mathcal{C}$ and morphisms $c_i : U'_i \to U$ such that $u(c_i) = c \circ u(f_i)$, and*
(5) *given $V \in \mathrm{Ob}(\mathcal{D})$ there exists a covering of V in $\mathcal{D}$ of the form $\{u(U_i) \to V\}_{i\in I}$.*

[6] In this generality we don't know $f \circ g'$ is equal to $g \circ f'$ as morphisms of topoi (there is a canonical 2-arrow from the first to the second which may not be an isomorphism).

Then the morphism of topoi

$$g : Sh(\mathcal{C}) \longrightarrow Sh(\mathcal{D})$$

associated to the cocontinuous functor u by Lemma 21.1 is an equivalence.

Proof. Assume u satisfies properties (1) – (5). We will show that the adjunction mappings

$$\mathcal{G} \longrightarrow g_*g^{-1}\mathcal{G} \quad \text{and} \quad g^{-1}g_*\mathcal{F} \longrightarrow \mathcal{F}$$

are isomorphisms.

Note that Lemma 21.5 applies and we have $g^{-1}\mathcal{G}(U) = \mathcal{G}(u(U))$ for any sheaf $\mathcal{G}$ on $\mathcal{D}$. Next, let $\mathcal{F}$ be a sheaf on $\mathcal{C}$, and let V be an object of $\mathcal{D}$. By definition we have $g_*\mathcal{F}(V) = \lim_{u(U)\to V} \mathcal{F}(U)$. Hence

$$g^{-1}g_*\mathcal{F}(U) = \lim_{U', u(U')\to u(U)} \mathcal{F}(U')$$

where the morphisms $\psi : u(U') \to u(U)$ need not be of the form $u(\alpha)$. The category of such pairs (U', ψ) has a final object, namely (U, id), which gives rise to the map from the limit into $\mathcal{F}(U)$. Let $(s_{(U',\psi)})$ be an element of the limit. We want to show that $s_{(U',\psi)}$ is uniquely determined by the value $s_{(U,\mathrm{id})} \in \mathcal{F}(U)$. By property (4) given any (U', ψ) there exists a covering $\{U'_i \to U'\}$ such that the compositions $u(U'_i) \to u(U') \to u(U)$ are of the form $u(c_i)$ for some $c_i : U'_i \to U$ in $\mathcal{C}$. Hence

$$s_{(U',\psi)}|_{U'_i} = c_i^*(s_{(U,\mathrm{id})}).$$

Since $\mathcal{F}$ is a sheaf it follows that indeed $s_{(U',\psi)}$ is determined by $s_{(U,\mathrm{id})}$. This proves uniqueness. For existence, assume given any $s \in \mathcal{F}(U)$, $\psi : u(U') \to u(U)$, $\{f_i : U'_i \to U'\}$ and $c_i : U'_i \to U$ such that $\psi \circ u(f_i) = u(c_i)$ as above. We claim there exists a (unique) element $s_{(U',\psi)} \in \mathcal{F}(U')$ such that

$$s_{(U',\psi)}|_{U'_i} = c_i^*(s).$$

Namely, a priori it is not clear the elements $c_i^*(s)|_{U'_i \times_{U'} U'_j}$ and $c_j^*(s)|_{U'_i \times_{U'} U'_j}$ agree, since the diagram

$$\begin{array}{ccc} U'_i \times_{U'} U'_j & \xrightarrow[\mathrm{pr}_2]{} & U'_j \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle c_j} \\ U'_i & \xrightarrow{c_i} & U \end{array}$$

need not commute. But condition (3) of the lemma guarantees that there exist coverings $\{f_{ijk} : U'_{ijk} \to U'_i \times_{U'} U'_j\}_{k\in K_{ij}}$ such that $c_i \circ \mathrm{pr}_1 \circ f_{ijk} = c_j \circ \mathrm{pr}_2 \circ f_{ijk}$. Hence

$$f_{ijk}^*\left(c_i^* s|_{U'_i \times_{U'} U'_j}\right) = f_{ijk}^*\left(c_j^* s|_{U'_i \times_{U'} U'_j}\right)$$

Hence $c_i^*(s)|_{U'_i \times_{U'} U'_j} = c_j^*(s)|_{U'_i \times_{U'} U'_j}$ by the sheaf condition for $\mathcal{F}$ and hence the existence of $s_{(U',\psi)}$ also by the sheaf condition for $\mathcal{F}$. The uniqueness guarantees that the collection $(s_{(U',\psi)})$ so obtained is an element of the limit with $s_{(U,\psi)} = s$. This proves that $g^{-1}g_*\mathcal{F} \to \mathcal{F}$ is an isomorphism.

Let $\mathcal{G}$ be a sheaf on $\mathcal{D}$. Let V be an object of $\mathcal{D}$. Then we see that

$$g_*g^{-1}\mathcal{G}(V) = \lim_{U,\psi:u(U)\to V} \mathcal{G}(u(U))$$

By the preceding paragraph we see that the value of the sheaf $g_*g^{-1}\mathcal{G}$ on an object V of the form $V = u(U)$ is equal to $\mathcal{G}(u(U))$. (Formally, this holds because we have $g^{-1}g_*g^{-1} \cong g^{-1}$, and the description of g^{-1} given at the beginning of the

proof; informally just by comparing limits here and above.) Hence the adjunction mapping $\mathcal{G} \to g_* g^{-1}\mathcal{G}$ has the property that it is a bijection on sections over any object of the form $u(U)$. Since by axiom (5) there exists a covering of V by objects of the form $u(U)$ we see easily that the adjunction map is an isomorphism. $\square$

It will be convenient to give cocontinuous functors as in Lemma 29.1 a name.

03CG **Definition 29.2.** Let $\mathcal{C}$, $\mathcal{D}$ be sites. A *special cocontinuous functor u from $\mathcal{C}$ to $\mathcal{D}$* is a cocontinuous functor $u : \mathcal{C} \to \mathcal{D}$ satisfying the assumptions and conclusions of Lemma 29.1.

03CH **Lemma 29.3.** *Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a special cocontinuous functor. For every object U of $\mathcal{C}$ we have a commutative diagram*

$$\begin{array}{ccc} \mathcal{C}/U & \xrightarrow{j_U} & \mathcal{C} \\ \downarrow & & \downarrow u \\ \mathcal{D}/u(U) & \xrightarrow{j_{u(U)}} & \mathcal{D} \end{array}$$

as in Lemma 28.4. The left vertical arrow is a special cocontinuous functor. Hence in the commutative diagram of topoi

$$\begin{array}{ccc} Sh(\mathcal{C}/U) & \xrightarrow{j_U} & Sh(\mathcal{C}) \\ \downarrow & & \downarrow u \\ Sh(\mathcal{D}/u(U)) & \xrightarrow{j_{u(U)}} & Sh(\mathcal{D}) \end{array}$$

the vertical arrows are equivalences.

Proof. We have seen the existence and commutativity of the diagrams in Lemma 28.4. We have to check hypotheses (1) – (5) of Lemma 29.1 for the induced functor $u : \mathcal{C}/U \to \mathcal{D}/u(U)$. This is completely mechanical.

Property (1). This is Lemma 28.4.

Property (2). Let $\{U_i'/U \to U'/U\}_{i\in I}$ be a covering of U'/U in $\mathcal{C}/U$. Because u is continuous we see that $\{u(U_i')/u(U) \to u(U')/u(U)\}_{i\in I}$ is a covering of $u(U')/u(U)$ in $\mathcal{D}/u(U)$. Hence (2) holds for $u : \mathcal{C}/U \to \mathcal{D}/u(U)$.

Property (3). Let $a, b : U''/U \to U'/U$ in $\mathcal{C}/U$ be morphisms such that $u(a) = u(b)$ in $\mathcal{D}/u(U)$. Because u satisfies (3) we see there exists a covering $\{f_i : U_i'' \to U''\}$ in $\mathcal{C}$ such that $a \circ f_i = b \circ f_i$. This gives a covering $\{f_i : U_i''/U \to U''/U\}$ in $\mathcal{C}/U$ such that $a \circ f_i = b \circ f_i$. Hence (3) holds for $u : \mathcal{C}/U \to \mathcal{D}/u(U)$.

Property (4). Let $U''/U, U'/U \in \mathrm{Ob}(\mathcal{C}/U)$ and a morphism $c : u(U'')/u(U) \to u(U')/u(U)$ in $\mathcal{D}/u(U)$ be given. Because u satisfies property (4) there exists a covering $\{f_i : U_i'' \to U''\}$ in $\mathcal{C}$ and morphisms $c_i : U_i'' \to U'$ such that $u(c_i) = c \circ u(f_i)$. We think of U_i'' as an object over U via the composition $U_i'' \to U'' \to U$. It may not be true that c_i is a morphism over U! But since $u(c_i)$ is a morphism over $u(U)$ we may apply property (3) for u and find coverings $\{f_{ik} : U_{ik}'' \to U_i''\}$ such that $c_{ik} = c_i \circ f_{ik} : U_{ik}'' \to U'$ are morphisms over U. Hence $\{f_i \circ f_{ik} : U_{ik}''/U \to U''/U\}$ is a covering in $\mathcal{C}/U$ such that $u(c_{ik}) = c \circ u(f_{ik})$. Hence (4) holds for $u : \mathcal{C}/U \to \mathcal{D}/u(U)$.

Property (5). Let $h : V \to u(U)$ be an object of $\mathcal{D}/u(U)$. Because u satisfies property (5) there exists a covering $\{c_i : u(U_i) \to V\}$ in $\mathcal{D}$. By property (4) we can find coverings $\{f_{ij} : U_{ij} \to U_i\}$ and morphisms $c_{ij} : U_{ij} \to U$ such that $u(c_{ij}) = h \circ c_i \circ u(f_{ij})$. Hence $\{u(U_{ij})/u(U) \to V/u(U)\}$ is a covering in $\mathcal{D}/u(U)$ of the desired shape and we conclude that (5) holds for $u : \mathcal{C}/U \to \mathcal{D}/u(U)$. □

03A1 **Lemma 29.4.** *Let $\mathcal{C}$ be a site. Let $\mathcal{C}' \subset Sh(\mathcal{C})$ be a full subcategory (with a set of objects) such that*

(1) *$h_U^\# \in \mathrm{Ob}(\mathcal{C}')$ for all $U \in \mathrm{Ob}(\mathcal{C})$, and*
(2) *$\mathcal{C}'$ is preserved under fibre products in $Sh(\mathcal{C})$.*

Declare a covering of $\mathcal{C}'$ to be any family $\{\mathcal{F}_i \to \mathcal{F}\}_{i \in I}$ of maps such that $\coprod_{i \in I} \mathcal{F}_i \to \mathcal{F}$ is a surjective map of sheaves. Then

(1) *$\mathcal{C}'$ is a site (after choosing a set of coverings, see Sets, Lemma 11.1),*
(2) *representable presheaves on $\mathcal{C}'$ are sheaves (i.e., the topology on $\mathcal{C}'$ is subcanonical, see Definition 12.2),*
(3) *the functor $v : \mathcal{C} \to \mathcal{C}'$, $U \mapsto h_U^\#$ is a special cocontinuous functor, hence induces an equivalence $g : Sh(\mathcal{C}) \to Sh(\mathcal{C}')$,*
(4) *for any $\mathcal{F} \in \mathrm{Ob}(\mathcal{C}')$ we have $g^{-1}h_{\mathcal{F}} = \mathcal{F}$, and*
(5) *for any $U \in \mathrm{Ob}(\mathcal{C})$ we have $g_* h_U^\# = h_{v(U)} = h_{h_U^\#}$.*

Proof. Warning: Some of the statements above may look be a bit confusing at first; this is because objects of $\mathcal{C}'$ can also be viewed as sheaves on $\mathcal{C}$! We omit the proof that the coverings of $\mathcal{C}'$ as described in the lemma satisfy the conditions of Definition 6.2.

Suppose that $\{\mathcal{F}_i \to \mathcal{F}\}$ is a surjective family of morphisms of sheaves. Let $\mathcal{G}$ be another sheaf. Part (2) of the lemma says that the equalizer of

$$\mathrm{Mor}_{Sh(\mathcal{C})}(\coprod_{i \in I} \mathcal{F}_i, \mathcal{G}) \rightrightarrows \mathrm{Mor}_{Sh(\mathcal{C})}(\coprod_{(i_0, i_1) \in I \times I} \mathcal{F}_{i_0} \times_{\mathcal{F}} \mathcal{F}_{i_1}, \mathcal{G})$$

is $\mathrm{Mor}_{Sh(\mathcal{C})}(\mathcal{F}, \mathcal{G})$. This is clear (for example use Lemma 11.3).

To prove (3) we have to check conditions (1) – (5) of Lemma 29.1. The fact that v is cocontinuous is equivalent to the description of surjective maps of sheaves in Lemma 11.2. The functor v is continuous because $U \mapsto h_U^\#$ commutes with fibre products, and transforms coverings into coverings (see Lemma 10.14, and Lemma 12.4). Properties (3), (4) of Lemma 29.1 are statements about morphisms $f : h_{U'}^\# \to h_U^\#$. Such a morphism is the same thing as an element of $h_U^\#(U')$. Hence (3) and (4) are immediate from the construction of the sheafification. Property (5) of Lemma 29.1 is Lemma 12.5. Denote $g : Sh(\mathcal{C}) \to Sh(\mathcal{C}')$ the equivalence of topoi associated with v by Lemma 29.1.

Let $\mathcal{F}$ be as in part (4) of the lemma. For any $U \in \mathrm{Ob}(\mathcal{C})$ we have

$$g^{-1}h_{\mathcal{F}}(U) = h_{\mathcal{F}}(v(U)) = \mathrm{Mor}_{Sh(\mathcal{C})}(h_U^\#, \mathcal{F}) = \mathcal{F}(U)$$

The first equality by Lemma 21.5. Thus part (4) holds.

Let $\mathcal{F} \in \mathrm{Ob}(\mathcal{C}')$. Let $U \in \mathrm{Ob}(\mathcal{C})$. Then

$$\begin{aligned} g_* h_U^\# (\mathcal{F}) &= \mathrm{Mor}_{Sh(\mathcal{C}')}(h_{\mathcal{F}}, g_* h_U^\#) \\ &= \mathrm{Mor}_{Sh(\mathcal{C})}(g^{-1} h_{\mathcal{F}}, h_U^\#) \\ &= \mathrm{Mor}_{Sh(\mathcal{C})}(\mathcal{F}, h_U^\#) \\ &= \mathrm{Mor}_{\mathcal{C}'}(\mathcal{F}, h_U^\#) \end{aligned}$$

as desired (where the third equality was shown above). □

Using this we can massage any topos to live over a site having all finite limits.

03CI **Lemma 29.5.** *Let $Sh(\mathcal{C})$ be a topos. Let $\{\mathcal{F}_i\}_{i \in I}$ be a set of sheaves on $\mathcal{C}$. There exists an equivalence of topoi $g : Sh(\mathcal{C}) \to Sh(\mathcal{C}')$ induced by a special cocontinuous functor $u : \mathcal{C} \to \mathcal{C}'$ of sites such that*

1. *$\mathcal{C}'$ has a subcanonical topology,*
2. *a family $\{V_j \to V\}$ of morphisms of $\mathcal{C}'$ is (combinatorially equivalent to) a covering of $\mathcal{C}'$ if and only if $\coprod h_{V_j} \to h_V$ is surjective,*
3. *$\mathcal{C}'$ has fibre products and a final object (i.e., $\mathcal{C}'$ has all finite limits),*
4. *every subsheaf of a representable sheaf on $\mathcal{C}'$ is representable, and*
5. *each $g_* \mathcal{F}_i$ is a representable sheaf.*

Proof. Consider the full subcategory $\mathcal{C}_1 \subset Sh(\mathcal{C})$ consisting of all $h_U^\#$ for all $U \in \mathrm{Ob}(\mathcal{C})$, the given sheaves $\mathcal{F}_i$ and the final sheaf $*$ (see Example 10.2). We are going to inductively define full subcategories

$$\mathcal{C}_1 \subset \mathcal{C}_2 \subset \mathcal{C}_2 \subset \ldots \subset Sh(\mathcal{C})$$

Namely, given $\mathcal{C}_n$ let $\mathcal{C}_{n+1}$ be the full subcategory consisting of all fibre products and subsheaves of objects of $\mathcal{C}_n$. (Note that $\mathcal{C}_{n+1}$ has a set of objects.) Set $\mathcal{C}' = \bigcup_{n \geq 1} \mathcal{C}_n$. A covering in $\mathcal{C}'$ is any family $\{\mathcal{G}_j \to \mathcal{G}\}_{j \in J}$ of morphisms of objects of $\mathcal{C}'$ such that $\coprod \mathcal{G}_j \to \mathcal{G}$ is surjective as a map of sheaves on $\mathcal{C}$. The functor $v : \mathcal{C} \to \mathcal{C}'$ is given by $U \mapsto h_U^\#$. Apply Lemma 29.4. □

Here is the goal of the current section.

03A2 **Lemma 29.6.** *Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $f : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be a morphism of topoi. Then there exists a site $\mathcal{C}'$ and a diagram of functors*

$$\mathcal{C} \xrightarrow[v]{} \mathcal{C}' \xleftarrow[u]{} \mathcal{D}$$

such that

1. *the functor v is a special cocontinuous functor,*
2. *the functor u commutes with fibre products, is continuous and defines a morphism of sites $\mathcal{C}' \to \mathcal{D}$, and*
3. *the morphism of topoi f agrees with the composition of morphisms of topoi*

$$Sh(\mathcal{C}) \longrightarrow Sh(\mathcal{C}') \longrightarrow Sh(\mathcal{D})$$

where the first arrow comes from v via Lemma 29.1 and the second arrow from u via Lemma 15.2.

This statement is closely related to [AGV71, Proposition 4.9.4. Exposé IV]. In order to get the whole result, one should also use [AGV71, Remarque 4.7.4, Exposé IV].

Proof. Consider the full subcategory $\mathcal{C}_1 \subset Sh(\mathcal{C})$ consisting of all $h_U^\#$ and all $f^{-1} h_V^\#$ for all $U \in \mathrm{Ob}(\mathcal{C})$ and all $V \in \mathrm{Ob}(\mathcal{D})$. Let $\mathcal{C}_{n+1}$ be a full subcategory consisting of all fibre products of objects of $\mathcal{C}_n$. Set $\mathcal{C}' = \bigcup_{n \geq 1} \mathcal{C}_n$. A covering in $\mathcal{C}'$ is any

family $\{\mathcal{F}_i \to \mathcal{F}\}_{i\in I}$ such that $\coprod_{i\in I} \mathcal{F}_i \to \mathcal{F}$ is surjective as a map of sheaves on $\mathcal{C}$. The functor $v : \mathcal{C} \to \mathcal{C}'$ is given by $U \mapsto h_U^\#$. The functor $u : \mathcal{D} \to \mathcal{C}'$ is given by $V \mapsto f^{-1}h_V^\#$.

Part (1) follows from Lemma 29.4.

Proof of (2) and (3) of the lemma. The functor u commutes with fibre products as both $V \mapsto h_V^\#$ and f^{-1} do. Moreover, since f^{-1} is exact and commutes with arbitrary colimits we see that it transforms a covering into a surjective family of morphisms of sheaves. Hence u is continuous. To see that it defines a morphism of sites we still have to see that u_s is exact. In order to do this we will show that $g^{-1} \circ u_s = f^{-1}$. Namely, then since g^{-1} is an equivalence and f^{-1} is exact we will conclude. Because g^{-1} is adjoint to g_*, and u_s is adjoint to u^s, and f^{-1} is adjoint to f_* it also suffices to prove that $u^s \circ g_* = f_*$. Let U be an object of $\mathcal{C}$ and let V be an object of $\mathcal{D}$. Then

$$\begin{aligned}(u^s g_* h_U^\#)(V) &= g_* h_U^\#(f^{-1}h_V^\#) \\ &= \mathrm{Mor}_{Sh(\mathcal{C})}(f^{-1}h_V^\#, h_U^\#) \\ &= \mathrm{Mor}_{Sh(\mathcal{D})}(h_V^\#, f_* h_U^\#) \\ &= f_* h_U^\#(V)\end{aligned}$$

The first equality because $u^s = u^p$. The second equality by Lemma 29.4 (5). The third equality by adjointness of f_* and f^{-1} and the final equality by properties of sheafification and the Yoneda lemma. We omit the verification that these identities are functorial in U and V. Hence we see that we have $u^s \circ g_* = f_*$ for sheaves of the form $h_U^\#$. This implies that $u^s \circ g_* = f_*$ and we win (some details omitted). □

03CJ **Remark 29.7.** Notation and assumptions as in Lemma 29.6. If the site $\mathcal{D}$ has a final object and fibre products then the functor $u : \mathcal{D} \to \mathcal{C}'$ satisfies all the assumptions of Proposition 14.7. Namely, in addition to the properties mentioned in the lemma u also transforms the final object of $\mathcal{D}$ into the final object of $\mathcal{C}'$. This is clear from the construction of u. Hence, if we first apply Lemmas 29.5 to $\mathcal{D}$ and then Lemma 29.6 to the resulting morphism of topoi $Sh(\mathcal{C}) \to Sh(\mathcal{D}')$ we obtain the following statement: Any morphism of topoi $f : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ fits into a commutative diagram

$$\begin{array}{ccc} Sh(\mathcal{C}) & \xrightarrow{f} & Sh(\mathcal{D}) \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle e} \\ Sh(\mathcal{C}') & \xrightarrow{f'} & Sh(\mathcal{D}') \end{array}$$

where the following properties hold:

(1) the morphisms e and g are equivalences given by special cocontinuous functors $\mathcal{C} \to \mathcal{C}'$ and $\mathcal{D} \to \mathcal{D}'$,
(2) the sites $\mathcal{C}'$ and $\mathcal{D}'$ have fibre products, final objects and have subcanonical topologies,
(3) the morphism $f' : \mathcal{C}' \to \mathcal{D}'$ comes from a morphism of sites corresponding to a functor $u : \mathcal{D}' \to \mathcal{C}'$ to which Proposition 14.7 applies, and
(4) given any set of sheaves $\mathcal{F}_i$ (resp. $\mathcal{G}_j$) on $\mathcal{C}$ (resp. $\mathcal{D}$) we may assume each of these is a representable sheaf on $\mathcal{C}'$ (resp. $\mathcal{D}'$).

It is often useful to replace $\mathcal{C}$ and $\mathcal{D}$ by $\mathcal{C}'$ and $\mathcal{D}'$.

03CK **Remark 29.8.** Notation and assumptions as in Lemma 29.6. Suppose that in addition the original morphism of topoi $Sh(\mathcal{C}) \to Sh(\mathcal{D})$ is an equivalence. Then the construction in the proof of Lemma 29.6 gives two functors

$$\mathcal{C} \to \mathcal{C}' \leftarrow \mathcal{D}$$

which are both special cocontinuous functors. Hence in this case we can actually factor the morphism of topoi as a composition

$$Sh(\mathcal{C}) \to Sh(\mathcal{C}') = Sh(\mathcal{D}') \leftarrow Sh(\mathcal{D})$$

as in Remark 29.7, but with the middle morphism an identity.

30. Localization of topoi

04GY We repeat some of the material on localization to the apparently more general case of topoi. In reality this is not more general since we may always enlarge the underlying sites to assume that we are localizing at objects of the site.

04GZ **Lemma 30.1.** *Let $\mathcal{C}$ be a site. Let $\mathcal{F}$ be a sheaf on $\mathcal{C}$. Then the category $Sh(\mathcal{C})/\mathcal{F}$ is a topos. There is a canonical morphism of topoi*

$$j_{\mathcal{F}} : Sh(\mathcal{C})/\mathcal{F} \longrightarrow Sh(\mathcal{C})$$

which is a localization as in Section 25 such that

1. *the functor $j_{\mathcal{F}}^{-1}$ is the functor $\mathcal{H} \mapsto \mathcal{H} \times \mathcal{F}/\mathcal{F}$, and*
2. *the functor $j_{\mathcal{F}!}$ is the forgetful functor $\mathcal{G}/\mathcal{F} \mapsto \mathcal{G}$.*

Proof. Apply Lemma 29.5. This means we may assume $\mathcal{C}$ is a site with subcanonical topology, and $\mathcal{F} = h_U = h_U^\#$ for some $U \in \mathrm{Ob}(\mathcal{C})$. Hence the material of Section 25 applies. In particular, there is an equivalence $Sh(\mathcal{C}/U) = Sh(\mathcal{C})/h_U^\#$ such that the composition

$$Sh(\mathcal{C}/U) \to Sh(\mathcal{C})/h_U^\# \to Sh(\mathcal{C})$$

is equal to $j_{U!}$, see Lemma 25.4. Denote $a : Sh(\mathcal{C})/h_U^\# \to Sh(\mathcal{C}/U)$ the inverse functor, so $j_{\mathcal{F}!} = j_{U!} \circ a$, $j_{\mathcal{F}}^{-1} = a^{-1} \circ j_U^{-1}$, and $j_{\mathcal{F},*} = j_{U,*} \circ a$. The description of $j_{\mathcal{F}!}$ follows from the above. The description of $j_{\mathcal{F}}^{-1}$ follows from Lemma 25.7. □

04H0 **Lemma 30.2.** *In the situation of Lemma 30.1, the functor $j_{\mathcal{F},*}$ is the one associates to $\varphi : \mathcal{G} \to \mathcal{F}$ the sheaf*

$$U \longmapsto \{\alpha : \mathcal{F}|_U \to \mathcal{G}|_U \text{ such that } \alpha \text{ is a right inverse to } \varphi|_U\}.$$

Proof. For any $\varphi : \mathcal{G} \to \mathcal{F}$, let us use the notation $\mathcal{G}/\mathcal{F}$ to denote the corresponding object of $Sh(\mathcal{C})/\mathcal{F}$. We have

$$(j_{\mathcal{F},*}(\mathcal{G}/\mathcal{F}))(U) = \mathrm{Mor}_{Sh(\mathcal{C})}(h_U^\#, j_{\mathcal{F},*}(\mathcal{G}/\mathcal{F})) = \mathrm{Mor}_{Sh(\mathcal{C})/\mathcal{F}}(j_{\mathcal{F}}^{-1}h_U^\#, (\mathcal{G}/\mathcal{F})).$$

By Lemma 30.1 this set is the fiber over the element $h_U^\# \times \mathcal{F} \to \mathcal{F}$ under the map of sets

$$\mathrm{Mor}_{Sh(\mathcal{C})}(h_U^\# \times \mathcal{F}, \mathcal{G}) \xrightarrow{\varphi\circ} \mathrm{Mor}_{Sh(\mathcal{C})}(h_U^\# \times \mathcal{F}, \mathcal{F}).$$

By the adjunction in Lemma 26.2, we have

$$\begin{aligned}\mathrm{Mor}_{Sh(\mathcal{C})}(h_U^\# \times \mathcal{F}, \mathcal{G}) &= \mathrm{Mor}_{Sh(\mathcal{C})}(h_U^\#, \mathcal{H}om(\mathcal{F}, \mathcal{G})) \\ &= \mathrm{Mor}_{Sh(\mathcal{C}/U)}(\mathcal{F}|_{\mathcal{C}/U}, \mathcal{G}|_{\mathcal{C}/U}), \\ \mathrm{Mor}_{Sh(\mathcal{C})}(h_U^\# \times \mathcal{F}, \mathcal{F}) &= \mathrm{Mor}_{Sh(\mathcal{C})}(h_U^\#, \mathcal{H}om(\mathcal{F}, \mathcal{F})) \\ &= \mathrm{Mor}_{Sh(\mathcal{C}/U)}(\mathcal{F}|_{\mathcal{C}/U}, \mathcal{F}|_{\mathcal{C}/U}),\end{aligned}$$

and under the adjunction, the map $\varphi\circ$ becomes the map

$$\mathrm{Mor}_{Sh(\mathcal{C}/U)}(\mathcal{F}|_{\mathcal{C}/U}, \mathcal{G}|_{\mathcal{C}/U}) \longrightarrow \mathrm{Mor}_{Sh(\mathcal{C}/U)}(\mathcal{F}|_{\mathcal{C}/U}, \mathcal{F}|_{\mathcal{C}/U}), \quad \psi \longmapsto \varphi|_{\mathcal{C}/U} \circ \psi,$$

the element $h_U^\# \times \mathcal{F} \to \mathcal{F}$ becomes $\mathrm{id}_{\mathcal{F}|_{\mathcal{C}/U}}$. Therefore $(j_{\mathcal{F},*}\mathcal{G}/\mathcal{F})(U)$ is isomorphic to the fiber of $\mathrm{id}_{\mathcal{F}|_{\mathcal{C}/U}}$ under the map

$$\mathrm{Mor}_{Sh(\mathcal{C}/U)}(\mathcal{F}|_{\mathcal{C}/U}, \mathcal{G}|_{\mathcal{C}/U}) \xrightarrow{\varphi|_{\mathcal{C}/U}\circ} \mathrm{Mor}_{Sh(\mathcal{C}/U)}(\mathcal{F}|_{\mathcal{C}/U}, \mathcal{F}|_{\mathcal{C}/U}),$$

which is $\{\alpha : \mathcal{F}|_U \to \mathcal{G}|_U$ such that α is a right inverse to $\varphi|_U\}$ as desired. □

0791 **Lemma 30.3.** *Let $\mathcal{C}$ be a site. Let $\mathcal{F}$ be a sheaf on $\mathcal{C}$. Let $\mathcal{C}/\mathcal{F}$ be the category of pairs (U, s) where $U \in \mathrm{Ob}(\mathcal{C})$ and $s \in \mathcal{F}(U)$. Let a covering in $\mathcal{C}/\mathcal{F}$ be a family $\{(U_i, s_i) \to (U, s)\}$ such that $\{U_i \to U\}$ is a covering of $\mathcal{C}$. Then $j : \mathcal{C}/\mathcal{F} \to \mathcal{C}$ is a continuous and cocontinuous functor of sites which induces a morphism of topoi $j : Sh(\mathcal{C}/\mathcal{F}) \to Sh(\mathcal{C})$. In fact, there is an equivalence $Sh(\mathcal{C}/\mathcal{F}) = Sh(\mathcal{C})/\mathcal{F}$ which turns j into $j_{\mathcal{F}}$.*

Proof. We omit the verification that $\mathcal{C}/\mathcal{F}$ is a site and that j is continuous and cocontinuous. By Lemma 21.5 there exists a morphism of topoi j as indicated, with $j^{-1}\mathcal{G}(U, s) = \mathcal{G}(U)$, and there is a left adjoint $j_!$ to j^{-1}. A morphism $\varphi : * \to j^{-1}\mathcal{G}$ on $\mathcal{C}/\mathcal{F}$ is the same thing as a rule which assigns to every pair (U, s) a section $\varphi(s) \in \mathcal{G}(U)$ compatible with restriction maps. Hence this is the same thing as a morphism $\varphi : \mathcal{F} \to \mathcal{G}$ over $\mathcal{C}$. We conclude that $j_!* = \mathcal{F}$. In particular, for every $\mathcal{H} \in Sh(\mathcal{C}/\mathcal{F})$ there is a canonical map

$$j_!\mathcal{H} \to j_!* = \mathcal{F}$$

i.e., we obtain a functor $j'_! : Sh(\mathcal{C}/\mathcal{F}) \to Sh(\mathcal{C})/\mathcal{F}$. An inverse to this functor is the rule which assigns to an object $\varphi : \mathcal{G} \to \mathcal{F}$ of $Sh(\mathcal{C})/\mathcal{F}$ the sheaf

$$a(\mathcal{G}/\mathcal{F}) : (U, s) \longmapsto \{t \in \mathcal{G}(U) \mid \varphi(t) = s\}$$

We omit the verification that $a(\mathcal{G}/\mathcal{F})$ is a sheaf and that a is inverse to $j'_!$. □

04IP **Definition 30.4.** Let $\mathcal{C}$ be a site. Let $\mathcal{F}$ be a sheaf on $\mathcal{C}$.

(1) The topos $Sh(\mathcal{C})/\mathcal{F}$ is called the *localization of the topos $Sh(\mathcal{C})$ at $\mathcal{F}$*.
(2) The morphism of topoi $j_{\mathcal{F}} : Sh(\mathcal{C})/\mathcal{F} \to Sh(\mathcal{C})$ of Lemma 30.1 is called the *localization morphism*.

We are going to show that whenever the sheaf $\mathcal{F}$ is equal to $h_U^\#$ for some object U of the site, then the localization of the topos is equal to the category of sheaves on the localization of the site at U. Moreover, we are going to check that any functorialities are compatible with this identification.

04IQ **Lemma 30.5.** *Let $\mathcal{C}$ be a site. Let $\mathcal{F} = h_U^\#$ for some object U of $\mathcal{C}$. Then $j_{\mathcal{F}} : Sh(\mathcal{C})/\mathcal{F} \to Sh(\mathcal{C})$ constructed in Lemma 30.1 agrees with the morphism of topoi*

$j_U : Sh(\mathcal{C}/U) \to Sh(\mathcal{C})$ constructed in Section 25 via the identification $Sh(\mathcal{C}/U) = Sh(\mathcal{C})/h_U^\#$ of Lemma 25.4.

Proof. We have seen in Lemma 25.4 that the composition $Sh(\mathcal{C}/U) \to Sh(\mathcal{C})/h_U^\# \to Sh(\mathcal{C})$ is $j_{U!}$. The functor $Sh(\mathcal{C})/h_U^\# \to Sh(\mathcal{C})$ is $j_{\mathcal{F}!}$ by Lemma 30.1. Hence $j_{\mathcal{F}!} = j_{U!}$ via the identification. So $j_{\mathcal{F}}^{-1} = j_U^{-1}$ (by adjointness) and so $j_{\mathcal{F},*} = j_{U,*}$ (by adjointness again). □

04IR **Lemma 30.6.** *Let $\mathcal{C}$ be a site. If $s : \mathcal{G} \to \mathcal{F}$ is a morphism of sheaves on $\mathcal{C}$ then there exists a natural commutative diagram of morphisms of topoi*

$$\begin{array}{ccc} Sh(\mathcal{C})/\mathcal{G} & \xrightarrow{\quad j \quad} & Sh(\mathcal{C})/\mathcal{F} \\ & {\scriptstyle j_{\mathcal{G}}} \searrow \quad \swarrow {\scriptstyle j_{\mathcal{F}}} & \\ & Sh(\mathcal{C}) & \end{array}$$

where $j = j_{\mathcal{G}/\mathcal{F}}$ is the localization of the topos $Sh(\mathcal{C})/\mathcal{F}$ at the object $\mathcal{G}/\mathcal{F}$. In particular we have

$$j^{-1}(\mathcal{H} \to \mathcal{F}) = (\mathcal{H} \times_{\mathcal{F}} \mathcal{G} \to \mathcal{G})$$

and

$$j_!(\mathcal{E} \xrightarrow{e} \mathcal{F}) = (\mathcal{E} \xrightarrow{s \circ e} \mathcal{G}).$$

Proof. The description of j^{-1} and $j_!$ comes from the description of those functors in Lemma 30.1. The equality of functors $j_{\mathcal{G}!} = j_{\mathcal{F}!} \circ j_!$ is clear from the description of these functors (as forgetful functors). By adjointness we also obtain the equalities $j_{\mathcal{G}}^{-1} = j^{-1} \circ j_{\mathcal{F}}^{-1}$, and $j_{\mathcal{G},*} = j_{\mathcal{F},*} \circ j_*$. □

04IS **Lemma 30.7.** *Assume $\mathcal{C}$ and $s : \mathcal{G} \to \mathcal{F}$ are as in Lemma 30.6. If $\mathcal{G} = h_V^\#$ and $\mathcal{F} = h_U^\#$ and $s : \mathcal{G} \to \mathcal{F}$ comes from a morphism $V \to U$ of $\mathcal{C}$ then the diagram in Lemma 30.6 is identified with diagram (25.8.1) via the identifications $Sh(\mathcal{C}/V) = Sh(\mathcal{C})/h_V^\#$ and $Sh(\mathcal{C}/U) = Sh(\mathcal{C})/h_U^\#$ of Lemma 25.4.*

Proof. This is true because the descriptions of j^{-1} agree. See Lemma 25.9 and Lemma 30.6. □

31. Localization and morphisms of topoi

04IT This section is the analogue of Section 28 for morphisms of topoi.

04H1 **Lemma 31.1.** *Let $f : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be a morphism of topoi. Let $\mathcal{G}$ be a sheaf on $\mathcal{D}$. Set $\mathcal{F} = f^{-1}\mathcal{G}$. Then there exists a commutative diagram of topoi*

$$\begin{array}{ccc} Sh(\mathcal{C})/\mathcal{F} & \xrightarrow{j_{\mathcal{F}}} & Sh(\mathcal{C}) \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Sh(\mathcal{D})/\mathcal{G} & \xrightarrow{j_{\mathcal{G}}} & Sh(\mathcal{D}). \end{array}$$

The morphism f' is characterized by the property that

$$(f')^{-1}(\mathcal{H} \xrightarrow{\varphi} \mathcal{G}) = (f^{-1}\mathcal{H} \xrightarrow{f^{-1}\varphi} \mathcal{F})$$

and we have $f'_* j_{\mathcal{F}}^{-1} = j_{\mathcal{G}}^{-1} f_*$.

Proof. Since the statement is about topoi and does not refer to the underlying sites we may change sites at will. Hence by the discussion in Remark 29.7 we may assume that f is given by a continuous functor $u : \mathcal{D} \to \mathcal{C}$ satisfying the assumptions of Proposition 14.7 between sites having all finite limits and subcanonical topologies, and such that $\mathcal{G} = h_V$ for some object V of $\mathcal{D}$. Then $\mathcal{F} = f^{-1}h_V = h_{u(V)}$ by Lemma 13.5. By Lemma 28.1 we obtain a commutative diagram of morphisms of topoi

$$\begin{array}{ccc} Sh(\mathcal{C}/U) & \xrightarrow[j_U]{} & Sh(\mathcal{C}) \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Sh(\mathcal{D}/V) & \xrightarrow{j_V} & Sh(\mathcal{D}), \end{array}$$

and we have $f'_* j_U^{-1} = j_V^{-1} f_*$. By Lemma 30.5 we may identify $j_{\mathcal{F}}$ and j_U and $j_{\mathcal{G}}$ and j_V. The description of $(f')^{-1}$ is given in Lemma 28.1. □

04IU **Lemma 31.2.** *Let $f : \mathcal{C} \to \mathcal{D}$ be a morphism of sites given by the continuous functor $u : \mathcal{D} \to \mathcal{C}$. Let V be an object of $\mathcal{D}$. Set $U = u(V)$. Set $\mathcal{G} = h_V^\#$, and $\mathcal{F} = h_U^\# = f^{-1}h_V^\#$ (see Lemma 13.5). Then the diagram of morphisms of topoi of Lemma 31.1 agrees with the diagram of morphisms of topoi of Lemma 28.1 via the identifications $j_{\mathcal{F}} = j_U$ and $j_{\mathcal{G}} = j_V$ of Lemma 30.5.*

Proof. This is not a complete triviality as the choice of morphism of sites giving rise to f made in the proof of Lemma 31.1 may be different from the morphisms of sites given to us in the lemma. But in both cases the functor $(f')^{-1}$ is described by the same rule. Hence they agree and the associated morphism of topoi is the same. Some details omitted. □

04IV **Lemma 31.3.** *Let $f : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be a morphism of topoi. Let $\mathcal{G} \in Sh(\mathcal{D})$, $\mathcal{F} \in Sh(\mathcal{C})$ and $s : \mathcal{F} \to f^{-1}\mathcal{G}$ a morphism of sheaves. There exists a commutative diagram of topoi*

$$\begin{array}{ccc} Sh(\mathcal{C})/\mathcal{F} & \xrightarrow[j_{\mathcal{F}}]{} & Sh(\mathcal{C}) \\ {\scriptstyle f_s}\downarrow & & \downarrow{\scriptstyle f} \\ Sh(\mathcal{D})/\mathcal{G} & \xrightarrow{j_{\mathcal{G}}} & Sh(\mathcal{D}). \end{array}$$

We have $f_s = f' \circ j_{\mathcal{F}/f^{-1}\mathcal{G}}$ where $f' : Sh(\mathcal{C})/f^{-1}\mathcal{G} \to Sh(\mathcal{D})/\mathcal{F}$ is as in Lemma 31.1 and $j_{\mathcal{F}/f^{-1}\mathcal{G}} : Sh(\mathcal{C})/\mathcal{F} \to Sh(\mathcal{C})/f^{-1}\mathcal{G}$ is as in Lemma 30.6. The functor $(f_s)^{-1}$ is described by the rule

$$(f_s)^{-1}(\mathcal{H} \xrightarrow{\varphi} \mathcal{G}) = (f^{-1}\mathcal{H} \times_{f^{-1}\varphi, f^{-1}\mathcal{G}, s} \mathcal{F} \to \mathcal{F}).$$

Finally, given any morphisms $b : \mathcal{G}' \to \mathcal{G}$, $a : \mathcal{F}' \to \mathcal{F}$ and $s' : \mathcal{F}' \to f^{-1}\mathcal{G}'$ such that

$$\begin{array}{ccc} \mathcal{F}' & \xrightarrow[s']{} & f^{-1}\mathcal{G}' \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle f^{-1}b} \\ \mathcal{F} & \xrightarrow{s} & f^{-1}\mathcal{G} \end{array}$$

commutes, then the diagram

$$\begin{array}{ccc} Sh(\mathcal{C})/\mathcal{F}' & \xrightarrow{j_{\mathcal{F}'/\mathcal{F}}} & Sh(\mathcal{C})/\mathcal{F} \\ {\scriptstyle f_{s'}}\downarrow & & \downarrow{\scriptstyle f_s} \\ Sh(\mathcal{D})/\mathcal{G}' & \xrightarrow{j_{\mathcal{G}'/\mathcal{G}}} & Sh(\mathcal{D})/\mathcal{G}. \end{array}$$

commutes.

Proof. The commutativity of the first square follows from the commutativity of the diagram in Lemma 30.6 and the commutativity of the diagram in Lemma 31.1. The description of f_s^{-1} follows on combining the descriptions of $(f')^{-1}$ in Lemma 31.1 with the description of $(j_{\mathcal{F}/f^{-1}\mathcal{G}})^{-1}$ in Lemma 30.6. The commutativity of the last square then follows from the equality

$$f^{-1}\mathcal{H} \times_{f^{-1}\mathcal{G},s} \mathcal{F} \times_{\mathcal{F}} \mathcal{F}' = f^{-1}(\mathcal{H} \times_{\mathcal{G}} \mathcal{G}') \times_{f^{-1}\mathcal{G}',s'} \mathcal{F}'$$

which is formal. □

04IW **Lemma 31.4.** *Let $f : \mathcal{C} \to \mathcal{D}$ be a morphism of sites given by the continuous functor $u : \mathcal{D} \to \mathcal{C}$. Let V be an object of $\mathcal{D}$. Let $c : U \to u(V)$ be a morphism. Set $\mathcal{G} = h_V^\#$ and $\mathcal{F} = h_U^\# = f^{-1}h_V^\#$. Let $s : \mathcal{F} \to f^{-1}\mathcal{G}$ be the map induced by c. Then the diagram of morphisms of topoi of Lemma 28.3 agrees with the diagram of morphisms of topoi of Lemma 31.3 via the identifications $j_{\mathcal{F}} = j_U$ and $j_{\mathcal{G}} = j_V$ of Lemma 30.5.*

Proof. This follows on combining Lemmas 30.7 and 31.2. □

32. Points

00Y3

00Y4 **Definition 32.1.** Let $\mathcal{C}$ be a site. A *point of the topos* $Sh(\mathcal{C})$ is a morphism of topoi p from $Sh(pt)$ to $Sh(\mathcal{C})$.

We will define a point of a site in terms of a functor $u : \mathcal{C} \to Sets$. It will turn out later that u will define a morphism of sites which gives rise to a point of the topos associated to $\mathcal{C}$, see Lemma 32.8.

Let $\mathcal{C}$ be a site. Let $p = u$ be a functor $u : \mathcal{C} \to Sets$. This curious language is introduced because it seems funny to talk about neighbourhoods of functors; so we think of a "point" p as a geometric thing which is given by a categorical datum, namely the functor u. The fact that p is actually equal to u does not matter. A *neighbourhood* of p is a pair (U, x) with $U \in \mathrm{Ob}(\mathcal{C})$ and $x \in u(U)$. A *morphism of neighbourhoods* $(V, y) \to (U, x)$ is given by a morphism $\alpha : V \to U$ of $\mathcal{C}$ such that $u(\alpha)(y) = x$. Note that the category of neighbourhoods isn't a "big" category.

We define the *stalk* of a presheaf $\mathcal{F}$ at p as

04EH (32.1.1) $$\mathcal{F}_p = \mathrm{colim}_{\{(U,x)\}^{opp}} \mathcal{F}(U).$$

The colimit is over the opposite of the category of neighbourhoods of p. In other words, an element of $\mathcal{F}_p$ is given by a triple (U, x, s), where (U, x) is a neighbourhood of p and $s \in \mathcal{F}(U)$. Equality of triples is the equivalence relation generated by $(U, x, s) \sim (V, y, \alpha^* s)$ when α is as above.

Note that if $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves of sets, then we get a canonical map of stalks $\varphi_p : \mathcal{F}_p \to \mathcal{G}_p$. Thus we obtain a *stalk functor*

$$PSh(\mathcal{C}) \longrightarrow Sets, \quad \mathcal{F} \longmapsto \mathcal{F}_p.$$

We have defined the stalk functor using any functor $p = u : \mathcal{C} \to Sets$. No conditions are necessary for the definition to work[7]. On the other hand, it is probably better not to use this notion unless p actually is a point (see definition below), since in general the stalk functor does not have good properties.

00Y5 **Definition 32.2.** Let $\mathcal{C}$ be a site. A *point p of the site $\mathcal{C}$* is given by a functor $u : \mathcal{C} \to Sets$ such that

(1) For every covering $\{U_i \to U\}$ of $\mathcal{C}$ the map $\coprod u(U_i) \to u(U)$ is surjective.
(2) For every covering $\{U_i \to U\}$ of $\mathcal{C}$ and every morphism $V \to U$ the maps $u(U_i \times_U V) \to u(U_i) \times_{u(U)} u(V)$ are bijective.
(3) The stalk functor $Sh(\mathcal{C}) \to Sets$, $\mathcal{F} \mapsto \mathcal{F}_p$ is left exact.

The conditions should be familiar since they are modeled after those of Definitions 13.1 and 14.1. Note that (3) implies that $*_p = \{*\}$, see Example 10.2. Hence $u(U) \neq \emptyset$ for at least some U (because the empty colimit produces the empty set). We will show below (Lemma 32.7) that this does give rise to a point of the topos $Sh(\mathcal{C})$. Before we do so, we prove some lemmas for general functors u.

00Y6 **Lemma 32.3.** *Let $\mathcal{C}$ be a site. Let $p = u : \mathcal{C} \to Sets$ be a functor. There are functorial isomorphisms $(h_U)_p = u(U)$ for $U \in \mathrm{Ob}(\mathcal{C})$.*

Proof. An element of $(h_U)_p$ is given by a triple (V, y, f), where $V \in \mathrm{Ob}(\mathcal{C})$, $y \in u(V)$ and $f \in h_U(V) = \mathrm{Mor}_{\mathcal{C}}(V, U)$. Two such (V, y, f), (V', y', f') determine the same object if there exists a morphism $\phi : V \to V'$ such that $u(\phi)(y) = y'$ and $f' \circ \phi = f$, and in general you have to take chains of identities like this to get the correct equivalence relation. In any case, every (V, y, f) is equivalent to the element $(U, u(f)(y), \mathrm{id}_U)$. If ϕ exists as above, then the triples (V, y, f), (V', y', f') determine the same triple $(U, u(f)(y), \mathrm{id}_U) = (U, u(f')(y'), \mathrm{id}_U)$. This proves that the map $u(U) \to (h_U)_p$, $x \mapsto$ class of (U, x, id_U) is bijective. □

Let $\mathcal{C}$ be a site. Let $p = u : \mathcal{C} \to Sets$ be a functor. In analogy with the constructions in Section 5 given a set E we define a presheaf $u^p E$ by the rule

04EI (32.3.1) $$U \longmapsto u^p E(U) = \mathrm{Mor}_{Sets}(u(U), E) = \mathrm{Map}(u(U), E).$$

This defines a functor $u^p : Sets \to PSh(\mathcal{C})$, $E \mapsto u^p E$.

00Y7 **Lemma 32.4.** *For any functor $u : \mathcal{C} \to Sets$. The functor u^p is a right adjoint to the stalk functor on presheaves.*

Proof. Let $\mathcal{F}$ be a presheaf on $\mathcal{C}$. Let E be a set. A morphism $\mathcal{F} \to u^p E$ is given by a compatible system of maps $\mathcal{F}(U) \to \mathrm{Map}(u(U), E)$, i.e., a compatible system of maps $\mathcal{F}(U) \times u(U) \to E$. And by definition of $\mathcal{F}_p$ a map $\mathcal{F}_p \to E$ is given by a rule associating with each triple (U, x, σ) an element in E such that equivalent triples map to the same element, see discussion surrounding Equation (32.1.1). This also means a compatible system of maps $\mathcal{F}(U) \times u(U) \to E$. □

In analogy with Section 13 we have the following lemma.

[7] One should try to avoid the case where $u(U) = \emptyset$ for all U.

00Y8 **Lemma 32.5.** *Let $\mathcal{C}$ be a site. Let $p = u : \mathcal{C} \to Sets$ be a functor. Suppose that for every covering $\{U_i \to U\}$ of $\mathcal{C}$*

(1) *the map $\coprod u(U_i) \to u(U)$ is surjective, and*
(2) *the maps $u(U_i \times_U U_j) \to u(U_i) \times_{u(U)} u(U_j)$ are surjective.*

Then we have

(1) *the presheaf $u^p E$ is a sheaf for all sets E, denote it $u^s E$,*
(2) *the stalk functor $Sh(\mathcal{C}) \to Sets$ and the functor $u^s : Sets \to Sh(\mathcal{C})$ are adjoint, and*
(3) *we have $\mathcal{F}_p = \mathcal{F}_p^\#$ for every presheaf of sets $\mathcal{F}$.*

Proof. The first assertion is immediate from the definition of a sheaf, assumptions (1) and (2), and the definition of $u^p E$. The second is a restatement of the adjointness of u^p and the stalk functor (Lemma 32.4) restricted to sheaves. The third assertion follows as, for any set E, we have

$$\mathrm{Map}(\mathcal{F}_p, E) = \mathrm{Mor}_{PSh(\mathcal{C})}(\mathcal{F}, u^p E) = \mathrm{Mor}_{Sh(\mathcal{C})}(\mathcal{F}^\#, u^s E) = \mathrm{Map}(\mathcal{F}_p^\#, E)$$

by the adjointness property of sheafification. □

In particular Lemma 32.5 holds when $p = u$ is a point. In this case we think of the sheaf $u^s E$ as the "skyscraper" sheaf with value E at p.

00Y9 **Definition 32.6.** Let p be a point of the site $\mathcal{C}$ given by the functor u. For a set E we define $p_* E = u^s E$ the sheaf described in Lemma 32.5 above. We sometimes call this a *skyscraper sheaf*.

In particular we have the following adjointness property of skyscraper sheaves and stalks:

$$\mathrm{Mor}_{Sh(\mathcal{C})}(\mathcal{F}, p_* E) = \mathrm{Map}(\mathcal{F}_p, E)$$

This motivates the notation $p^{-1}\mathcal{F} = \mathcal{F}_p$ which we will sometimes use.

00YA **Lemma 32.7.** *Let $\mathcal{C}$ be a site.*

(1) *Let p be a point of the site $\mathcal{C}$. Then the pair of functors (p_*, p^{-1}) introduced above define a morphism of topoi $Sh(pt) \to Sh(\mathcal{C})$.*
(2) *Let $p = (p_*, p^{-1})$ be a point of the topos $Sh(\mathcal{C})$. Then the functor $u : U \mapsto p^{-1}(h_U^\#)$ gives rise to a point p' of the site $\mathcal{C}$ whose associated morphism of topoi $(p'_*, (p')^{-1})$ is equal to p.*

Proof. Proof of (1). By the above the functors p_* and p^{-1} are adjoint. The functor p^{-1} is required to be exact by Definition 32.2. Hence the conditions imposed in Definition 15.1 are all satisfied and we see that (1) holds.

Proof of (2). Let $\{U_i \to U\}$ be a covering of $\mathcal{C}$. Then $\coprod (h_{U_i})^\# \to h_U^\#$ is surjective, see Lemma 12.4. Since p^{-1} is exact (by definition of a morphism of topoi) we conclude that $\coprod u(U_i) \to u(U)$ is surjective. This proves part (1) of Definition 32.2. Sheafification is exact, see Lemma 10.14. Hence if $U \times_V W$ exists in $\mathcal{C}$, then

$$h^\#_{U \times_V W} = h^\#_U \times_{h^\#_V} h^\#_W$$

and we see that $u(U \times_V W) = u(U) \times_{u(V)} u(W)$ since p^{-1} is exact. This proves part (2) of Definition 32.2. Let $p' = u$, and let $\mathcal{F}_{p'}$ be the stalk functor defined by Equation (32.1.1) using u. There is a canonical comparison map $c : \mathcal{F}_{p'} \to \mathcal{F}_p = p^{-1}\mathcal{F}$. Namely, given a triple (U, x, σ) representing an element ξ of $\mathcal{F}_{p'}$

we think of σ as a map $\sigma : h_U^\# \to \mathcal{F}$ and we can set $c(\xi) = p^{-1}(\sigma)(x)$ since $x \in u(U) = p^{-1}(h_U^\#)$. By Lemma 32.3 we see that $(h_U)_{p'} = u(U)$. Since conditions (1) and (2) of Definition 32.2 hold for p' we also have $(h_U^\#)_{p'} = (h_U)_{p'}$ by Lemma 32.5. Hence we have

$$(h_U^\#)_{p'} = (h_U)_{p'} = u(U) = p^{-1}(h_U^\#)$$

We claim this bijection equals the comparison map $c : (h_U^\#)_{p'} \to p^{-1}(h_U^\#)$ (verification omitted). Any sheaf on $\mathcal{C}$ is a coequalizer of maps of coproducts of sheaves of the form $h_U^\#$, see Lemma 12.5. The stalk functor $\mathcal{F} \mapsto \mathcal{F}_{p'}$ and the functor p^{-1} commute with arbitrary colimits (as they are both left adjoints). We conclude c is an isomorphism for every sheaf $\mathcal{F}$. Thus the stalk functor $\mathcal{F} \mapsto \mathcal{F}_{p'}$ is isomorphic to p^{-1} and we in particular see that it is exact. This proves condition (3) of Definition 32.2 holds and p' is a point. The final assertion has already been shown above, since we saw that $p^{-1} = (p')^{-1}$. □

Actually a point always corresponds to a morphism of sites as we show in the following lemma.

04EL **Lemma 32.8.** *Let $\mathcal{C}$ be a site. Let p be a point of $\mathcal{C}$ given by $u : \mathcal{C} \to Sets$. Let S_0 be an infinite set such that $u(U) \subset S_0$ for all $U \in \mathrm{Ob}(\mathcal{C})$. Let $\mathcal{S}$ be the site constructed out of the powerset $S = \mathcal{P}(S_0)$ in Remark 15.3. Then*

(1) *there is an equivalence $i : Sh(pt) \to Sh(\mathcal{S})$,*
(2) *the functor $u : \mathcal{C} \to \mathcal{S}$ induces a morphism of sites $f : \mathcal{S} \to \mathcal{C}$, and*
(3) *the composition*

$$Sh(pt) \to Sh(\mathcal{S}) \to Sh(\mathcal{C})$$

is the morphism of topoi (p_*, p^{-1}) of Lemma 32.7.

Proof. Part (1) we saw in Remark 15.3. Moreover, recall that the equivalence associates to the set E the sheaf i_*E on $\mathcal{S}$ defined by the rule $V \mapsto \mathrm{Mor}_{Sets}(V, E)$. Part (2) is clear from the definition of a point of $\mathcal{C}$ (Definition 32.2) and the definition of a morphism of sites (Definition 14.1). Finally, consider f_*i_*E. By construction we have

$$f_*i_*E(U) = i_*E(u(U)) = \mathrm{Mor}_{Sets}(u(U), E)$$

which is equal to $p_*E(U)$, see Equation (32.3.1). This proves (3). □

Contrary to what happens in the topological case it is not always true that the stalk of the skyscraper sheaf with value E is E. Here is what is true in general.

05UX **Lemma 32.9.** *Let $\mathcal{C}$ be a site. Let $p : Sh(pt) \to Sh(\mathcal{C})$ be a point of the topos associated to $\mathcal{C}$. For any set E there are canonical maps*

$$E \longrightarrow (p_*E)_p \longrightarrow E$$

whose composition is id_E.

Proof. There is always an adjunction map $(p_*E)_p = p^{-1}p_*E \to E$. This map is an isomorphism when $E = \{*\}$ because p_* and p^{-1} are both left exact, hence transform the final object into the final object. Hence given $e \in E$ we can consider

the map $i_e : \{*\} \to E$ which gives

$$\begin{array}{ccc} p^{-1}p_*\{*\} & \xrightarrow{p^{-1}p_*i_e} & p^{-1}p_*E \\ \cong\downarrow & & \downarrow \\ \{*\} & \xrightarrow{i_e} & E \end{array}$$

whence the map $E \to (p_*E)_p = p^{-1}p_*E$ as desired. □

05UY **Lemma 32.10.** *Let $\mathcal{C}$ be a site. Let $p : Sh(pt) \to Sh(\mathcal{C})$ be a point of the topos associated to $\mathcal{C}$. The functor $p_* : Sets \to Sh(\mathcal{C})$ has the following properties: It commutes with arbitrary limits, it is left exact, it is faithful, it transforms surjections into surjections, it commutes with coequalizers, it reflects injections, it reflects surjections, and it reflects isomorphisms.*

Proof. Because p_* is a right adjoint it commutes with arbitrary limits and it is left exact. The fact that $p^{-1}p_*E \to E$ is a canonically split surjection implies that p_* is faithful, reflects injections, reflects surjections, and reflects isomorphisms. By Lemma 32.7 we may assume that p comes from a point $u : \mathcal{C} \to Sets$ of the underlying site $\mathcal{C}$. In this case the sheaf p_*E is given by

$$p_*E(U) = \mathrm{Mor}_{Sets}(u(U), E)$$

see Equation (32.3.1) and Definition 32.6. It follows immediately from this formula that p_* transforms surjections into surjections and coequalizers into coequalizers. □

33. Constructing points

05UZ In this section we give criteria for when a functor from a site to the category of sets defines a point of that site.

0F4E **Lemma 33.1.** *Let $\mathcal{C}$ be a site. Let $p = u : \mathcal{C} \to Sets$ be a functor. If the category of neighbourhoods of p is cofiltered, then the stalk functor (32.1.1) is left exact.*

Proof. Let $\mathcal{I} \to Sh(\mathcal{C})$, $i \mapsto \mathcal{F}_i$ be a finite diagram of sheaves. We have to show that the stalk of the limit of this system agrees with the limit of the stalks. Let $\mathcal{F}$ be the limit of the system as a *presheaf*. According to Lemma 10.1 this is a sheaf and it is the limit in the category of sheaves. Hence we have to show that $\mathcal{F}_p = \lim_{\mathcal{I}} \mathcal{F}_{i,p}$. Recall also that $\mathcal{F}$ has a simple description, see Section 4. Thus we have to show that

$$\lim_i \mathrm{colim}_{\{(U,x)\}^{opp}} \mathcal{F}_i(U) = \mathrm{colim}_{\{(U,x)\}^{opp}} \lim_i \mathcal{F}_i(U).$$

This holds, by Categories, Lemma 19.2, because the opposite of the category of neighbourhoods is filtered by assumption. □

00YB **Lemma 33.2.** *Let $\mathcal{C}$ be a site. Assume that $\mathcal{C}$ has a final object X and fibred products. Let $p = u : \mathcal{C} \to Sets$ be a functor such that*

1. *$u(X)$ is a singleton set, and*
2. *for every pair of morphisms $U \to W$ and $V \to W$ with the same target the map $u(U \times_W V) \to u(U) \times_{u(W)} u(V)$ is bijective.*

Then the the category of neighbourhoods of p is cofiltered and consequently the stalk functor $Sh(\mathcal{C}) \to Sets$, $\mathcal{F} \to \mathcal{F}_p$ commutes with finite limits.

Proof. Please note the analogy with Lemma 5.2. The assumptions on $\mathcal{C}$ imply that $\mathcal{C}$ has finite limits. See Categories, Lemma 18.4. Assumption (1) implies that the category of neighbourhoods is nonempty. Suppose (U, x) and (V, y) are neighbourhoods. Then $u(U \times V) = u(U \times_X V) = u(U) \times_{u(X)} u(V) = u(U) \times u(V)$ by (2). Hence there exists a neighbourhood $(U \times_X V, z)$ mapping to both (U, x) and (V, y). Let $a, b : (V, y) \to (U, x)$ be two morphisms in the category of neighbourhoods. Let W be the equalizer of $a, b : V \to U$. As in the proof of Categories, Lemma 18.4 we may write W in terms of fibre products:

$$W = (V \times_{a,U,b} V) \times_{(pr_1,pr_2),V\times V,\Delta} V$$

The bijectivity in (2) guarantees there exists an element $z \in u(W)$ which maps to $((y, y), y)$. Then $(W, z) \to (V, y)$ equalizes a, b as desired. The "consequently" clause is Lemma 33.1. □

00YC **Proposition 33.3.** *Let $\mathcal{C}$ be a site. Assume that finite limits exist in $\mathcal{C}$. (I.e., $\mathcal{C}$ has fibre products, and a final object.) A point p of such a site $\mathcal{C}$ is given by a functor $u : \mathcal{C} \to Sets$ such that*

(1) *u commutes with finite limits, and*
(2) *if $\{U_i \to U\}$ is a covering, then $\coprod_i u(U_i) \to u(U)$ is surjective.*

Proof. Suppose first that p is a point (Definition 32.2) given by a functor u. Condition (2) is satisfied directly from the definition of a point. By Lemma 32.3 we have $(h_U)_p = u(U)$. By Lemma 32.5 we have $(h_U^\#)_p = (h_U)_p$. Thus we see that u is equal to the composition of functors

$$\mathcal{C} \xrightarrow{h} PSh(\mathcal{C}) \xrightarrow{\#} Sh(\mathcal{C}) \xrightarrow{()_p} Sets$$

Each of these functors is left exact, and hence we see u satisfies (1).

Conversely, suppose that u satisfies (1) and (2). In this case we immediately see that u satisfies the first two conditions of Definition 32.2. And its stalk functor is exact, because it is a left adjoint by Lemma 32.5 and it commutes with finite limits by Lemma 33.2. □

00YD **Remark 33.4.** In fact, let $\mathcal{C}$ be a site. Assume $\mathcal{C}$ has a final object X and fibre products. Let $p = u : \mathcal{C} \to Sets$ be a functor such that

(1) $u(X) = \{*\}$ a singleton, and
(2) for every pair of morphisms $U \to W$ and $V \to W$ with the same target the map $u(U \times_W V) \to u(U) \times_{u(W)} u(V)$ is surjective.
(3) for every covering $\{U_i \to U\}$ the map $\coprod u(U_i) \to u(U)$ is surjective.

Then, in general, p is **not** a point of $\mathcal{C}$. An example is the category $\mathcal{C}$ with two objects $\{U, X\}$ and exactly one non-identity arrow, namely $U \to X$. We endow $\mathcal{C}$ with the trivial topology, i.e., the only coverings are $\{U \to U\}$ and $\{X \to X\}$. A sheaf $\mathcal{F}$ is the same thing as a presheaf and consists of a triple $(A, B, A \to B)$: namely $A = \mathcal{F}(X)$, $B = \mathcal{F}(U)$ and $A \to B$ is the restriction mapping corresponding to $U \to X$. Note that $U \times_X U = U$ so fibre products exist. Consider the functor $u = p$ with $u(X) = \{*\}$ and $u(U) = \{*_1, *_2\}$. This satisfies (1), (2), and (3), but the corresponding stalk functor (32.1.1) is the functor

$$(A, B, A \to B) \longmapsto B \amalg_A B$$

which isn't exact. Namely, consider $(\emptyset, \{1\}, \emptyset \to \{1\}) \to (\{1\}, \{1\}, \{1\} \to \{1\})$ which is an injective map of sheaves, but is transformed into the noninjective map of sets

$$\{1\} \amalg \{1\} \longrightarrow \{1\} \amalg_{\{1\}} \{1\}$$

by the stalk functor.

00YE **Example 33.5.** Let X be a topological space. Let X_{Zar} be the site of Example 6.4. Let $x \in X$ be a point. Consider the functor

$$u : X_{Zar} \longrightarrow Sets, \quad U \mapsto \begin{cases} \emptyset & \text{if} \quad x \notin U \\ \{*\} & \text{if} \quad x \in U \end{cases}$$

This functor commutes with product and fibred products, and turns coverings into surjective families of maps. Hence we obtain a point p of the site X_{Zar}. It is immediately verified that the stalk functor agrees with the stalk at x defined in Sheaves, Section 11.

04EJ **Example 33.6.** Let X be a topological space. What are the points of the topos $Sh(X)$? To see this, let X_{Zar} be the site of Example 6.4. By Lemma 32.7 a point of $Sh(X)$ corresponds to a point of this site. Let p be a point of the site X_{Zar} given by the functor $u : X_{Zar} \to Sets$. We are going to use the characterization of such a u in Proposition 33.3. This implies immediately that $u(\emptyset) = \emptyset$ and $u(U \cap V) = u(U) \times u(V)$. In particular we have $u(U) = u(U) \times u(U)$ via the diagonal map which implies that $u(U)$ is either a singleton or empty. Moreover, if $U = \bigcup U_i$ is an open covering then

$$u(U) = \emptyset \Rightarrow \forall i,\ u(U_i) = \emptyset \quad \text{and} \quad u(U) \neq \emptyset \Rightarrow \exists i,\ u(U_i) \neq \emptyset.$$

We conclude that there is a unique largest open $W \subset X$ with $u(W) = \emptyset$, namely the union of all the opens U with $u(U) = \emptyset$. Let $Z = X \setminus W$. If $Z = Z_1 \cup Z_2$ with $Z_i \subset Z$ closed, then $W = (X \setminus Z_1) \cap (X \setminus Z_2)$ so $\emptyset = u(W) = u(X \setminus Z_1) \times u(X \setminus Z_2)$ and we conclude that $u(X \setminus Z_1) = \emptyset$ or that $u(X \setminus Z_2) = \emptyset$. This means that $X \setminus Z_1 = W$ or that $X \setminus Z_2 = W$. In other words, Z is irreducible. Now we see that u is described by the rule

$$u : X_{Zar} \longrightarrow Sets, \quad U \mapsto \begin{cases} \emptyset & \text{if} \quad Z \cap U = \emptyset \\ \{*\} & \text{if} \quad Z \cap U \neq \emptyset \end{cases}$$

Note that for any irreducible closed $Z \subset X$ this functor satisfies assumptions (1), (2) of Proposition 33.3 and hence defines a point. In other words we see that points of the site X_{Zar} are in one-to-one correspondence with irreducible closed subsets of X. In particular, if X is a sober topological space, then points of X_{Zar} and points of X are in one to one correspondence, see Example 33.5.

00YF **Example 33.7.** Consider the site $\mathcal{T}_G$ described in Example 6.5 and Section 9. The forgetful functor $u : \mathcal{T}_G \to Sets$ commutes with products and fibred products and turns coverings into surjective families. Hence it defines a point of $\mathcal{T}_G$. We identify $Sh(\mathcal{T}_G)$ and G-$Sets$. The stalk functor

$$p^{-1} : Sh(\mathcal{T}_G) = G\text{-}Sets \longrightarrow Sets$$

is the forgetful functor. The pushforward p_* is the functor

$$Sets \longrightarrow Sh(\mathcal{T}_G) = G\text{-}Sets$$

which maps a set S to the G-set $\mathrm{Map}(G, S)$ with action $g \cdot \psi = \psi \circ R_g$ where R_g is right multiplication. In particular we have $p^{-1}p_*S = \mathrm{Map}(G, S)$ as a set and the maps $S \to \mathrm{Map}(G, S) \to S$ of Lemma 32.9 are the obvious ones.

08RH **Example 33.8.** Let $\mathcal{C}$ be a category endowed with the chaotic topology (Example 6.6). For every object U_0 of $\mathcal{C}$ the functor $u : U \mapsto \mathrm{Mor}_{\mathcal{C}}(U_0, U)$ defines a point p of $\mathcal{C}$. Namely, conditions (1) and (2) of Definition 32.2 are immediate as the only coverings are given by identity maps. Condition (2) holds because $\mathcal{F}_p = \mathcal{F}(U_0)$ and since the topology is discrete taking sections over U_0 is an exact functor.

34. Points and morphisms of topoi

05V0 In this section we make a few remarks about points and morphisms of topoi.

0F4F **Lemma 34.1.** *Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. Let $v : \mathcal{D} \to Sets$ be a functor and set $w = v \circ u$. Denote q, resp., p the stalk functor (32.1.1) associated to v, resp. w. Then $(u_p\mathcal{F})_q = \mathcal{F}_p$ functorially in the presheaf $\mathcal{F}$ on $\mathcal{C}$.*

Proof. This is a simple categorical fact. We have

$$\begin{aligned}(u_p\mathcal{F})_q &= \mathrm{colim}_{(V,y)} \mathrm{colim}_{U,\phi:V\to u(U)} \mathcal{F}(U) \\ &= \mathrm{colim}_{(V,y,U,\phi:V\to u(U))} \mathcal{F}(U) \\ &= \mathrm{colim}_{(U,x)} \mathcal{F}(U) \\ &= \mathcal{F}_p\end{aligned}$$

The first equality holds by the definition of u_p and the definition of the stalk functor. Observe that $y \in v(V)$. In the second equality we simply combine colimits. To see the third equality we apply Categories, Lemma 17.5 to the functor F of diagram categories defined by the rule

$$F((V, y, U, \phi : V \to u(U))) = (U, v(\phi)(y)).$$

This makes sense because $w(U) = v(u(U))$. Let us check the hypotheses of Categories, Lemma 17.5. Observe that F has a right inverse, namely $(U, x) \mapsto (u(U), x, U, \mathrm{id} : u(U) \to u(U))$. Again this makes sense because $x \in w(U) = v(u(U))$. On the other hand, there is always a morphism

$$(V, y, U, \phi : V \to u(U)) \longrightarrow (u(U), v(\phi)(y), U, \mathrm{id} : u(U) \to u(U))$$

in the fibre category over (U, x) which shows the fibre categories are connected. The fourth and final equality is clear. □

05V1 **Lemma 34.2.** *Let $f : \mathcal{D} \to \mathcal{C}$ be a morphism of sites given by a continuous functor $u : \mathcal{C} \to \mathcal{D}$. Let q be a point of $\mathcal{D}$ given by the functor $v : \mathcal{D} \to Sets$, see Definition 32.2. Then the functor $v \circ u : \mathcal{C} \to Sets$ defines a point p of $\mathcal{C}$ and moreover there is a canonical identification*

$$(f^{-1}\mathcal{F})_q = \mathcal{F}_p$$

for any sheaf $\mathcal{F}$ on $\mathcal{C}$.

First proof Lemma 34.2. Note that since u is continuous and since v defines a point, it is immediate that $v \circ u$ satisfies conditions (1) and (2) of Definition 32.2. Let us prove the displayed equality. Let $\mathcal{F}$ be a sheaf on $\mathcal{C}$. Then

$$(f^{-1}\mathcal{F})_q = (u_s\mathcal{F})_q = (u_p\mathcal{F})_q = \mathcal{F}_p$$

The first equality since $f^{-1} = u_s$, the second equality by Lemma 32.5, and the third by Lemma 34.1. Hence now we see that p also satisfies condition (3) of Definition 32.2 because it is a composition of exact functors. This finishes the proof. □

Second proof Lemma 34.2. By Lemma 32.8 we may factor (q_*, q^{-1}) as

$$Sh(pt) \xrightarrow{i} Sh(\mathcal{S}) \xrightarrow{h} Sh(\mathcal{D})$$

where the second morphism of topoi comes from a morphism of sites $h : \mathcal{S} \to \mathcal{D}$ induced by the functor $v : \mathcal{D} \to \mathcal{S}$ (which makes sense as $\mathcal{S} \subset Sets$ is a full subcategory containing every object in the image of v). By Lemma 14.4 the composition $v \circ u : \mathcal{C} \to \mathcal{S}$ defines a morphism of sites $g : \mathcal{S} \to \mathcal{C}$. In particular, the functor $v \circ u : \mathcal{C} \to \mathcal{S}$ is continuous which by the definition of the coverings in $\mathcal{S}$, see Remark 15.3, means that $v \circ u$ satisfies conditions (1) and (2) of Definition 32.2. On the other hand, we see that

$$g_* i_* E(U) = i_* E(v(u(U)) = \mathrm{Mor}_{Sets}(v(u(U)), E)$$

by the construction of i in Remark 15.3. Note that this is the same as the formula for which is equal to $(v \circ u)^p E$, see Equation (32.3.1). By Lemma 32.5 the functor $g_* i_* = (v \circ u)^p = (v \circ u)^s$ is right adjoint to the stalk functor $\mathcal{F} \mapsto \mathcal{F}_q$. Hence we see that the stalk functor p^{-1} is canonically isomorphic to $i^{-1} \circ g^{-1}$. Hence it is exact and we conclude that p is a point. Finally, as we have $g = f \circ h$ by construction we see that $p^{-1} = i^{-1} \circ h^{-1} \circ f^{-1} = q^{-1} \circ f^{-1}$, i.e., we have the displayed formula of the lemma. □

05V2 **Lemma 34.3.** *Let $f : Sh(\mathcal{D}) \to Sh(\mathcal{C})$ be a morphism of topoi. Let $q : Sh(pt) \to Sh(\mathcal{D})$ be a point. Then $p = f \circ q$ is a point of the topos $Sh(\mathcal{C})$ and we have a canonical identification*

$$(f^{-1}\mathcal{F})_q = \mathcal{F}_p$$

for any sheaf $\mathcal{F}$ on $\mathcal{C}$.

Proof. This is immediate from the definitions and the fact that we can compose morphisms of topoi. □

35. Localization and points

04EK In this section we show that points of a localization $\mathcal{C}/U$ are constructed in a simple manner from the points of $\mathcal{C}$.

04H2 **Lemma 35.1.** *Let $\mathcal{C}$ be a site. Let p be a point of $\mathcal{C}$ given by $u : \mathcal{C} \to Sets$. Let U be an object of $\mathcal{C}$ and let $x \in u(U)$. The functor*

$$v : \mathcal{C}/U \longrightarrow Sets, \quad (\varphi : V \to U) \longmapsto \{y \in u(V) \mid u(\varphi)(y) = x\}$$

defines a point q of the site $\mathcal{C}/U$ such that the diagram

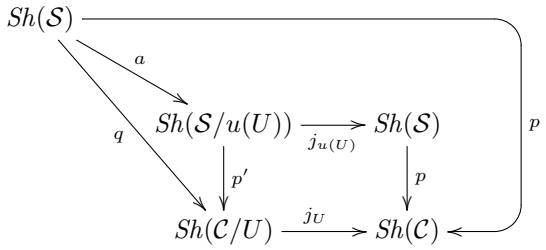
$$\begin{array}{ccc} & & Sh(pt) \\ & \overset{q}{\swarrow} & \downarrow p \\ Sh(\mathcal{C}/U) & \xrightarrow{j_U} & Sh(\mathcal{C}) \end{array}$$

commutes. In other words $\mathcal{F}_p = (j_U^{-1}\mathcal{F})_q$ for any sheaf on $\mathcal{C}$.

Proof. Choose S and $\mathcal{S}$ as in Lemma 32.8. We may identify $Sh(pt) = Sh(\mathcal{S})$ as in that lemma, and we may write $p = f : Sh(\mathcal{S}) \to Sh(\mathcal{C})$ for the morphism of topoi induced by u. By Lemma 28.1 we get a commutative diagram of topoi

$$\begin{array}{ccc} Sh(\mathcal{S}/u(U)) & \xrightarrow[j_{u(U)}]{} & Sh(\mathcal{S}) \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ Sh(\mathcal{C}/U) & \xrightarrow{j_U} & Sh(\mathcal{C}), \end{array}$$

where p' is given by the functor $u' : \mathcal{C}/U \to \mathcal{S}/u(U)$, $V/U \mapsto u(V)/u(U)$. Consider the functor $j_x : \mathcal{S} \cong \mathcal{S}/x$ obtained by assigning to a set E the set E endowed with the constant map $E \to u(U)$ with value x. Then j_x is a fully faithful cocontinuous functor which has a continuous right adjoint $v_x : (\psi : E \to u(U)) \mapsto \psi^{-1}(\{x\})$. Note that $j_{u(U)} \circ j_x = \mathrm{id}_{\mathcal{S}}$, and $v_x \circ u' = v$. These observations imply that we have the following commutative diagram of topoi



Namely:

(1) The morphism $a : Sh(\mathcal{S}) \to Sh(\mathcal{S}/u(U))$ is the morphism of topoi associated to the cocontinuous functor j_x, which equals the morphism associated to the continuous functor v_x, see Lemma 21.1 and Section 22.
(2) The composition $p \circ j_{u(U)} \circ a = p$ since $j_{u(U)} \circ j_x = \mathrm{id}_{\mathcal{S}}$.
(3) The composition $p' \circ a$ gives a morphism of topoi. Moreover, it is the morphism of topoi associated to the continuous functor $v_x \circ u' = v$. Hence v does indeed define a point q of $\mathcal{C}/U$ which fits into the diagram above by construction.

This ends the proof of the lemma. □

04H3 **Lemma 35.2.** *Let $\mathcal{C}$, p, u, U be as in Lemma 35.1. The construction of Lemma 35.1 gives a one to one correspondence between points q of $\mathcal{C}/U$ lying over p and elements x of $u(U)$.*

Proof. Let q be a point of $\mathcal{C}/U$ given by the functor $v : \mathcal{C}/U \to Sets$ such that $j_U \circ q = p$ as morphisms of topoi. Recall that $u(V) = p^{-1}(h_V^\#)$ for any object V of $\mathcal{C}$, see Lemma 32.7. Similarly $v(V/U) = q^{-1}(h_{V/U}^\#)$ for any object V/U of $\mathcal{C}/U$. Consider the following two diagrams

$$\begin{array}{ccc} \mathrm{Mor}_{\mathcal{C}/U}(W/U, V/U) & \longrightarrow & \mathrm{Mor}_{\mathcal{C}}(W, V) \\ \downarrow & & \downarrow \\ \mathrm{Mor}_{\mathcal{C}/U}(W/U, U/U) & \longrightarrow & \mathrm{Mor}_{\mathcal{C}}(W, U) \end{array} \qquad \begin{array}{ccc} h_{V/U}^\# & \longrightarrow & j_U^{-1}(h_V^\#) \\ \downarrow & & \downarrow \\ h_{U/U}^\# & \longrightarrow & j_U^{-1}(h_U^\#) \end{array}$$

The right hand diagram is the sheafification of the diagram of presheaves on $\mathcal{C}/U$ which maps W/U to the left hand diagram of sets. (There is a small technical point to make here, namely, that we have $(j_U^{-1}h_V)^\# = j_U^{-1}(h_V^\#)$ and similarly for h_U, see Lemma 20.4.) Note that the left hand diagram of sets is cartesian. Since sheafification is exact (Lemma 10.14) we conclude that the right hand diagram is cartesian.

Apply the exact functor q^{-1} to the right hand diagram to get a cartesian diagram

$$\begin{array}{ccc} v(V/U) & \longrightarrow & u(V) \\ \downarrow & & \downarrow \\ v(U/U) & \longrightarrow & u(U) \end{array}$$

of sets. Here we have used that $q^{-1} \circ j^{-1} = p^{-1}$. Since U/U is a final object of $\mathcal{C}/U$ we see that $v(U/U)$ is a singleton. Hence the image of $v(U/U)$ in $u(U)$ is an element x, and the top horizontal map gives a bijection $v(V/U) \to \{y \in u(V) \mid y \mapsto x \text{ in } u(U)\}$ as desired. □

04H4 **Lemma 35.3.** *Let $\mathcal{C}$ be a site. Let p be a point of $\mathcal{C}$ given by $u : \mathcal{C} \to Sets$. Let U be an object of $\mathcal{C}$. For any sheaf $\mathcal{G}$ on $\mathcal{C}/U$ we have*

$$(j_{U!}\mathcal{G})_p = \coprod_q \mathcal{G}_q$$

where the coproduct is over the points q of $\mathcal{C}/U$ associated to elements $x \in u(U)$ as in Lemma 35.1.

Proof. We use the description of $j_{U!}\mathcal{G}$ as the sheaf associated to the presheaf $V \mapsto \coprod_{\varphi \in \mathrm{Mor}_\mathcal{C}(V,U)} \mathcal{G}(V/_\varphi U)$ of Lemma 25.2. Also, the stalk of $j_{U!}\mathcal{G}$ at p is equal to the stalk of this presheaf, see Lemma 32.5. Hence we see that

$$(j_{U!}\mathcal{G})_p = \mathrm{colim}_{(V,y)} \coprod_{\varphi : V \to U} \mathcal{G}(V/_\varphi U)$$

To each element (V, y, φ, s) of this colimit, we can assign $x = u(\varphi)(y) \in u(U)$. Hence we obtain

$$(j_{U!}\mathcal{G})_p = \coprod_{x \in u(U)} \mathrm{colim}_{(\varphi : V \to U, y),\ u(\varphi)(y) = x} \mathcal{G}(V/_\varphi U).$$

This is equal to the expression of the lemma by our construction of the points q. □

04H5 **Remark 35.4.** Warning: The result of Lemma 35.3 has no analogue for $j_{U,*}$.

36. 2-morphisms of topoi

04I9 This is a brief section concerning the notion of a 2-morphism of topoi.

04IA **Definition 36.1.** Let $f, g : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be two morphisms of topoi. A *2-morphism from f to g* is given by a transformation of functors $t : f_* \to g_*$.

Pictorially we sometimes represent t as follows:

$$Sh(\mathcal{C}) \underset{g}{\overset{f}{\Downarrow t}} Sh(\mathcal{D})$$

Note that since f^{-1} is adjoint to f_* and g^{-1} is adjoint to g_* we see that t induces also a transformation of functors $t : g^{-1} \to f^{-1}$ (usually denoted by the same symbol) uniquely characterized by the condition that the diagram

$$\begin{array}{ccc} \mathrm{Mor}_{Sh(\mathcal{D})}(\mathcal{G}, f_*\mathcal{F}) & = & \mathrm{Mor}_{Sh(\mathcal{C})}(f^{-1}\mathcal{G}, \mathcal{F}) \\ {\scriptstyle t\circ -}\downarrow & & \downarrow{\scriptstyle -\circ t} \\ \mathrm{Mor}_{Sh(\mathcal{D})}(\mathcal{G}, g_*\mathcal{F}) & = & \mathrm{Mor}_{Sh(\mathcal{C})}(g^{-1}\mathcal{G}, \mathcal{F}) \end{array}$$

commutes. Because of set theoretic difficulties (see Remark 15.4) we do not obtain a 2-category of topoi. But we can still define horizontal and vertical composition and show that the axioms of a strict 2-category listed in Categories, Section 29 hold. Namely, vertical composition of 2-morphisms is clear (just compose transformations of functors), composition of 1-morphisms has been defined in Definition 15.1, and horizontal composition of

$$Sh(\mathcal{C}) \underset{g}{\overset{f}{\Downarrow t}} Sh(\mathcal{D}) \underset{g'}{\overset{f'}{\Downarrow s}} Sh(\mathcal{E})$$

is defined by the transformation of functors $s \star t$ introduced in Categories, Definition 28.1. Explicitly, $s \star t$ is given by

$$f'_*f_*\mathcal{F} \xrightarrow{f'_*t} f'_*g_*\mathcal{F} \xrightarrow{s} g'_*g_*\mathcal{F} \quad \text{or} \quad f'_*f_*\mathcal{F} \xrightarrow{s} g'_*f_*\mathcal{F} \xrightarrow{g'_*t} g'_*g_*\mathcal{F}$$

(these maps are equal). Since these definitions agree with the ones in Categories, Section 28 it follows from Categories, Lemma 28.2 that the axioms of a strict 2-category hold with these definitions.

37. Morphisms between points

00YG

00YH **Lemma 37.1.** *Let $\mathcal{C}$ be a site. Let $u, u' : \mathcal{C} \to Sets$ be two functors, and let $t : u' \to u$ be a transformation of functors. Then we obtain a canonical transformation of stalk functors $t_{stalk} : \mathcal{F}_{p'} \to \mathcal{F}_p$ which agrees with t via the identifications of Lemma 32.3.*

Proof. Omitted. □

00YI **Definition 37.2.** Let $\mathcal{C}$ be a site. Let p, p' be points of $\mathcal{C}$ given by functors $u, u' : \mathcal{C} \to Sets$. A *morphism* $f : p \to p'$ is given by a transformation of functors

$$f_u : u' \to u.$$

Note how the transformation of functors goes the other way. This makes sense, as we will see later, by thinking of the morphism f as a kind of 2-arrow pictorially as follows:

$$Sets = Sh(pt) \underset{p'}{\overset{p}{\Downarrow f}} Sh(\mathcal{C})$$

Namely, we will see later that f_u induces a canonical transformation of functors $p_* \to p'_*$ between the skyscraper sheaf constructions.

This is a fairly important notion, and deserves a more complete treatment here. List of desiderata

(1) Describe the automorphisms of the point of $\mathcal{T}_G$ described in Example 33.7.
(2) Describe $\mathrm{Mor}(p, p')$ in terms of $\mathrm{Mor}(p_*, p'_*)$.
(3) Specialization of points in topological spaces. Show that if $x' \in \overline{\{x\}}$ in the topological space X, then there is a morphism $p \to p'$, where p (resp. p') is the point of X_{Zar} associated to x (resp. x').

38. Sites with enough points

00YJ

00YK **Definition 38.1.** Let $\mathcal{C}$ be a site.

(1) A family of points $\{p_i\}_{i\in I}$ is called *conservative* if every map of sheaves $\phi : \mathcal{F} \to \mathcal{G}$ which is an isomorphism on all the fibres $\mathcal{F}_{p_i} \to \mathcal{G}_{p_i}$ is an isomorphism.
(2) We say that $\mathcal{C}$ *has enough points* if there exists a conservative family of points.

It turns out that you can then check "exactness" at the stalks.

00YL **Lemma 38.2.** *Let $\mathcal{C}$ be a site and let $\{p_i\}_{i\in I}$ be a conservative family of points. Then*

(1) *Given any map of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$ we have $\forall i, \varphi_{p_i}$ injective implies φ injective.*
(2) *Given any map of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$ we have $\forall i, \varphi_{p_i}$ surjective implies φ surjective.*
(3) *Given any pair of maps of sheaves $\varphi_1, \varphi_2 : \mathcal{F} \to \mathcal{G}$ we have $\forall i, \varphi_{1,p_i} = \varphi_{2,p_i}$ implies $\varphi_1 = \varphi_2$.*
(4) *Given a finite diagram $\mathcal{G} : \mathcal{J} \to Sh(\mathcal{C})$, a sheaf $\mathcal{F}$ and morphisms $q_j : \mathcal{F} \to \mathcal{G}_j$ then $(\mathcal{F}, q_j)$ is a limit of the diagram if and only if for each i the stalk $(\mathcal{F}_{p_i}, (q_j)_{p_i})$ is one.*
(5) *Given a finite diagram $\mathcal{F} : \mathcal{J} \to Sh(\mathcal{C})$, a sheaf $\mathcal{G}$ and morphisms $e_j : \mathcal{F}_j \to \mathcal{G}$ then $(\mathcal{G}, e_j)$ is a colimit of the diagram if and only if for each i the stalk $(\mathcal{G}_{p_i}, (e_j)_{p_i})$ is one.*

Proof. We will use over and over again that all the stalk functors commute with any finite limits and colimits and hence with products, fibred products, etc. We will also use that injective maps are the monomorphisms and the surjective maps are the epimorphisms. A map of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$ is injective if and only if $\mathcal{F} \to \mathcal{F} \times_{\mathcal{G}} \mathcal{F}$ is an isomorphism. Hence (1). Similarly, $\varphi : \mathcal{F} \to \mathcal{G}$ is surjective if and only if $\mathcal{G} \amalg_{\mathcal{F}} \mathcal{G} \to \mathcal{G}$ is an isomorphism. Hence (2). The maps $a, b : \mathcal{F} \to \mathcal{G}$ are equal if and only if $\mathcal{F} \times_{a,\mathcal{G},b} \mathcal{F} \to \mathcal{F} \times \mathcal{F}$ is an isomorphism. Hence (3). The assertions (4) and (5) follow immediately from the definitions and the remarks at the start of this proof. □

00YM **Lemma 38.3.** *Let $\mathcal{C}$ be a site and let $\{(p_i, u_i)\}_{i\in I}$ be a family of points. The family is conservative if and only if for every sheaf $\mathcal{F}$ and every $U \in \mathrm{Ob}(\mathcal{C})$ and every pair of distinct sections $s, s' \in \mathcal{F}(U)$, $s \neq s'$ there exists an i and $x \in u_i(U)$ such that the triples (U, x, s) and (U, x, s') define distinct elements of $\mathcal{F}_{p_i}$.*

Proof. Suppose that the family is conservative and that $\mathcal{F}$, U, and s, s' are as in the lemma. The sections s, s' define maps $a, a' : (h_U)^\# \to \mathcal{F}$ which are distinct. Hence, by Lemma 38.2 there is an i such that $a_{p_i} \neq a'_{p_i}$. Recall that $(h_U)^\#_{p_i} = u_i(U)$, by Lemmas 32.3 and 32.5. Hence there exists an $x \in u_i(U)$ such that $a_{p_i}(x) \neq a'_{p_i}(x)$ in $\mathcal{F}_{p_i}$. Unwinding the definitions you see that (U, x, s) and (U, x, s') are as in the statement of the lemma.

To prove the converse, assume the condition on the existence of points of the lemma. Let $\phi : \mathcal{F} \to \mathcal{G}$ be a map of sheaves which is an isomorphism at all the stalks. We have to show that ϕ is both injective and surjective, see Lemma 11.2. Injectivity is an immediate consequence of the assumption. To show surjectivity we have to show that $\mathcal{G} \amalg_{\mathcal{F}} \mathcal{G} \to \mathcal{G}$ is an isomorphism (Categories, Lemma 13.3). Since this map is clearly surjective, it suffices to check injectivity which follows as $\mathcal{G} \amalg_{\mathcal{F}} \mathcal{G} \to \mathcal{G}$ is injective on all stalks by assumption. □

In the following lemma the points $q_{i,x}$ are exactly all the points of $\mathcal{C}/U$ lying over the point p_i according to Lemma 35.2.

04H6 **Lemma 38.4.** *Let $\mathcal{C}$ be a site. Let U be an object of $\mathcal{C}$. let $\{(p_i, u_i)\}_{i \in I}$ be a family of points of $\mathcal{C}$. For $x \in u_i(U)$ let $q_{i,x}$ be the point of $\mathcal{C}/U$ constructed in Lemma 35.1. If $\{p_i\}$ is a conservative family of points, then $\{q_{i,x}\}_{i \in I, x \in u_i(U)}$ is a conservative family of points of $\mathcal{C}/U$. In particular, if $\mathcal{C}$ has enough points, then so does every localization $\mathcal{C}/U$.*

Proof. We know that $j_{U!}$ induces an equivalence $j_{U!} : Sh(\mathcal{C}/U) \to Sh(\mathcal{C})/h^\#_U$, see Lemma 25.4. Moreover, we know that $(j_{U!}\mathcal{G})_{p_i} = \coprod_x \mathcal{G}_{q_{i,x}}$, see Lemma 35.3. Hence the result follows formally. □

The following lemma tells us we can check the existence of points locally on the site.

06UL **Lemma 38.5.** *Let $\mathcal{C}$ be a site. Let $\{U_i\}_{i \in I}$ be a family of objects of $\mathcal{C}$. Assume*

1. *$\coprod h^\#_{U_i} \to *$ is a surjective map of sheaves, and*
2. *each localization $\mathcal{C}/U_i$ has enough points.*

Then $\mathcal{C}$ has enough points.

Proof. For each $i \in I$ let $\{p_j\}_{j \in J_i}$ be a conservative family of points of $\mathcal{C}/U_i$. For $j \in J_i$ denote $q_j : Sh(pt) \to Sh(\mathcal{C})$ the composition of p_j with the localization morphism $Sh(\mathcal{C}/U_i) \to Sh(\mathcal{C})$. Then q_j is a point, see Lemma 34.3. We claim that the family of points $\{q_j\}_{j \in \coprod J_i}$ is conservative. Namely, let $\mathcal{F} \to \mathcal{G}$ be a map of sheaves on $\mathcal{C}$ such that $\mathcal{F}_{q_j} \to \mathcal{G}_{q_j}$ is an isomorphism for all $j \in \coprod J_i$. Let W be an object of $\mathcal{C}$. By assumption (1) there exists a covering $\{W_a \to W\}$ and morphisms $W_a \to U_{i(a)}$. Since $(\mathcal{F}|_{\mathcal{C}/U_{i(a)}})_{p_j} = \mathcal{F}_{q_j}$ and $(\mathcal{G}|_{\mathcal{C}/U_{i(a)}})_{p_j} = \mathcal{G}_{q_j}$ by Lemma 34.3 we see that $\mathcal{F}|_{U_{i(a)}} \to \mathcal{G}|_{U_{i(a)}}$ is an isomorphism since the family of points $\{p_j\}_{j \in J_{i(a)}}$ is conservative. Hence $\mathcal{F}(W_a) \to \mathcal{G}(W_a)$ is bijective for each a. Similarly $\mathcal{F}(W_a \times_W W_b) \to \mathcal{G}(W_a \times_W W_b)$ is bijective for each a, b. By the sheaf condition this shows that $\mathcal{F}(W) \to \mathcal{G}(W)$ is bijective, i.e., $\mathcal{F} \to \mathcal{G}$ is an isomorphism. □

0F4G **Lemma 38.6.** *Let $u : \mathcal{C} \to \mathcal{D}$ be a continuous functor of sites. Let $\{(q_i, v_i)\}_{i \in I}$ be a conservative family of points of $\mathcal{D}$. If each functor $u_i = v_i \circ u$ defines a point of $\mathcal{C}$, then u defines a morphism of sites $f : \mathcal{D} \to \mathcal{C}$.*

Proof. Denote p_i the stalk functor (32.1.1) on $PSh(\mathcal{C})$ corresponding to the functor u_i. We have

$$(f^{-1}\mathcal{F})_{q_i} = (u_s\mathcal{F})_{q_i} = (u_p\mathcal{F})_{q_i} = \mathcal{F}_{p_i}$$

The first equality since $f^{-1} = u_s$, the second equality by Lemma 32.5, and the third by Lemma 34.1. Hence if p_i is a point, then pulling back by f and then taking stalks at q_i is an exact functor. Since the family of points $\{q_i\}$ is conservative, this implies that f^{-1} is an exact functor and we see that f is a morphism of sites by Definition 14.1. □

39. Criterion for existence of points

00YN This section corresponds to Deligne's appendix to [AGV71, Exposé VI]. In fact it is almost literally the same.

Let $\mathcal{C}$ be a site. Suppose that $(I, \geq)$ is a directed set, and that $(U_i, f_{ii'})$ is an inverse system over I, see Categories, Definition 21.2. Given the data $(I, \geq, U_i, f_{ii'})$ we define

$$u : \mathcal{C} \longrightarrow Sets, \quad u(V) = \operatorname{colim}_i \operatorname{Mor}_{\mathcal{C}}(U_i, V)$$

Let $\mathcal{F} \mapsto \mathcal{F}_p$ be the stalk functor associated to u as in Section 32. It is direct from the definition that actually

$$\mathcal{F}_p = \operatorname{colim}_i \mathcal{F}(U_i)$$

in this special case. Note that u commutes with all finite limits (I mean those that are representable in $\mathcal{C}$) because each of the functors $V \mapsto \operatorname{Mor}_{\mathcal{C}}(U_i, V)$ do, see Categories, Lemma 19.2.

We say that a system $(I, \geq, U_i, f_{ii'})$ is a *refinement* of $(J, \geq, V_j, g_{jj'})$ if $J \subset I$, the ordering on J induced from that of I and $V_j = U_j$, $g_{jj'} = f_{jj'}$ (in words, the inverse system over J is induced by that over I). Let u be the functor associated to $(I, \geq, U_i, f_{ii'})$ and let u' be the functor associated to $(J, \geq, V_j, g_{jj'})$. This induces a transformation of functors

$$u' \longrightarrow u$$

simply because the colimits for u' are over a subsystem of the systems in the colimits for u. In particular we get an associated transformation of stalk functors $\mathcal{F}_{p'} \to \mathcal{F}_p$, see Lemma 37.1.

00YO **Lemma 39.1.** *Let $\mathcal{C}$ be a site. Let $(J, \geq, V_j, g_{jj'})$ be a system as above with associated pair of functors (u', p'). Let $\mathcal{F}$ be a sheaf on $\mathcal{C}$. Let $s, s' \in \mathcal{F}_{p'}$ be distinct elements. Let $\{W_k \to W\}$ be a finite covering of $\mathcal{C}$. Let $f \in u'(W)$. There exists a refinement $(I, \geq, U_i, f_{ii'})$ of $(J, \geq, V_j, g_{jj'})$ such that s, s' map to distinct elements of $\mathcal{F}_p$ and that the image of f in $u(W)$ is in the image of one of the $u(W_k)$.*

Proof. There exists a $j_0 \in J$ such that f is defined by $f' : V_{j_0} \to W$. For $j \geq j_0$ we set $V_{j,k} = V_j \times_{f' \circ f_{jj_0}, W} W_k$. Then $\{V_{j,k} \to V_j\}$ is a finite covering in the site $\mathcal{C}$. Hence $\mathcal{F}(V_j) \subset \prod_k \mathcal{F}(V_{j,k})$. By Categories, Lemma 19.2 once again we see that

$$\mathcal{F}_{p'} = \operatorname{colim}_j \mathcal{F}(V_j) \longrightarrow \prod\nolimits_k \operatorname{colim}_j \mathcal{F}(V_{j,k})$$

is injective. Hence there exists a k such that s and s' have distinct image in $\operatorname{colim}_j \mathcal{F}(V_{j,k})$. Let $J_0 = \{j \in J, j \geq j_0\}$ and $I = J \amalg J_0$. We order I so that no element of the second summand is smaller than any element of the first, but otherwise using the ordering on J. If $j \in I$ is in the first summand then we use V_j and if $j \in I$ is in the second summand then we use $V_{j,k}$. We omit the definition of

the transition maps of the inverse system. By the above it follows that s, s' have distinct image in $\mathcal{F}_p$. Moreover, the restriction of f' to $V_{j,k}$ factors through W_k by construction. □

00YP **Lemma 39.2.** *Let $\mathcal{C}$ be a site. Let $(J, \geq, V_j, g_{jj'})$ be a system as above with associated pair of functors (u', p'). Let $\mathcal{F}$ be a sheaf on $\mathcal{C}$. Let $s, s' \in \mathcal{F}_{p'}$ be distinct elements. There exists a refinement $(I, \geq, U_i, f_{ii'})$ of $(J, \geq, V_j, g_{jj'})$ such that s, s' map to distinct elements of $\mathcal{F}_p$ and such that for every finite covering $\{W_k \to W\}$ of the site $\mathcal{C}$, and any $f \in u'(W)$ the image of f in $u(W)$ is in the image of one of the $u(W_k)$.*

Proof. Let E be the set of pairs $(\{W_k \to W\}, f \in u'(W))$. Consider pairs $(E' \subset E, (I, \geq, U_i, f_{ii'}))$ such that

(1) $(I, \geq, U_i, g_{ii'})$ is a refinement of $(J, \geq, V_j, g_{jj'})$,
(2) s, s' map to distinct elements of $\mathcal{F}_p$, and
(3) for every pair $(\{W_k \to W\}, f \in u'(W)) \in E'$ we have that the image of f in $u(W)$ is in the image of one of the $u(W_k)$.

We order such pairs by inclusion in the first factor and by refinement in the second. Denote $\mathcal{S}$ the class of all pairs $(E' \subset E, (I, \geq, U_i, f_{ii'}))$ as above. We claim that the hypothesis of Zorn's lemma holds for $\mathcal{S}$. Namely, suppose that $(E'_a, (I_a, \geq, U_i, f_{ii'}))_{a \in A}$ is a totally ordered subset of $\mathcal{S}$. Then we can define $E' = \bigcup_{a \in A} E'_a$ and we can set $I = \bigcup_{a \in A} I_a$. We claim that the corresponding pair $(E', (I, \geq, U_i, f_{ii'}))$ is an element of $\mathcal{S}$. Conditions (1) and (3) are clear. For condition (2) you note that

$$u = \operatorname{colim}_{a \in A} u_a \text{ and correspondingly } \mathcal{F}_p = \operatorname{colim}_{a \in A} \mathcal{F}_{p_a}$$

The distinctness of the images of s, s' in this stalk follows from the description of a directed colimit of sets, see Categories, Section 19. We will simply write $(E', (I, \ldots)) = \bigcup_{a \in A} (E'_a, (I_a, \ldots))$ in this situation.

OK, so Zorn's Lemma would apply if $\mathcal{S}$ was a set, and this would, combined with Lemma 39.1 above easily prove the lemma. It doesn't since $\mathcal{S}$ is a class. In order to circumvent this we choose a well ordering on E. For $e \in E$ set $E'_e = \{e' \in E \mid e' \leq e\}$. Using transfinite recursion we construct pairs $(E'_e, (I_e, \ldots)) \in \mathcal{S}$ such that $e_1 \leq e_2 \Rightarrow (E'_{e_1}, (I_{e_1}, \ldots)) \leq (E'_{e_2}, (I_{e_2}, \ldots))$. Let $e \in E$, say $e = (\{W_k \to W\}, f \in u'(W))$. If e has a predecessor $e-1$, then we let $(I_e, \ldots)$ be a refinement of $(I_{e-1}, \ldots)$ as in Lemma 39.1 with respect to the system $e = (\{W_k \to W\}, f \in u'(W))$. If e does not have a predecessor, then we let $(I_e, \ldots)$ be a refinement of $\bigcup_{e' < e} (I_{e'}, \ldots)$ with respect to the system $e = (\{W_k \to W\}, f \in u'(W))$. Finally, the union $\bigcup_{e \in E} I_e$ will be a solution to the problem posed in the lemma. □

00YQ **Proposition 39.3.** *Let $\mathcal{C}$ be a site. Assume that*

(1) *finite limits exist in $\mathcal{C}$, and*
(2) *every covering $\{U_i \to U\}_{i \in I}$ has a refinement by a finite covering of $\mathcal{C}$.*

Then $\mathcal{C}$ has enough points.

[AGV71, Exposé VI, Appendix by Deligne, Proposition 9.0]

Proof. We have to show that given any sheaf $\mathcal{F}$ on $\mathcal{C}$, any $U \in \mathrm{Ob}(\mathcal{C})$, and any distinct sections $s, s' \in \mathcal{F}(U)$, there exists a point p such that s, s' have distinct image in $\mathcal{F}_p$. See Lemma 38.3. Consider the system $(J, \geq, V_j, g_{jj'})$ with $J = \{1\}$, $V_1 = U$, $g_{11} = \mathrm{id}_U$. Apply Lemma 39.2. By the result of that lemma we get a system $(I, \geq, U_i, f_{ii'})$ refining our system such that $s_p \neq s'_p$ and such that moreover

for every finite covering $\{W_k \to W\}$ of the site $\mathcal{C}$ the map $\coprod_k u(W_k) \to u(W)$ is surjective. Since every covering of $\mathcal{C}$ can be refined by a finite covering we conclude that $\coprod_k u(W_k) \to u(W)$ is surjective for *any* covering $\{W_k \to W\}$ of the site $\mathcal{C}$. This implies that $u = p$ is a point, see Proposition 33.3 (and the discussion at the beginning of this section which guarantees that u commutes with finite limits). □

0DW0 **Lemma 39.4.** *Let $\mathcal{C}$ be a site. Let I be a set and for $i \in I$ let U_i be an object of $\mathcal{C}$ such that*

(1) *$\coprod h_{U_i}$ surjects onto the final object of $Sh(\mathcal{C})$, and*
(2) *$\mathcal{C}/U_i$ satisfies the hypotheses of Proposition 39.3.*

Then $\mathcal{C}$ has enough points.

Proof. By assumption (2) and the proposition $\mathcal{C}/U_i$ has enough points. The points of $\mathcal{C}/U_i$ give points of $\mathcal{C}$ via the procedure of Lemma 34.2. Thus it suffices to show: if $\phi : \mathcal{F} \to \mathcal{G}$ is a map of sheaves on $\mathcal{C}$ such that $\phi|_{\mathcal{C}/U_i}$ is an isomorphism for all i, then ϕ is an isomorphism. By assumption (1) for every object W of $\mathcal{C}$ there is a covering $\{W_j \to W\}_{j \in J}$ such that for $j \in J$ there is an $i \in I$ and a morphism $f_j : W_j \to U_i$. Then the maps $\mathcal{F}(W_j) \to \mathcal{G}(W_j)$ are bijective and similarly for $\mathcal{F}(W_j \times_W W_{j'}) \to \mathcal{G}(W_j \times_W W_{j'})$. The sheaf condition tells us that $\mathcal{F}(W) \to \mathcal{G}(W)$ is bijective as desired. □

40. Weakly contractible objects

090J A *weakly contractible* object of a site is one that satisfies the equivalent conditions of the following lemma.

090K **Lemma 40.1.** *Let $\mathcal{C}$ be a site. Let U be an object of $\mathcal{C}$. The following conditions are equivalent*

(1) *For every covering $\{U_i \to U\}$ there exists a map of sheaves $h_U^\# \to \coprod h_{U_i}^\#$ right inverse to the sheafification of $\coprod h_{U_i} \to h_U$.*
(2) *For every surjection of sheaves of sets $\mathcal{F} \to \mathcal{G}$ the map $\mathcal{F}(U) \to \mathcal{G}(U)$ is surjective.*

Proof. Assume (1) and let $\mathcal{F} \to \mathcal{G}$ be a surjective map of sheaves of sets. For $s \in \mathcal{G}(U)$ there exists a covering $\{U_i \to U\}$ and $t_i \in \mathcal{F}(U_i)$ mapping to $s|_{U_i}$, see Definition 11.1. Think of t_i as a map $t_i : h_{U_i}^\# \to \mathcal{F}$ via (12.3.1). Then precomposing $\coprod t_i : \coprod h_{U_i}^\# \to \mathcal{F}$ with the map $h_U^\# \to \coprod h_{U_i}^\#$ we get from (1) we obtain a section $t \in \mathcal{F}(U)$ mapping to s. Thus (2) holds.

Assume (2) holds. Let $\{U_i \to U\}$ be a covering. Then $\coprod h_{U_i}^\# \to h_U^\#$ is surjective (Lemma 12.4). Hence by (2) there exists a section s of $\coprod h_{U_i}^\#$ mapping to the section id_U of $h_U^\#$. This section corresponds to a map $h_U^\# \to \coprod h_{U_i}^\#$ which is right inverse to the sheafification of $\coprod h_{U_i} \to h_U$ which proves (1). □

090L **Definition 40.2.** Let $\mathcal{C}$ be a site.

(1) We say an object U of $\mathcal{C}$ is *weakly contractible* if the equivalent conditions of Lemma 40.1 hold.
(2) We say a site has *enough weakly contractible objects* if every object U of $\mathcal{C}$ has a covering $\{U_i \to U\}$ with U_i weakly contractible for all i.

(3) More generally, if P is a property of objects of $\mathcal{C}$ we say that $\mathcal{C}$ has *enough P objects* if every object U of $\mathcal{C}$ has a covering $\{U_i \to U\}$ such that U_i has P for all i.

The small étale site of $\mathbf{A}^1_{\mathbf{C}}$ does not have any weakly contractible objects. On the other hand, the small pro-étale site of any scheme has enough contractible objects.

41. Exactness properties of pushforward

04D5 Let f be a morphism of topoi. The functor f_* in general is only left exact. There are many additional conditions one can impose on this functor to single out particular classes of morphisms of topoi. We collect them here and note some of the logical dependencies. Some parts of the following lemma are purely category theoretical (i.e., they do not depend on having a morphism of topoi, just having a pair of adjoint functors is enough).

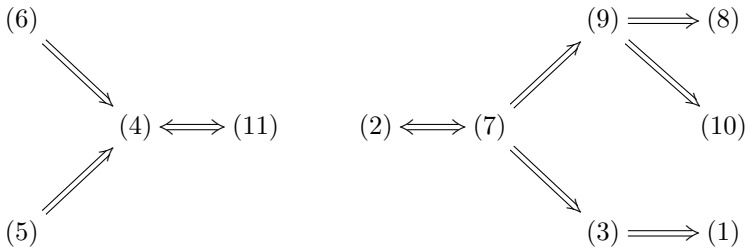
04D6 **Lemma 41.1.** *Let $f : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be a morphism of topoi. Consider the following properties (on sheaves of sets):*

(1) *f_* is faithful,*
(2) *f_* is fully faithful,*
(3) *$f^{-1}f_*\mathcal{F} \to \mathcal{F}$ is surjective for all $\mathcal{F}$ in $Sh(\mathcal{C})$,*
(4) *f_* transforms surjections into surjections,*
(5) *f_* commutes with coequalizers,*
(6) *f_* commutes with pushouts,*
(7) *$f^{-1}f_*\mathcal{F} \to \mathcal{F}$ is an isomorphism for all $\mathcal{F}$ in $Sh(\mathcal{C})$,*
(8) *f_* reflects injections,*
(9) *f_* reflects surjections,*
(10) *f_* reflects bijections, and*
(11) *for any surjection $\mathcal{F} \to f^{-1}\mathcal{G}$ there exists a surjection $\mathcal{G}' \to \mathcal{G}$ such that $f^{-1}\mathcal{G}' \to f^{-1}\mathcal{G}$ factors through $\mathcal{F} \to f^{-1}\mathcal{G}$.*

Then we have the following implications

(a) *(2) $\Rightarrow$ (1),*
(b) *(3) $\Rightarrow$ (1),*
(c) *(7) $\Rightarrow$ (1), (2), (3), (8), (9), (10).*
(d) *(3) $\Leftrightarrow$ (9),*
(e) *(6) $\Rightarrow$ (4) and (5) $\Rightarrow$ (4),*
(f) *(4) $\Leftrightarrow$ (11),*
(g) *(9) $\Rightarrow$ (8), (10), and*
(h) *(2) $\Leftrightarrow$ (7).*

Picture



Proof. Proof of (a): This is immediate from the definitions.

Proof of (b). Suppose that $a, b : \mathcal{F} \to \mathcal{F}'$ are maps of sheaves on $\mathcal{C}$. If $f_*a = f_*b$, then $f^{-1}f_*a = f^{-1}f_*b$. Consider the commutative diagram

$$\begin{array}{ccc} \mathcal{F} & \rightrightarrows & \mathcal{F}' \\ \uparrow & & \uparrow \\ f^{-1}f_*\mathcal{F} & \rightrightarrows & f^{-1}f_*\mathcal{F}' \end{array}$$

If the bottom two arrows are equal and the vertical arrows are surjective then the top two arrows are equal. Hence (b) follows.

Proof of (c). Suppose that $a : \mathcal{F} \to \mathcal{F}'$ is a map of sheaves on $\mathcal{C}$. Consider the commutative diagram

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & \mathcal{F}' \\ \uparrow & & \uparrow \\ f^{-1}f_*\mathcal{F} & \longrightarrow & f^{-1}f_*\mathcal{F}' \end{array}$$

If (7) holds, then the vertical arrows are isomorphisms. Hence if f_*a is injective (resp. surjective, resp. bijective) then the bottom arrow is injective (resp. surjective, resp. bijective) and hence the top arrow is injective (resp. surjective, resp. bijective). Thus we see that (7) implies (8), (9), (10). It is clear that (7) implies (3). The implications (7) $\Rightarrow$ (2), (1) follow from (a) and (h) which we will see below.

Proof of (d). Assume (3). Suppose that $a : \mathcal{F} \to \mathcal{F}'$ is a map of sheaves on $\mathcal{C}$ such that f_*a is surjective. As f^{-1} is exact this implies that $f^{-1}f_*a : f^{-1}f_*\mathcal{F} \to f^{-1}f_*\mathcal{F}'$ is surjective. Combined with (3) this implies that a is surjective. This means that (9) holds. Assume (9). Let $\mathcal{F}$ be a sheaf on $\mathcal{C}$. We have to show that the map $f^{-1}f_*\mathcal{F} \to \mathcal{F}$ is surjective. It suffices to show that $f_*f^{-1}f_*\mathcal{F} \to f_*\mathcal{F}$ is surjective. And this is true because there is a canonical map $f_*\mathcal{F} \to f_*f^{-1}f_*\mathcal{F}$ which is a one-sided inverse.

Proof of (e). We use Categories, Lemma 13.3 without further mention. If $\mathcal{F} \to \mathcal{F}'$ is surjective then $\mathcal{F}' \amalg_{\mathcal{F}} \mathcal{F}' \to \mathcal{F}'$ is an isomorphism. Hence (6) implies that

$$f_*\mathcal{F}' \amalg_{f_*\mathcal{F}} f_*\mathcal{F}' = f_*(\mathcal{F}' \amalg_{\mathcal{F}} \mathcal{F}') \longrightarrow f_*\mathcal{F}'$$

is an isomorphism also. And this in turn implies that $f_*\mathcal{F} \to f_*\mathcal{F}'$ is surjective. Hence we see that (6) implies (4). If $\mathcal{F} \to \mathcal{F}'$ is surjective then $\mathcal{F}'$ is the coequalizer of the two projections $\mathcal{F} \times_{\mathcal{F}'} \mathcal{F} \to \mathcal{F}$ by Lemma 11.3. Hence if (5) holds, then $f_*\mathcal{F}'$ is the coequalizer of the two projections

$$f_*(\mathcal{F} \times_{\mathcal{F}'} \mathcal{F}) = f_*\mathcal{F} \times_{f_*\mathcal{F}'} f_*\mathcal{F} \longrightarrow f_*\mathcal{F}$$

which clearly means that $f_*\mathcal{F} \to f_*\mathcal{F}'$ is surjective. Hence (5) implies (4) as well.

Proof of (f). Assume (4). Let $\mathcal{F} \to f^{-1}\mathcal{G}$ be a surjective map of sheaves on $\mathcal{C}$. By (4) we see that $f_*\mathcal{F} \to f_*f^{-1}\mathcal{G}$ is surjective. Let $\mathcal{G}'$ be the fibre product

$$\begin{array}{ccc} f_*\mathcal{F} & \longrightarrow & f_*f^{-1}\mathcal{G} \\ \uparrow & & \uparrow \\ \mathcal{G}' & \longrightarrow & \mathcal{G} \end{array}$$

so that $\mathcal{G}' \to \mathcal{G}$ is surjective also. Consider the commutative diagram

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & f^{-1}\mathcal{G} \\ \uparrow & & \uparrow \\ f^{-1}f_*\mathcal{F} & \longrightarrow & f^{-1}f_*f^{-1}\mathcal{G} \\ \uparrow & & \uparrow \\ f^{-1}\mathcal{G}' & \longrightarrow & f^{-1}\mathcal{G} \end{array}$$

and we see the required result. Conversely, assume (11). Let $a : \mathcal{F} \to \mathcal{F}'$ be surjective map of sheaves on $\mathcal{C}$. Consider the fibre product diagram

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & \mathcal{F}' \\ \uparrow & & \uparrow \\ \mathcal{F}'' & \longrightarrow & f^{-1}f_*\mathcal{F}' \end{array}$$

Because the lower horizontal arrow is surjective and by (11) we can find a surjection $\gamma : \mathcal{G}' \to f_*\mathcal{F}'$ such that $f^{-1}\gamma$ factors through $\mathcal{F}'' \to f^{-1}f_*\mathcal{F}'$:

$$\begin{array}{ccccc} & & \mathcal{F} & \longrightarrow & \mathcal{F}' \\ & & \uparrow & & \uparrow \\ f^{-1}\mathcal{G}' & \longrightarrow & \mathcal{F}'' & \longrightarrow & f^{-1}f_*\mathcal{F}' \end{array}$$

Pushing this down using f_* we get a commutative diagram

$$\begin{array}{ccccc} & & f_*\mathcal{F} & \longrightarrow & f_*\mathcal{F}' \\ & & \uparrow & & \uparrow \\ f_*f^{-1}\mathcal{G}' & \longrightarrow & f_*\mathcal{F}'' & \longrightarrow & f_*f^{-1}f_*\mathcal{F}' \\ \uparrow & & & & \uparrow \\ \mathcal{G}' & & \longrightarrow & & f_*\mathcal{F}' \end{array}$$

which proves that (4) holds.

Proof of (g). Assume (9). We use Categories, Lemma 13.3 without further mention. Let $a : \mathcal{F} \to \mathcal{F}'$ be a map of sheaves on $\mathcal{C}$ such that f_*a is injective. This means that $f_*\mathcal{F} \to f_*\mathcal{F} \times_{f_*\mathcal{F}'} f_*\mathcal{F} = f_*(\mathcal{F} \times_{\mathcal{F}'} \mathcal{F})$ is an isomorphism. Thus by (9) we see that $\mathcal{F} \to \mathcal{F} \times_{\mathcal{F}'} \mathcal{F}$ is surjective, i.e., an isomorphism. Thus a is injective, i.e., (8) holds. Since (10) is trivially equivalent to (8) + (9) we are done with (g).

Proof of (h). This is Categories, Lemma 24.4. □

Here is a condition on a morphism of sites which guarantees that the functor f_* transforms surjective maps into surjective maps.

04D7 **Lemma 41.2.** *Let $f : \mathcal{D} \to \mathcal{C}$ be a morphism of sites associated to the continuous functor $u : \mathcal{C} \to \mathcal{D}$. Assume that for any object U of $\mathcal{C}$ and any covering $\{V_j \to u(U)\}$ in $\mathcal{D}$ there exists a covering $\{U_i \to U\}$ in $\mathcal{C}$ such that the map of sheaves*

$$\coprod h^\#_{u(U_i)} \to h^\#_{u(U)}$$

factors through the map of sheaves

$$\coprod h^\#_{V_j} \to h^\#_{u(U)}.$$

Then f_* transforms surjective maps of sheaves into surjective maps of sheaves.

Proof. Let $a : \mathcal{F} \to \mathcal{G}$ be a surjective map of sheaves on $\mathcal{D}$. Let U be an object of $\mathcal{C}$ and let $s \in f_*\mathcal{G}(U) = \mathcal{G}(u(U))$. By assumption there exists a covering $\{V_j \to u(U)\}$ and sections $s_j \in \mathcal{F}(V_j)$ with $a(s_j) = s|_{V_j}$. Now we may think of the sections s, s_j and a as giving a commutative diagram of maps of sheaves

$$\begin{array}{ccc} \coprod h^\#_{V_j} & \xrightarrow{\coprod s_j} & \mathcal{F} \\ \downarrow & & \downarrow a \\ h^\#_{u(U)} & \xrightarrow{s} & \mathcal{G} \end{array}$$

By assumption there exists a covering $\{U_i \to U\}$ such that we can enlarge the commutative diagram above as follows

$$\begin{array}{ccccc} & & \coprod h^\#_{V_j} & \xrightarrow{\coprod s_j} & \mathcal{F} \\ & \nearrow & \downarrow & & \downarrow a \\ \coprod h^\#_{u(U_i)} & \longrightarrow & h^\#_{u(U)} & \xrightarrow{s} & \mathcal{G} \end{array}$$

Because $\mathcal{F}$ is a sheaf the map from the left lower corner to the right upper corner corresponds to a family of sections $s_i \in \mathcal{F}(u(U_i))$, i.e., sections $s_i \in f_*\mathcal{F}(U_i)$. The commutativity of the diagram implies that $a(s_i)$ is equal to the restriction of s to U_i. In other words we have shown that f_*a is a surjective map of sheaves. □

04D8 **Example 41.3.** Assume $f : \mathcal{D} \to \mathcal{C}$ satisfies the assumptions of Lemma 41.2. Then it is in general not the case that f_* commutes with coequalizers or pushouts. Namely, suppose that f is the morphism of sites associated to the morphism of topological spaces $X = \{1, 2\} \to Y = \{*\}$ (see Example 14.2), where Y is a singleton space, and $X = \{1, 2\}$ is a discrete space with two points. A sheaf $\mathcal{F}$ on X is given by a pair (A_1, A_2) of sets. Then $f_*\mathcal{F}$ corresponds to the set $A_1 \times A_2$. Hence if $a = (a_1, a_2), b = (b_1, b_2) : (A_1, A_2) \to (B_1, B_2)$ are maps of sheaves on X, then the coequalizer of a, b is (C_1, C_2) where C_i is the coequalizer of a_i, b_i, and the coequalizer of f_*a, f_*b is the coequalizer of

$$a_1 \times a_2, b_1 \times b_2 : A_1 \times A_2 \longrightarrow B_1 \times B_2$$

which is in general different from $C_1 \times C_2$. Namely, if $A_2 = \emptyset$ then $A_1 \times A_2 = \emptyset$, and hence the coequalizer of the displayed arrows is $B_1 \times B_2$, but in general $C_1 \neq B_1$. A similar example works for pushouts.

The following lemma gives a criterion for when a morphism of sites has a functor f_* which reflects injections and surjections. Note that this also implies that f_* is faithful and that the map $f^{-1}f_*\mathcal{F} \to \mathcal{F}$ is always surjective.

04D9 **Lemma 41.4.** *Let $f : \mathcal{D} \to \mathcal{C}$ be a morphism of sites given by the functor $u : \mathcal{C} \to \mathcal{D}$. Assume that for every object V of $\mathcal{D}$ there exist objects U_i of $\mathcal{C}$ and morphisms $u(U_i) \to V$ such that $\{u(U_i) \to V\}$ is a covering of $\mathcal{D}$. In this case the functor $f_* : Sh(\mathcal{D}) \to Sh(\mathcal{C})$ reflects injections and surjections.*

Proof. Let $\alpha : \mathcal{F} \to \mathcal{G}$ be maps of sheaves on $\mathcal{D}$. By assumption for every object V of $\mathcal{D}$ we get $\mathcal{F}(V) \subset \prod \mathcal{F}(u(U_i)) = \prod f_*\mathcal{F}(U_i)$ by the sheaf condition for some $U_i \in \mathrm{Ob}(\mathcal{C})$ and similarly for $\mathcal{G}$. Hence it is clear that if $f_*\alpha$ is injective, then α is injective. In other words f_* reflects injections.

Suppose that $f_*\alpha$ is surjective. Then for $V, U_i, u(U_i) \to V$ as above and a section $s \in \mathcal{G}(V)$, there exist coverings $\{U_{ij} \to U_i\}$ such that $s|_{u(U_{ij})}$ is in the image of $\mathcal{F}(u(U_{ij}))$. Since $\{u(U_{ij}) \to V\}$ is a covering (as u is continuous and by the axioms of a site) we conclude that s is locally in the image. Thus α is surjective. In other words f_* reflects surjections. □

08LS **Example 41.5.** We construct a morphism $f : \mathcal{D} \to \mathcal{C}$ satisfying the assumptions of Lemma 41.4. Namely, let $\varphi : G \to H$ be a morphism of finite groups. Consider the sites $\mathcal{D} = \mathcal{T}_G$ and $\mathcal{C} = \mathcal{T}_H$ of countable G-sets and H-sets and coverings countable families of jointly surjective maps (Example 6.5). Let $u : \mathcal{T}_H \to \mathcal{T}_G$ be the functor described in Section 16 and $f : \mathcal{T}_G \to \mathcal{T}_H$ the corresponding morphism of sites. If φ is injective, then every countable G-set is, as a G-set, the quotient of a countable H-set (this fails if φ isn't injective). Thus f satisfies the hypothesis of Lemma 41.4. If the sheaf $\mathcal{F}$ on $\mathcal{T}_G$ corresponds to the G-set S, then the canonical map

$$f^{-1}f_*\mathcal{F} \longrightarrow \mathcal{F}$$

corresponds to the map

$$\mathrm{Map}_G(H, S) \longrightarrow S, \quad a \longmapsto a(1_H)$$

If φ is injective but not surjective, then this map is surjective (as it should according to Lemma 41.4) but not injective in general (for example take $G = \{1\}$, $H = \{1, \sigma\}$, and $S = \{1, 2\}$). Moreover, the functor f_* does not commute with coequalizers or pushouts (for $G = \{1\}$ and $H = \{1, \sigma\}$).

42. Almost cocontinuous functors

04B4 Let $\mathcal{C}$ be a site. The category $PSh(\mathcal{C})$ has an initial object, namely the presheaf which assigns the empty set to each object of $\mathcal{C}$. Let us denote this presheaf by $\emptyset$. It follows from the properties of sheafification that the sheafification $\emptyset^\#$ of $\emptyset$ is an initial object of the category $Sh(\mathcal{C})$ of sheaves on $\mathcal{C}$.

04B5 **Definition 42.1.** Let $\mathcal{C}$ be a site. We say an object U of $\mathcal{C}$ is *sheaf theoretically empty* if $\emptyset^\# \to h_U^\#$ is an isomorphism of sheaves.

The following lemma makes this notion more explicit.

04B6 **Lemma 42.2.** *Let $\mathcal{C}$ be a site. Let U be an object of $\mathcal{C}$. The following are equivalent:*

1. *U is sheaf theoretically empty,*
2. *$\mathcal{F}(U)$ is a singleton for each sheaf $\mathcal{F}$,*
3. *$\emptyset^\#(U)$ is a singleton,*
4. *$\emptyset^\#(U)$ is nonempty, and*
5. *the empty family is a covering of U in $\mathcal{C}$.*

Moreover, if U is sheaf theoretically empty, then for any morphism $U' \to U$ of $\mathcal{C}$ the object U' is sheaf theoretically empty.

Proof. For any sheaf $\mathcal{F}$ we have $\mathcal{F}(U) = \mathrm{Mor}_{Sh(\mathcal{C})}(h_U^\#, \mathcal{F})$. Hence, we see that (1) and (2) are equivalent. It is clear that (2) implies (3) implies (4). If every covering of U is given by a nonempty family, then $\emptyset^+(U)$ is empty by definition of the plus construction. Note that $\emptyset^+ = \emptyset^\#$ as $\emptyset$ is a separated presheaf, see Theorem 10.10. Thus we see that (4) implies (5). If (5) holds, then $\mathcal{F}(U)$ is a singleton for every sheaf $\mathcal{F}$ by the sheaf condition for $\mathcal{F}$, see Remark 7.2. Thus (5) implies (2) and (1) – (5) are equivalent. The final assertion of the lemma follows from Axiom (3) of Definition 6.2 applied the empty covering of U. □

04B7 **Definition 42.3.** Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. We say u is *almost cocontinuous* if for every object U of $\mathcal{C}$ and every covering $\{V_j \to u(U)\}_{j\in J}$ there exists a covering $\{U_i \to U\}_{i\in I}$ in $\mathcal{C}$ such that for each i in I we have at least one of the following two conditions

(1) $u(U_i)$ is sheaf theoretically empty, or
(2) the morphism $u(U_i) \to u(U)$ factors through V_j for some $j \in J$.

The motivation for this definition comes from a closed immersion $i : Z \to X$ of topological spaces. As discussed in Example 21.9 the continuous functor $X_{Zar} \to Z_{Zar}$, $U \mapsto Z \cap U$ is not cocontinuous. But it is almost cocontinuous in the sense defined above. We know that i_* while not exact on sheaves of sets, is exact on sheaves of abelian groups, see Sheaves, Remark 32.5. And this holds in general for continuous and almost cocontinuous functors.

04B8 **Lemma 42.4.** *Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. Assume that u is continuous and almost cocontinuous. Let $\mathcal{G}$ be a presheaf on $\mathcal{D}$ such that $\mathcal{G}(V)$ is a singleton whenever V is sheaf theoretically empty. Then $(u^p\mathcal{G})^\# = u^p(\mathcal{G}^\#)$.*

Proof. Let $U \in \mathrm{Ob}(\mathcal{C})$. We have to show that $(u^p\mathcal{G})^\#(U) = u^p(\mathcal{G}^\#)(U)$. It suffices to show that $(u^p\mathcal{G})^+(U) = u^p(\mathcal{G}^+)(U)$ since $\mathcal{G}^+$ is another presheaf for which the assumption of the lemma holds. We have

$$u^p(\mathcal{G}^+)(U) = \mathcal{G}^+(u(U)) = \mathrm{colim}_{\mathcal{V}}\, \check{H}^0(\mathcal{V}, \mathcal{G})$$

where the colimit is over the coverings $\mathcal{V}$ of $u(U)$ in $\mathcal{D}$. On the other hand, we see that

$$u^p(\mathcal{G})^+(U) = \mathrm{colim}_{\mathcal{U}}\, \check{H}^0(u(\mathcal{U}), \mathcal{G})$$

where the colimit is over the category of coverings $\mathcal{U} = \{U_i \to U\}_{i\in I}$ of U in $\mathcal{C}$ and $u(\mathcal{U}) = \{u(U_i) \to u(U)\}_{i\in I}$. The condition that u is continuous means that each $u(\mathcal{U})$ is a covering. Write $I = I_1 \amalg I_2$, where

$$I_2 = \{i \in I \mid u(U_i) \text{ is sheaf theoretically empty}\}$$

Then $u(\mathcal{U})' = \{u(U_i) \to u(U)\}_{i\in I_1}$ is still a covering of because each of the other pieces can be covered by the empty family and hence can be dropped by Axiom (2) of Definition 6.2. Moreover, $\check{H}^0(u(\mathcal{U}), \mathcal{G}) = \check{H}^0(u(\mathcal{U})', \mathcal{G})$ by our assumption on $\mathcal{G}$. Finally, the condition that u is almost cocontinuous implies that for every covering $\mathcal{V}$ of $u(U)$ there exists a covering $\mathcal{U}$ of U such that $u(\mathcal{U})'$ refines $\mathcal{V}$. It follows that the two colimits displayed above have the same value as desired. □

04B9 **Lemma 42.5.** *Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. Assume that u is continuous and almost cocontinuous. Then $u^s = u^p : Sh(\mathcal{D}) \to Sh(\mathcal{C})$ commutes with pushouts and coequalizers (and more generally finite connected colimits).*

Proof. Let $\mathcal{I}$ be a finite connected index category. Let $\mathcal{I} \to Sh(\mathcal{D})$, $i \mapsto \mathcal{G}_i$ by a diagram. We know that the colimit of this diagram is the sheafification of the colimit in the category of presheaves, see Lemma 10.13. Denote colim^{Psh} the colimit in the category of presheaves. Since $\mathcal{I}$ is finite and connected we see that $\text{colim}_i^{Psh} \mathcal{G}_i$ is a presheaf satisfying the assumptions of Lemma 42.4 (because a finite connected colimit of singleton sets is a singleton). Hence that lemma gives

$$\begin{aligned} u^s(\text{colim}_i \mathcal{G}_i) &= u^s((\text{colim}_i^{Psh} \mathcal{G}_i)^\#) \\ &= (u^p(\text{colim}_i^{Psh} \mathcal{G}_i))^\# \\ &= (\text{colim}_i^{PSh} u^p(\mathcal{G}_i))^\# \\ &= \text{colim}_i u^s(\mathcal{G}_i) \end{aligned}$$

as desired. □

04BA **Lemma 42.6.** *Let $f : \mathcal{D} \to \mathcal{C}$ be a morphism of sites associated to the continuous functor $u : \mathcal{C} \to \mathcal{D}$. If u is almost cocontinuous then f_* commutes with pushouts and coequalizers (and more generally finite connected colimits).*

Proof. This is a special case of Lemma 42.5. □

43. Subtopoi

08LT Here is the definition.

08LU **Definition 43.1.** Let $\mathcal{C}$ and $\mathcal{D}$ be sites. A morphism of topoi $f : Sh(\mathcal{D}) \to Sh(\mathcal{C})$ is called an *embedding* if f_* is fully faithful.

According to Lemma 41.1 this is equivalent to asking the adjunction map $f^{-1}f_*\mathcal{F} \to \mathcal{F}$ to be an isomorphism for every sheaf $\mathcal{F}$ on $\mathcal{D}$.

08LV **Definition 43.2.** Let $\mathcal{C}$ be a site. A strictly full subcategory $E \subset Sh(\mathcal{C})$ is a *subtopos* if there exists an embedding of topoi $f : Sh(\mathcal{D}) \to Sh(\mathcal{C})$ such that E is equal to the essential image of the functor f_*.

The subtopoi constructed in the following lemma will be dubbed "open" in the definition later on.

08LW **Lemma 43.3.** *Let $\mathcal{C}$ be a site. Let $\mathcal{F}$ be a sheaf on $\mathcal{C}$. The following are equivalent*

1. *$\mathcal{F}$ is a subobject of the final object of $Sh(\mathcal{C})$, and*
2. *the topos $Sh(\mathcal{C})/\mathcal{F}$ is a subtopos of $Sh(\mathcal{C})$.*

Proof. We have seen in Lemma 30.1 that $Sh(\mathcal{C})/\mathcal{F}$ is a topos. In fact, we recall the proof. First we apply Lemma 29.5 to see that we may assume $\mathcal{C}$ is a site with a subcanonical topology, fibre products, a final object X, and an object U with $\mathcal{F} = h_U$. The proof of Lemma 30.1 shows that the morphism of topoi $j_\mathcal{F} : Sh(\mathcal{C})/\mathcal{F} \to Sh(\mathcal{C})$ is equal (modulo certain identifications) to the localization morphism $j_U : Sh(\mathcal{C}/U) \to Sh(\mathcal{C})$.

Assume (2). This means that $j_U^{-1}j_{U,*}\mathcal{G} \to \mathcal{G}$ is an isomorphism for all sheaves $\mathcal{G}$ on $\mathcal{C}/U$. For any object Z/U of $\mathcal{C}/U$ we have

$$(j_{U,*}h_{Z/U})(U) = \text{Mor}_{\mathcal{C}/U}(U \times_X U/U, Z/U)$$

by Lemma 27.2. Setting $\mathcal{G} = h_{Z/U}$ in the equality above we obtain

$$\text{Mor}_{\mathcal{C}/U}(U \times_X U/U, Z/U) = \text{Mor}_{\mathcal{C}/U}(U, Z/U)$$

for all Z/U. By Yoneda's lemma (Categories, Lemma 3.5) this implies $U \times_X U = U$. By Categories, Lemma 13.3 $U \to X$ is a monomorphism, in other words (1) holds.

Assume (1). Then $j_U^{-1} j_{U,*} = \text{id}$ by Lemma 27.4. □

08LX **Definition 43.4.** Let $\mathcal{C}$ be a site. A strictly full subcategory $E \subset Sh(\mathcal{C})$ is an *open subtopos* if there exists a subsheaf $\mathcal{F}$ of the final object of $Sh(\mathcal{C})$ such that E is the subtopos $Sh(\mathcal{C})/\mathcal{F}$ described in Lemma 43.3.

This means there is a bijection between the collection of open subtopoi of $Sh(\mathcal{C})$ and the set of subobjects of the final object of $Sh(\mathcal{C})$. Given an open subtopos there is a "closed" complement.

08LY **Lemma 43.5.** *Let $\mathcal{C}$ be a site. Let $\mathcal{F}$ be a subsheaf of the final object $*$ of $Sh(\mathcal{C})$. The full subcategory of sheaves $\mathcal{G}$ such that $\mathcal{F} \times \mathcal{G} \to \mathcal{F}$ is an isomorphism is a subtopos of $Sh(\mathcal{C})$.*

Proof. We apply Lemma 29.5 to see that we may assume $\mathcal{C}$ is a site with the properties listed in that lemma. In particular $\mathcal{C}$ has a final object X (so that $* = h_X$) and an object U with $\mathcal{F} = h_U$.

Let $\mathcal{D} = \mathcal{C}$ as a category but a covering is a family $\{V_j \to V\}$ of morphisms such that $\{V_i \to V\} \cup \{U \times_X V \to V\}$ is a covering. By our choice of $\mathcal{C}$ this means exactly that

$$h_{U \times_X V} \amalg \coprod h_{V_i} \longrightarrow h_V$$

is surjective. We claim that $\mathcal{D}$ is a site, i.e., the coverings satisfy the conditions (1), (2), (3) of Definition 6.2. Condition (1) holds. For condition (2) suppose that $\{V_i \to V\}$ and $\{V_{ij} \to V_i\}$ are coverings of $\mathcal{D}$. Then the composition

$$\coprod \left(h_{U \times_X V_i} \amalg \coprod h_{V_{ij}} \right) \longrightarrow h_{U \times_X V} \amalg \coprod h_{V_i} \longrightarrow h_V$$

is surjective. Since each of the morphisms $U \times_X V_i \to V$ factors through $U \times_X V$ we see that

$$h_{U \times_X V} \amalg \coprod h_{V_{ij}} \longrightarrow h_V$$

is surjective, i.e., $\{V_{ij} \to V\}$ is a covering of V in $\mathcal{D}$. Condition (3) follows similarly as a base change of a surjective map of sheaves is surjective.

Note that the (identity) functor $u : \mathcal{C} \to \mathcal{D}$ is continuous and commutes with fibre products and final objects. Hence we obtain a morphism $f : \mathcal{D} \to \mathcal{C}$ of sites (Proposition 14.7). Observe that f_* is the identity functor on underlying presheaves, hence fully faithful. To finish the proof we have to show that the essential image of f_* is the full subcategory $E \subset Sh(\mathcal{C})$ singled out in the lemma. To do this, note that $\mathcal{G} \in \text{Ob}(Sh(\mathcal{C}))$ is in E if and only if $\mathcal{G}(U \times_X V)$ is a singleton for all objects V of $\mathcal{C}$. Thus such a sheaf satisfies the sheaf property for all coverings of $\mathcal{D}$ (argument omitted). Conversely, if $\mathcal{G}$ satisfies the sheaf property for all coverings of $\mathcal{D}$, then $\mathcal{G}(U \times_X V)$ is a singleton, as in $\mathcal{D}$ the object $U \times_X V$ is covered by the empty covering. □

08LZ **Definition 43.6.** Let $\mathcal{C}$ be a site. A strictly full subcategory $E \subset Sh(\mathcal{C})$ is an *closed subtopos* if there exists a subsheaf $\mathcal{F}$ of the final object of $Sh(\mathcal{C})$ such that E is the subtopos described in Lemma 43.5.

All right, and now we can define what it means to have a closed immersion and an open immersion of topoi.

08M0 **Definition 43.7.** Let $f : Sh(\mathcal{D}) \to Sh(\mathcal{C})$ be a morphism of topoi.

(1) We say f is an *open immersion* if f is an embedding and the essential image of f_* is an open subtopos.
(2) We say f is a *closed immersion* if f is an embedding and the essential image of f_* is a closed subtopos.

08M1 **Lemma 43.8.** *Let $i : Sh(\mathcal{D}) \to Sh(\mathcal{C})$ be a closed immersion of topoi. Then i_* is fully faithful, transforms surjections into surjections, commutes with coequalizers, commutes with pushouts, reflects injections, reflects surjections, and reflects bijections.*

Proof. Let $\mathcal{F}$ be a subsheaf of the final object $*$ of $Sh(\mathcal{C})$ and let $E \subset Sh(\mathcal{C})$ be the full subcategory consisting of those $\mathcal{G}$ such that $\mathcal{F} \times \mathcal{G} \to \mathcal{F}$ is an isomorphism. By Lemma 43.5 the functor i_* is isomorphic to the inclusion functor $\iota : E \to Sh(\mathcal{C})$.

Let $j_{\mathcal{F}} : Sh(\mathcal{C})/\mathcal{F} \to Sh(\mathcal{C})$ be the localization functor (Lemma 30.1). Note that E can also be described as the collection of sheaves $\mathcal{G}$ such that $j_{\mathcal{F}}^{-1}\mathcal{G} = *$.

Let $a, b : \mathcal{G}_1 \to \mathcal{G}_2$ be two morphism of E. To prove ι commutes with coequalizers it suffices to show that the coequalizer of a, b in $Sh(\mathcal{C})$ lies in E. This is clear because the coequalizer of two morphisms $* \to *$ is $*$ and because $j_{\mathcal{F}}^{-1}$ is exact. Similarly for pushouts.

Thus i_* satisfies properties (5), (6), and (7) of Lemma 41.1 and hence the morphism i satisfies all properties mentioned in that lemma, in particular the ones mentioned in this lemma. □

44. Sheaves of algebraic structures

00YR In Sheaves, Section 15 we introduced a type of algebraic structure to be a pair $(\mathcal{A}, s)$, where $\mathcal{A}$ is a category, and $s : \mathcal{A} \to Sets$ is a functor such that

(1) s is faithful,
(2) $\mathcal{A}$ has limits and s commutes with limits,
(3) $\mathcal{A}$ has filtered colimits and s commutes with them, and
(4) s reflects isomorphisms.

For such a type of algebraic structure we saw that a presheaf $\mathcal{F}$ with values in $\mathcal{A}$ on a space X is a sheaf if and only if the associated presheaf of sets is a sheaf. Moreover, we worked out the notion of stalk, and given a continuous map $f : X \to Y$ we defined adjoint functors pushforward and pullback on sheaves of algebraic structures which agrees with pushforward and pullback on the underlying sheaves of sets. In addition extending a sheaf of algebraic structures from a basis to all opens of a space, works as expected.

Part of this material still works in the setting of sites and sheaves. Let $(\mathcal{A}, s)$ be a type of algebraic structure. Let $\mathcal{C}$ be a site. Let us denote $PSh(\mathcal{C}, \mathcal{A})$, resp. $Sh(\mathcal{C}, \mathcal{A})$ the category of presheaves, resp. sheaves with values in $\mathcal{A}$ on $\mathcal{C}$.

(α) A presheaf with values in $\mathcal{A}$ is a sheaf if and only if its underlying presheaf of sets is a sheaf. See the proof of Sheaves, Lemma 9.2.
(β) Given a presheaf $\mathcal{F}$ with values in $\mathcal{A}$ the presheaf $\mathcal{F}^\# = (\mathcal{F}^+)^+$ is a sheaf. This is true since the colimits in the sheafification process are filtered, and even colimits over directed sets (see Section 10, especially the proof of Lemma 10.14) and since s commutes with filtered colimits.

(γ) We get the following commutative diagram

$$\begin{array}{ccc} Sh(\mathcal{C}, \mathcal{A}) & \underset{\#}{\rightleftarrows} & PSh(\mathcal{C}, \mathcal{A}) \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle s} \\ Sh(\mathcal{C}) & \rightleftarrows & PSh(\mathcal{C}) \end{array}$$

(δ) We have $\mathcal{F} = \mathcal{F}^\#$ if and only if $\mathcal{F}$ is a sheaf of algebraic structures.
(ϵ) The functor $^\#$ is adjoint to the inclusion functor:

$$\mathrm{Mor}_{PSh(\mathcal{C},\mathcal{A})}(\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{Sh(\mathcal{C},\mathcal{A})}(\mathcal{G}^\#, \mathcal{F})$$

The proof is the same as the proof of Proposition 10.12.
(ζ) The functor $\mathcal{F} \mapsto \mathcal{F}^\#$ is left exact. The proof is the same as the proof of Lemma 10.14.

00YS **Definition 44.1.** Let $f : \mathcal{D} \to \mathcal{C}$ be a morphism of sites given by a functor $u : \mathcal{C} \to \mathcal{D}$. We define the *pushforward* functor for presheaves of algebraic structures by the rule $u^p\mathcal{F}(U) = \mathcal{F}(uU)$, and for sheaves of algebraic structures by the same rule, namely $f_*\mathcal{F}(U) = \mathcal{F}(uU)$.

The problem comes with trying the define the pullback. The reason is that the colimits defining the functor u_p in Section 5 may not be filtered. Thus the axioms above are not enough in general to define the pullback of a (pre)sheaf of algebraic structures. Nonetheless, in almost all cases the following lemma is sufficient to define pushforward, and pullback of (pre)sheaves of algebraic structures.

00YT **Lemma 44.2.** *Suppose the functor $u : \mathcal{C} \to \mathcal{D}$ satisfies the hypotheses of Proposition 14.7, and hence gives rise to a morphism of sites $f : \mathcal{D} \to \mathcal{C}$. In this case the pullback functor f^{-1} (resp. u_p) and the pushforward functor f_* (resp. u^p) extend to an adjoint pair of functors on the categories of sheaves (resp. presheaves) of algebraic structures. Moreover, these functors commute with taking the underlying sheaf (resp. presheaf) of sets.*

Proof. We have defined $f_* = u^p$ above. In the course of the proof of Proposition 14.7 we saw that all the colimits used to define u_p are filtered under the assumptions of the proposition. Hence we conclude from the definition of a type of algebraic structure that we may define u_p by exactly the same colimits as a functor on presheaves of algebraic structures. Adjointness of u_p and u^p is proved in exactly the same way as the proof of Lemma 5.4. The discussion of sheafification of presheaves of algebraic structures above then implies that we may define $f^{-1}(\mathcal{F}) = (u_p\mathcal{F})^\#$. □

We briefly discuss a method for dealing with pullback and pushforward for a general morphism of sites, and more generally for any morphism of topoi.

Let $\mathcal{C}$ be a site. In the case $\mathcal{A} = Ab$, we may think of an abelian (pre)sheaf on $\mathcal{C}$ as a quadruple $(\mathcal{F}, +, 0, i)$. Here the data are

(D1) $\mathcal{F}$ is a sheaf of sets,
(D2) $+ : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$ is a morphism of sheaves of sets,
(D3) $0 : * \to \mathcal{F}$ is a morphism from the singleton sheaf (see Example 10.2) to $\mathcal{F}$, and
(D4) $i : \mathcal{F} \to \mathcal{F}$ is a morphism of sheaves of sets.

These data have to satisfy the following axioms

(A1) $+$ is associative and commutative,
(A2) 0 is a unit for $+$, and
(A3) $+ \circ (1, i) = 0 \circ (\mathcal{F} \to *)$.

Compare Sheaves, Lemma 4.3. Let $f : \mathcal{D} \to \mathcal{C}$ be a morphism of sites. Note that since f^{-1} is exact we have $f^{-1}* = *$ and $f^{-1}(\mathcal{F} \times \mathcal{F}) = f^{-1}\mathcal{F} \times f^{-1}\mathcal{F}$. Thus we can define $f^{-1}\mathcal{F}$ simply as the quadruple $(f^{-1}\mathcal{F}, f^{-1}+, f^{-1}0, f^{-1}i)$. The axioms are going to be preserved because f^{-1} is a functor which commutes with finite limits. Finally it is not hard to check that f_* and f^{-1} are adjoint as usual.

In [AGV71] this method is used. They introduce something called an "*espèce the structure algébrique ≪définie par limites projectives finie≫*". For such an espèce you can use the method described above to define a pair of adjoint functors f^{-1} and f_* as above. This clearly works for most algebraic structures that one encounters in practice. Instead of formalizing this construction we simply list those algebraic structures for which this method works (to be verified case by case). In fact, this method works for any morphism of topoi.

00YV **Proposition 44.3.** *Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $f = (f^{-1}, f_*)$ be a morphism of topoi from $Sh(\mathcal{D}) \to Sh(\mathcal{C})$. The method introduced above gives rise to an adjoint pair of functors (f^{-1}, f_*) on sheaves of algebraic structures compatible with taking the underlying sheaves of sets for the following types of algebraic structures:*

(1) *pointed sets,*
(2) *abelian groups,*
(3) *groups,*
(4) *monoids,*
(5) *rings,*
(6) *modules over a fixed ring, and*
(7) *lie algebras over a fixed field.*

Moreover, in each of these cases the results above labeled (α), (β), (γ), (δ), (ϵ), and (ζ) hold.

Proof. The final statement of the proposition holds simply since each of the listed categories, endowed with the obvious forgetful functor, is indeed a type of algebraic structure in the sense explained at the beginning of this section. See Sheaves, Lemma 15.2.

Proof of (2). We think of a sheaf of abelian groups as a quadruple $(\mathcal{F}, +, 0, i)$ as explained in the discussion preceding the proposition. If $(\mathcal{F}, +, 0, i)$ lives on $\mathcal{C}$, then its pullback is defined as $(f^{-1}\mathcal{F}, f^{-1}+, f^{-1}0, f^{-1}i)$. If $(\mathcal{G}, +, 0, i)$ lives on $\mathcal{D}$, then its pushforward is defined as $(f_*\mathcal{G}, f_*+, f_*0, f_*i)$. This works because $f_*(\mathcal{G} \times \mathcal{G}) = f_*\mathcal{G} \times f_*\mathcal{G}$. Adjointness follows from adjointness of the set based functors, since

$$\mathrm{Mor}_{Ab(\mathcal{C})}((\mathcal{F}_1, +, 0, i), (\mathcal{F}_2, +, 0, i)) = \left\{ \begin{array}{c} \varphi \in \mathrm{Mor}_{Sh(\mathcal{C})}(\mathcal{F}_1, \mathcal{F}_2) \\ \varphi \text{ is compatible with } +, 0, i \end{array} \right\}$$

Details left to the reader.

This method also works for sheaves of rings by thinking of a sheaf of rings (with unit) as a sextuple $(\mathcal{O}, +, 0, i, \cdot, 1)$ satisfying a list of axioms that you can find in any elementary algebra book.

A sheaf of pointed sets is a pair $(\mathcal{F}, p)$, where $\mathcal{F}$ is a sheaf of sets, and $p : * \to \mathcal{F}$ is a map of sheaves of sets.

A sheaf of groups is given by a quadruple $(\mathcal{F}, \cdot, 1, i)$ with suitable axioms.

A sheaf of monoids is given by a pair $(\mathcal{F}, \cdot)$ with suitable axiom.

Let R be a ring. An sheaf of R-modules is given by a quintuple $(\mathcal{F}, +, 0, i, \{\lambda_r\}_{r\in R})$, where the quadruple $(\mathcal{F}, +, 0, i)$ is a sheaf of abelian groups as above, and $\lambda_r : \mathcal{F} \to \mathcal{F}$ is a family of morphisms of sheaves of sets such that $\lambda_r \circ 0 = 0$, $\lambda_r \circ + = + \circ (\lambda_r, \lambda_r)$, $\lambda_{r+r'} = + \circ \lambda_r \times \lambda_{r'} \circ (\mathrm{id}, \mathrm{id})$, $\lambda_{rr'} = \lambda_r \circ \lambda_{r'}$, $\lambda_1 = \mathrm{id}$, $\lambda_0 = 0 \circ (\mathcal{F} \to *)$. □

We will discuss the category of sheaves of modules over a sheaf of rings in Modules on Sites, Section 10.

00YU **Remark 44.4.** Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $u : \mathcal{D} \to \mathcal{C}$ be a continuous functor which gives rise to a morphism of sites $\mathcal{C} \to \mathcal{D}$. Note that even in the case of abelian groups we have not defined a pullback functor for presheaves of abelian groups. Since all colimits are representable in the category of abelian groups, we certainly may define a functor u_p^{ab} on abelian presheaves by the same colimits as we have used to define u_p on presheaves of sets. It will also be the case that u_p^{ab} is adjoint to u^p on the categories of abelian presheaves. However, it will not always be the case that u_p^{ab} agrees with u_p on the underlying presheaves of sets.

45. Pullback maps

06UM It sometimes happens that a site $\mathcal{C}$ does not have a final object. In this case we define the global section functor as follows.

06UN **Definition 45.1.** The *global sections* of a presheaf of sets $\mathcal{F}$ over a site $\mathcal{C}$ is the set

$$\Gamma(\mathcal{C}, \mathcal{F}) = \mathrm{Mor}_{PSh(\mathcal{C})}(*, \mathcal{F})$$

where $*$ is the final object in the category of presheaves on $\mathcal{C}$, i.e., the presheaf which associates to every object a singleton.

Of course the same definition applies to sheaves as well. Here is one way to compute global sections.

0792 **Lemma 45.2.** *Let $\mathcal{C}$ be a site. Let $a, b : V \to U$ be objects of $\mathcal{C}$ such that*

$$h_V^\# \rightrightarrows h_U^\# \longrightarrow *$$

is a coequalizer in $Sh(\mathcal{C})$. Then $\Gamma(\mathcal{C}, \mathcal{F})$ is the equalizer of $a^*, b^* : \mathcal{F}(U) \to \mathcal{F}(V)$.

Proof. Since $\mathrm{Mor}_{Sh(\mathcal{C})}(h_U^\#, \mathcal{F}) = \mathcal{F}(U)$ this is clear from the definitions. □

Now, let $f : Sh(\mathcal{D}) \to Sh(\mathcal{C})$ be a morphism of topoi. Then for any sheaf $\mathcal{F}$ on $\mathcal{C}$ there is a pullback map

$$f^{-1} : \Gamma(\mathcal{C}, \mathcal{F}) \longrightarrow \Gamma(\mathcal{D}, f^{-1}\mathcal{F})$$

Namely, as f^{-1} is exact it transforms $*$ into $*$. Hence a global section s of $\mathcal{F}$ over $\mathcal{C}$, which is a map of sheaves $s : * \to \mathcal{F}$, can be pulled back to $f^{-1}s : * = f^{-1}* \to f^{-1}\mathcal{F}$.

We can generalize this a bit by considering a pair of sheaves $\mathcal{F}$, $\mathcal{G}$ on $\mathcal{C}$, $\mathcal{D}$ together with a map $f^{-1}\mathcal{F} \to \mathcal{G}$. Then we compose the construction above with the obvious map $\Gamma(\mathcal{D}, f^{-1}\mathcal{F}) \to \Gamma(\mathcal{D}, \mathcal{G})$ to get a map

$$\Gamma(\mathcal{C}, \mathcal{F}) \longrightarrow \Gamma(\mathcal{D}, \mathcal{G})$$

This map is sometimes also called a pullback map.

A slightly more general construction which occurs frequently in nature is the following. Suppose that we have a commutative diagram of morphisms of topoi

$$\begin{array}{ccc} Sh(\mathcal{D}) & \xrightarrow{f} & Sh(\mathcal{C}) \\ & {\scriptstyle h}\searrow \quad \swarrow{\scriptstyle g} & \\ & Sh(\mathcal{B}) & \end{array}$$

Next, suppose that we have a sheaf $\mathcal{F}$ on $\mathcal{C}$. Then there is a *pullback map*

$$f^{-1} : g_*\mathcal{F} \longrightarrow h_* f^{-1}\mathcal{F}$$

Namely, it is just the map coming from the identification $g_* f_* f^{-1}\mathcal{F} = h_* f^{-1}\mathcal{F}$ together with g_* applied to the canonical map $\mathcal{F} \to f_* f^{-1}\mathcal{F}$. If g is the identity, then this map on global sections agrees with the pullback map above.

In the situation of the previous paragraph, suppose we have a pair of sheaves $\mathcal{F}$, $\mathcal{G}$ on $\mathcal{C}$, $\mathcal{D}$ together with a map $f^{-1}\mathcal{F} \to \mathcal{G}$, then we compose the pullback map above with h_* applied to $f^{-1}\mathcal{F} \to \mathcal{G}$ to get a map

$$g_*\mathcal{F} \longrightarrow h_*\mathcal{G}$$

Restricting to sections over an object of $\mathcal{B}$ one recovers the "pullback map" on global sections discussed above (with suitable choices of sites).

An even more general situation is where we have a commutative diagram of topoi

$$\begin{array}{ccc} Sh(\mathcal{D}) & \xrightarrow{f} & Sh(\mathcal{C}) \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ Sh(\mathcal{B}) & \xrightarrow{e} & Sh(\mathcal{A}) \end{array}$$

and a sheaf $\mathcal{G}$ on $\mathcal{C}$. Then there is a *base change map*

$$e^{-1} g_* \mathcal{G} \longrightarrow h_* f^{-1} \mathcal{G}.$$

Namely, this map is adjoint to a map $g_*\mathcal{G} \to e_* h_* f^{-1}\mathcal{G} = (e \circ h)_* f^{-1}\mathcal{G}$ which is the pullback map just described.

0F6X **Remark 45.3.** Consider a commutative diagram

$$\begin{array}{ccc} Sh(\mathcal{B}') & \xrightarrow{k} & Sh(\mathcal{B}) \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Sh(\mathcal{C}') & \xrightarrow{l} & Sh(\mathcal{C}) \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ Sh(\mathcal{D}') & \xrightarrow{m} & Sh(\mathcal{D}) \end{array}$$

of topoi. Then the base change maps for the two squares compose to give the base change map for the outer rectangle. More precisely, the composition

$$\begin{aligned} m^{-1} \circ (g \circ f)_* &= m^{-1} \circ g_* \circ f_* \\ &\to g'_* \circ l^{-1} \circ f_* \\ &\to g'_* \circ f'_* \circ k^{-1} \\ &= (g' \circ f')_* \circ k^{-1} \end{aligned}$$

is the base change map for the rectangle. We omit the verification.

0F6Y **Remark 45.4.** Consider a commutative diagram

$$\begin{array}{ccccc} Sh(\mathcal{C}'') & \xrightarrow{g'} & Sh(\mathcal{C}') & \xrightarrow{g} & Sh(\mathcal{C}) \\ \downarrow{\scriptstyle f''} & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Sh(\mathcal{D}'') & \xrightarrow{h'} & Sh(\mathcal{D}') & \xrightarrow{h} & Sh(\mathcal{D}) \end{array}$$

of ringed topoi. Then the base change maps for the two squares compose to give the base change map for the outer rectangle. More precisely, the composition

$$\begin{aligned} (h \circ h')^{-1} \circ f_* &= (h')^{-1} \circ h^{-1} \circ f_* \\ &\to (h')^{-1} \circ f'_* \circ g^{-1} \\ &\to f''_* \circ (g')^{-1} \circ g^{-1} \\ &= f''_* \circ (g \circ g')^{-1} \end{aligned}$$

is the base change map for the rectangle. We omit the verification.

46. Comparison with SGA4

0CMZ Our notation for the functors u^p and u_p from Section 5 and u^s and u_s from Section 13 is taken from [Art62, pages 14 and 42]. Having made these choices, the notation for the functor ${}_pu$ in Section 19 and ${}_su$ in Section 20 seems reasonable. In this section we compare our notation with that of SGA4.

Presheaves: Let $u : \mathcal{C} \to \mathcal{D}$ be a functor between categories. The functor u^p is denoted u^* in [AGV71, Exposee I, Section 5]. The functor u_p is denoted $u_!$ in [AGV71, Exposee I, Proposition 5.1]. The functor ${}_pu$ is denoted u_* in [AGV71, Exposee I, Proposition 5.1]. In other words, we have

$$u_p, u^p, {}_pu \quad (SP) \quad \text{versus} \quad u_!, u^*, u_* \quad (SGA4)$$

The reader should be cautioned that different notation is used for these functors in different parts of SGA4.

Sheaves and continuous functors: Suppose that $\mathcal{C}$ and $\mathcal{D}$ are sites and that $u : \mathcal{C} \to \mathcal{D}$ is a continuous functor (Definition 13.1). The functor u^s is denoted u_s in [AGV71, Exposee III, 1.11]. The functor u_s is denoted u^s in [AGV71, Exposee III, Proposition 1.2]. In other words, we have

$$u_s, u^s \quad (SP) \quad \text{versus} \quad u^s, u_s \quad (SGA4)$$

When u defines a morphism of sites $f : \mathcal{D} \to \mathcal{C}$ (Definition 14.1) we see that the associated morphism of topoi (Lemma 15.2) is the same as that in [AGV71, Exposee IV, (4.9.1.1)].

Sheaves and cocontinuous functors: Suppose that $\mathcal{C}$ and $\mathcal{D}$ are sites and that $u : \mathcal{C} \to \mathcal{D}$ is a cocontinuous functor (Definition 20.1). The functor ${}_su$ (Lemma 20.2) is denoted u_* in [AGV71, Exposee III, Proposition 2.3]. The functor $(u^p\ )^\#$ is denoted u^* in [AGV71, Exposee III, Proposition 2.3]. In other words, we have

$$(u^p\ )^\#, {}_su \quad (SP) \quad \text{versus} \quad u^*, u_* \quad (SGA4)$$

Thus the morphism of topoi associated to u in Lemma 21.1 is the same as that in [AGV71, Exposee IV, 4.7].

Morphisms of Topoi: If f is a morphism of topoi given by the functors (f^{-1}, f_*) then the functor f^{-1} is denoted f^* in [AGV71, Exposee IV, Definition 3.1]. We will use f^{-1} to denote pullback of sheaves of sets or more generally sheaves of algebraic structure (Section 44). We will use f^* to denote pullback of sheaves of modules for a morphism of ringed topoi (Modules on Sites, Definition 13.1).

47. Topologies

00YW In this section we define what a topology on a category is as defined in [AGV71]. One can develop all of the machinery of sheaves and topoi in this language. A modern exposition of this material can be found in [KS06]. However, the case of most interest for algebraic geometry is the topology defined by a site on its underlying category. Thus we strongly suggest the first time reader **skip this section and all other sections of this chapter**!

00YX **Definition 47.1.** Let $\mathcal{C}$ be a category. Let $U \in \mathrm{Ob}(\mathcal{C})$. A *sieve S on U* is a subpresheaf $S \subset h_U$.

In other words, a sieve on U picks out for each object $T \in \mathrm{Ob}(\mathcal{C})$ a subset $S(T)$ of the set of all morphisms $T \to U$. In fact, the only condition on the collection of subsets $S(T) \subset h_U(T) = \mathrm{Mor}_{\mathcal{C}}(T, U)$ is the following rule

00YY (47.1.1)
$$\left.\begin{array}{c}(\alpha : T \to U) \in S(T) \\ g : T' \to T\end{array}\right\} \Rightarrow (\alpha \circ g : T' \to U) \in S(T')$$

A good mental picture to keep in mind is to think of the map $S \to h_U$ as a "morphism from S to U".

00YZ **Lemma 47.2.** *Let $\mathcal{C}$ be a category. Let $U \in \mathrm{Ob}(\mathcal{C})$.*

1. *The collection of sieves on U is a set.*
2. *Inclusion defines a partial ordering on this set.*
3. *Unions and intersections of sieves are sieves.*

00Z0 4. *Given a family of morphisms $\{U_i \to U\}_{i \in I}$ of $\mathcal{C}$ with target U there exists a unique smallest sieve S on U such that each $U_i \to U$ belongs to $S(U_i)$.*

5. *The sieve $S = h_U$ is the maximal sieve.*
6. *The empty subpresheaf is the minimal sieve.*

Proof. By our definition of subpresheaf, the collection of all subpresheaves of a presheaf $\mathcal{F}$ is a subset of $\prod_{U \in \mathrm{Ob}(\mathcal{C})} \mathcal{P}(\mathcal{F}(U))$. And this is a set. (Here $\mathcal{P}(A)$ denotes the powerset of A.) Hence the collection of sieves on U is a set.

The partial ordering is defined by: $S \leq S'$ if and only if $S(T) \subset S'(T)$ for all $T \to U$. Notation: $S \subset S'$.

Given a collection of sieves S_i, $i \in I$ on U we can define $\bigcup S_i$ as the sieve with values $(\bigcup S_i)(T) = \bigcup S_i(T)$ for all $T \in \mathrm{Ob}(\mathcal{C})$. We define the intersection $\bigcap S_i$ in the same way.

Given $\{U_i \to U\}_{i \in I}$ as in the statement, consider the morphisms of presheaves $h_{U_i} \to h_U$. We simply define S as the union of the images (Definition 3.5) of these maps of presheaves.

The last two statements of the lemma are obvious. □

00Z1 **Definition 47.3.** Let $\mathcal{C}$ be a category. Given a family of morphisms $\{f_i : U_i \to U\}_{i \in I}$ of $\mathcal{C}$ with target U we say the sieve S on U described in Lemma 47.2 part (4) is the *sieve on U generated by the morphisms f_i*.

00Z2 **Definition 47.4.** Let $\mathcal{C}$ be a category. Let $f : V \to U$ be a morphism of $\mathcal{C}$. Let $S \subset h_U$ be a sieve. We define the *pullback of S by f* to be the sieve $S \times_U V$ of V defined by the rule

$$(\alpha : T \to V) \in (S \times_U V)(T) \Leftrightarrow (f \circ \alpha : T \to U) \in S(T)$$

We leave it to the reader to see that this is indeed a sieve (hint: use Equation 47.1.1). We also sometimes call $S \times_U V$ the *base change* of S by $f : V \to U$.

00Z3 **Lemma 47.5.** *Let $\mathcal{C}$ be a category. Let $U \in \mathrm{Ob}(\mathcal{C})$. Let S be a sieve on U. If $f : V \to U$ is in S, then $S \times_U V = h_V$ is maximal.*

Proof. Trivial from the definitions. □

00Z4 **Definition 47.6.** Let $\mathcal{C}$ be a category. A *topology on $\mathcal{C}$* is given by a rule which assigns to every $U \in \mathrm{Ob}(\mathcal{C})$ a subset $J(U)$ of the set of all sieves on U satisfying the following conditions

(1) For every morphism $f : V \to U$ in $\mathcal{C}$, and every element $S \in J(U)$ the pullback $S \times_U V$ is an element of $J(V)$.
(2) If S and S' are sieves on $U \in \mathrm{Ob}(\mathcal{C})$, if $S \in J(U)$, and if for all $f \in S(V)$ the pullback $S' \times_U V$ belongs to $J(V)$, then S' belongs to $J(U)$.
(3) For every $U \in \mathrm{Ob}(\mathcal{C})$ the maximal sieve $S = h_U$ belongs to $J(U)$.

In this case, the sieves belonging to $J(U)$ are called the *covering sieves*.

00Z5 **Lemma 47.7.** *Let $\mathcal{C}$ be a category. Let J be a topology on $\mathcal{C}$. Let $U \in \mathrm{Ob}(\mathcal{C})$.*

(1) *Finite intersections of elements of $J(U)$ are in $J(U)$.*
(2) *If $S \in J(U)$ and $S' \supset S$, then $S' \in J(U)$.*

Proof. Let $S, S' \in J(U)$. Consider $S'' = S \cap S'$. For every $V \to U$ in $S(U)$ we have

$$S' \times_U V = S'' \times_U V$$

simply because $V \to U$ already is in S. Hence by the second axiom of the definition we see that $S'' \in J(U)$.

Let $S \in J(U)$ and $S' \supset S$. For every $V \to U$ in $S(U)$ we have $S' \times_U V = h_V$ by Lemma 47.5. Thus $S' \times_U V \in J(V)$ by the third axiom. Hence $S' \in J(U)$ by the second axiom. □

00Z6 **Definition 47.8.** Let $\mathcal{C}$ be a category. Let J, J' be two topologies on $\mathcal{C}$. We say that J is *finer* or *stronger* than J' if and only if for every object U of $\mathcal{C}$ we have $J'(U) \subset J(U)$. In this case we also say that J' is *coarser* or *weaker* than J.

In other words, any covering sieve of J' is a covering sieve of J. There exists a finest topology on $\mathcal{C}$, namely that topology where any sieve is a covering sieve. This is called the *discrete topology* of $\mathcal{C}$. There also exists a coarsest topology. Namely, the topology where $J(U) = \{h_U\}$ for all objects U. This is called the *chaotic* or *indiscrete topology*.

00Z7 **Lemma 47.9.** *Let $\mathcal{C}$ be a category. Let $\{J_i\}_{i\in I}$ be a set of topologies.*

(1) *The rule $J(U) = \bigcap J_i(U)$ defines a topology on $\mathcal{C}$.*
(2) *There is a coarsest topology finer than all of the topologies J_i.*

Proof. The first part is direct from the definitions. The second follows by taking the intersection of all topologies finer than all of the J_i. □

At this point we can define without any motivation what a sheaf is.

00Z8 **Definition 47.10.** Let $\mathcal{C}$ be a category endowed with a topology J. Let $\mathcal{F}$ be a presheaf of sets on $\mathcal{C}$. We say that $\mathcal{F}$ is a *sheaf* on $\mathcal{C}$ if for every $U \in \mathrm{Ob}(\mathcal{C})$ and for every covering sieve S of U the canonical map

$$\mathrm{Mor}_{PSh(\mathcal{C})}(h_U, \mathcal{F}) \longrightarrow \mathrm{Mor}_{PSh(\mathcal{C})}(S, \mathcal{F})$$

is bijective.

Recall that the left hand side of the displayed formula equals $\mathcal{F}(U)$. In other words, $\mathcal{F}$ is a sheaf if and only if a section of $\mathcal{F}$ over U is the same thing as a compatible collection of sections $s_{T,\alpha} \in \mathcal{F}(T)$ parametrized by $(\alpha : T \to U) \in S(T)$, and this for every covering sieve S on U.

00Z9 **Lemma 47.11.** *Let $\mathcal{C}$ be a category. Let $\{\mathcal{F}_i\}_{i\in I}$ be a collection of presheaves of sets on $\mathcal{C}$. For each $U \in \mathrm{Ob}(\mathcal{C})$ denote $J(U)$ the set of sieves S with the following property: For every morphism $V \to U$, the maps*

$$\mathrm{Mor}_{PSh(\mathcal{C})}(h_V, \mathcal{F}_i) \longrightarrow \mathrm{Mor}_{PSh(\mathcal{C})}(S \times_U V, \mathcal{F}_i)$$

are bijective for all $i \in I$. Then J defines a topology on $\mathcal{C}$. This topology is the finest topology in which all of the $\mathcal{F}_i$ are sheaves.

Proof. If we show that J is a topology, then the last statement of the lemma immediately follows. The first and third axioms of a topology are immediately verified. Thus, assume that we have an object U, and sieves S, S' of U such that $S \in J(U)$, and for all $V \to U$ in $S(V)$ we have $S' \times_U V \in J(V)$. We have to show that $S' \in J(U)$. In other words, we have to show that for any $f : W \to U$, the maps

$$\mathcal{F}_i(W) = \mathrm{Mor}_{PSh(\mathcal{C})}(h_W, \mathcal{F}_i) \longrightarrow \mathrm{Mor}_{PSh(\mathcal{C})}(S' \times_U W, \mathcal{F}_i)$$

are bijective for all $i \in I$. Pick an element $i \in I$ and pick an element $\varphi \in \mathrm{Mor}_{PSh(\mathcal{C})}(S' \times_U W, \mathcal{F}_i)$. We will construct a section $s \in \mathcal{F}_i(W)$ mapping to φ.

Suppose $\alpha : V \to W$ is an element of $S \times_U W$. According to the definition of pullbacks we see that the composition $f \circ \alpha : V \to W \to U$ is in S. Hence $S' \times_U V$ is in $J(W)$ by assumption on the pair of sieves S, S'. Now we have a commutative diagram of presheaves

$$\begin{array}{ccc} S' \times_U V & \longrightarrow & h_V \\ \downarrow & & \downarrow \\ S' \times_U W & \longrightarrow & h_W \end{array}$$

The restriction of φ to $S' \times_U V$ corresponds to an element $s_{V,\alpha} \in \mathcal{F}_i(V)$. This we see from the definition of J, and because $S' \times_U V$ is in $J(W)$. We leave it to the reader to check that the rule $(V, \alpha) \mapsto s_{V,\alpha}$ defines an element $\psi \in \mathrm{Mor}_{PSh(\mathcal{C})}(S \times_U W, \mathcal{F}_i)$. Since $S \in J(U)$ we see immediately from the definition of J that ψ corresponds to an element s of $\mathcal{F}_i(W)$.

We leave it to the reader to verify that the construction $\varphi \mapsto s$ is inverse to the natural map displayed above. □

00ZA **Definition 47.12.** Let $\mathcal{C}$ be a category. The finest topology on $\mathcal{C}$ such that all representable presheaves are sheaves, see Lemma 47.11, is called the *canonical topology* of $\mathcal{C}$.

48. The topology defined by a site

00ZB Suppose that $\mathcal{C}$ is a category, and suppose that $\mathrm{Cov}_1(\mathcal{C})$ and $\mathrm{Cov}_2(\mathcal{C})$ are sets of coverings that define the structure of a site on $\mathcal{C}$. In this situation it can happen that the categories of sheaves (of sets) for $\mathrm{Cov}_1(\mathcal{C})$ and $\mathrm{Cov}_2(\mathcal{C})$ are the same, see for example Lemma 8.7.

It turns out that the category of sheaves on $\mathcal{C}$ with respect to some topology J determines and is determined by the topology J. This is a nontrivial statement which we will address later, see Theorem 50.2.

Accepting this for the moment it makes sense to study the topology determined by a site.

00ZC **Lemma 48.1.** *Let $\mathcal{C}$ be a site with coverings $Cov(\mathcal{C})$. For every object U of $\mathcal{C}$, let $J(U)$ denote the set of sieves S on U with the following property: there exists a covering $\{f_i : U_i \to U\}_{i \in I} \in Cov(\mathcal{C})$ so that the sieve S' generated by the f_i (see Definition 47.3) is contained in S.*

1. *This J is a topology on $\mathcal{C}$.*
2. *A presheaf $\mathcal{F}$ is a sheaf for this topology (see Definition 47.10) if and only if it is a sheaf on the site (see Definition 7.1).*

Proof. To prove the first assertion we just note that axioms (1), (2) and (3) of the definition of a site (Definition 6.2) directly imply the axioms (3), (2) and (1) of the definition of a topology (Definition 47.6). As an example we prove J has property (2). Namely, let U be an object of $\mathcal{C}$, let S, S' be sieves on U such that $S \in J(U)$, and such that for every $V \to U$ in $S(V)$ we have $S' \times_U V \in J(V)$. By definition of $J(U)$ we can find a covering $\{f_i : U_i \to U\}$ of the site such that S the image of $h_{U_i} \to h_U$ is contained in S. Since each $S' \times_U U_i$ is in $J(U_i)$ we see that there are coverings $\{U_{ij} \to U_i\}$ of the site such that $h_{U_{ij}} \to h_{U_i}$ is contained in $S' \times_U U_i$. By definition of the base change this means that $h_{U_{ij}} \to h_U$ is contained in the subpresheaf $S' \subset h_U$. By axiom (2) for sites we see that $\{U_{ij} \to U\}$ is a covering of U and we conclude that $S' \in J(U)$ by definition of J.

Let $\mathcal{F}$ be a presheaf. Suppose that $\mathcal{F}$ is a sheaf in the topology J. We will show that $\mathcal{F}$ is a sheaf on the site as well. Let $\{f_i : U_i \to U\}_{i \in I}$ be a covering of the site. Let $s_i \in \mathcal{F}(U_i)$ be a family of sections such that $s_i|_{U_i \times_U U_j} = s_j|_{U_i \times_U U_j}$ for all i, j. We have to show that there exists a unique section $s \in \mathcal{F}(U)$ restricting back to the s_i on the U_i. Let $S \subset h_U$ be the sieve generated by the f_i. Note that $S \in J(U)$

by definition. In stead of constructing s, by the sheaf condition in the topology, it suffices to construct an element

$$\varphi \in \mathrm{Mor}_{PSh(\mathcal{C})}(S, \mathcal{F}).$$

Take $\alpha \in S(T)$ for some object $T \in \mathcal{U}$. This means exactly that $\alpha : T \to U$ is a morphism which factors through f_i for some $i \in I$ (and maybe more than 1). Pick such an index i and a factorization $\alpha = f_i \circ \alpha_i$. Define $\varphi(\alpha) = \alpha_i^* s_i$. If i', $\alpha = f_i \circ \alpha'_{i'}$ is a second choice, then $\alpha_i^* s_i = (\alpha'_{i'})^* s_{i'}$ exactly because of our condition $s_i|_{U_i \times_U U_j} = s_j|_{U_i \times_U U_j}$ for all i, j. Thus $\varphi(\alpha)$ is well defined. We leave it to the reader to verify that φ, which in turn determines s is correct in the sense that s restricts back to s_i.

Let $\mathcal{F}$ be a presheaf. Suppose that $\mathcal{F}$ is a sheaf on the site $(\mathcal{C}, \mathrm{Cov}(\mathcal{C}))$. We will show that $\mathcal{F}$ is a sheaf for the topology J as well. Let U be an object of $\mathcal{C}$. Let S be a covering sieve on U with respect to the topology J. Let

$$\varphi \in \mathrm{Mor}_{PSh(\mathcal{C})}(S, \mathcal{F}).$$

We have to show there is a unique element in $\mathcal{F}(U) = \mathrm{Mor}_{PSh(\mathcal{C})}(h_U, \mathcal{F})$ which restricts back to φ. By definition there exists a covering $\{f_i : U_i \to U\}_{i \in I} \in \mathrm{Cov}(\mathcal{C})$ such that $f_i : U_i \in U$ belongs to $S(U_i)$. Hence we can set $s_i = \varphi(f_i) \in \mathcal{F}(U_i)$. Then it is a pleasant exercise to see that $s_i|_{U_i \times_U U_j} = s_j|_{U_i \times_U U_j}$ for all i, j. Thus we obtain the desired section s by the sheaf condition for $\mathcal{F}$ on the site $(\mathcal{C}, \mathrm{Cov}(\mathcal{C}))$. Details left to the reader. □

00ZD **Definition 48.2.** Let $\mathcal{C}$ be a site with coverings $\mathrm{Cov}(\mathcal{C})$. The *topology associated to* $\mathcal{C}$ is the topology J constructed in Lemma 48.1 above.

Let $\mathcal{C}$ be a category. Let $\mathrm{Cov}_1(\mathcal{C})$ and $\mathrm{Cov}_2(\mathcal{C})$ be two coverings defining the structure of a site on $\mathcal{C}$. It may very well happen that the topologies defined by these are the same. If this happens then we say $\mathrm{Cov}_1(\mathcal{C})$ and $\mathrm{Cov}_2(\mathcal{C})$ *define the same topology* on $\mathcal{C}$. And if this happens then the categories of sheaves are the same, by Lemma 48.1.

It is usually the case that we only care about the topology defined by a collection of coverings, and we view the possibility of choosing different sets of coverings as a tool to study the topology.

00ZE **Remark 48.3.** Enlarging the class of coverings. Clearly, if $\mathrm{Cov}(\mathcal{C})$ defines the structure of a site on $\mathcal{C}$ then we may add to $\mathcal{C}$ any set of families of morphisms with fixed target tautologically equivalent (see Definition 8.2) to elements of $\mathrm{Cov}(\mathcal{C})$ without changing the topology.

00ZF **Remark 48.4.** Shrinking the class of coverings. Let $\mathcal{C}$ be a site. Consider the set

$$\mathcal{S} = P(\mathrm{Arrows}(\mathcal{C})) \times \mathrm{Ob}(\mathcal{C})$$

where $P(\mathrm{Arrows}(\mathcal{C}))$ is the power set of the set of morphisms, i.e., the set of all sets of morphisms. Let $\mathcal{S}_\tau \subset \mathcal{S}$ be the subset consisting of those $(T, U) \in \mathcal{S}$ such that (a) all $\varphi \in T$ have target U, (b) the collection $\{\varphi\}_{\varphi \in T}$ is tautologically equivalent (see Definition 8.2) to some covering in $\mathrm{Cov}(\mathcal{C})$. Clearly, considering the elements of $\mathcal{S}_\tau$ as the coverings, we do not get exactly the notion of a site as defined in Definition 6.2. The structure $(\mathcal{C}, \mathcal{S}_\tau)$ we get satisfies slightly modified conditions. The modified conditions are:

(0') $\mathrm{Cov}(\mathcal{C}) \subset P(\mathrm{Arrows}(\mathcal{C})) \times \mathrm{Ob}(\mathcal{C})$,

(1') If $V \to U$ is an isomorphism then $(\{V \to U\}, U) \in \mathrm{Cov}(\mathcal{C})$.
(2') If $(T, U) \in \mathrm{Cov}(\mathcal{C})$ and for $f : U' \to U$ in T we are given $(T_f, U') \in \mathrm{Cov}(\mathcal{C})$, then setting $T' = \{f \circ f' \mid f \in T,\ f' \in T_f\}$, we get $(T', U) \in \mathrm{Cov}(\mathcal{C})$.
(3') If $(T, U) \in \mathrm{Cov}(\mathcal{C})$ and $g : V \to U$ is a morphism of $\mathcal{C}$ then
  (a) $U' \times_{f,U,g} V$ exists for $f : U' \to U$ in T, and
  (b) setting $T' = \{\mathrm{pr}_2 : U' \times_{f,U,g} V \to V \mid f : U' \to U \in T\}$ for some choice of fibre products we get $(T', V) \in \mathrm{Cov}(\mathcal{C})$.

And it is easy to verify that, given a structure satisfying (0') – (3') above, then after suitably enlarging $\mathrm{Cov}(\mathcal{C})$ (compare Sets, Section 11) we get a site. Obviously there is little difference between this notion and the actual notion of a site, at least from the point of view of the topology. There are two benefits: because of condition (0') above the coverings automatically form a set, and because of (0') the totality of all structures of this type forms a set as well. The price you pay for this is that you have to keep writing "tautologically equivalent" everywhere.

49. Sheafification in a topology

00ZG In this section we explain the analogue of the sheafification construction in a topology.

Let $\mathcal{C}$ be a category. Let J be a topology on $\mathcal{C}$. Let $\mathcal{F}$ be a presheaf of sets. For every $U \in \mathrm{Ob}(\mathcal{C})$ we define

$$L\mathcal{F}(U) = \mathrm{colim}_{S \in J(U)^{opp}} \mathrm{Mor}_{PSh(\mathcal{C})}(S, \mathcal{F})$$

as a colimit. Here we think of $J(U)$ as a partially ordered set, ordered by inclusion, see Lemma 47.2. The transition maps in the system are defined as follows. If $S \subset S'$ are in $J(U)$, then $S \to S'$ is a morphism of presheaves. Hence there is a natural restriction mapping

$$\mathrm{Mor}_{PSh(\mathcal{C})}(S', \mathcal{F}) \longrightarrow \mathrm{Mor}_{PSh(\mathcal{C})}(S, \mathcal{F}).$$

Thus we see that $S \mapsto \mathrm{Mor}_{PSh(\mathcal{C})}(S, \mathcal{F})$ is a directed system as in Categories, Definition 21.2 provided we reverse the ordering on $J(U)$ (which is what the superscript ^{opp} is supposed to indicate). In particular, since $h_U \in J(U)$ there is a canonical map

$$\ell : \mathcal{F}(U) \longrightarrow L\mathcal{F}(U)$$

coming from the identification $\mathcal{F}(U) = \mathrm{Mor}_{PSh(\mathcal{C})}(h_U, \mathcal{F})$. In addition, the colimit defining $L\mathcal{F}(U)$ is directed since for any pair of covering sieves S, S' on U the sieve $S \cap S'$ is a covering sieve too, see Lemma 47.2.

Let $f : V \to U$ be a morphism in $\mathcal{C}$. Let $S \in J(U)$. There is a commutative diagram

$$\begin{array}{ccc} S \times_U V & \longrightarrow & h_V \\ \downarrow & & \downarrow \\ S & \longrightarrow & h_U \end{array}$$

We can use the left vertical map to get canonical restriction maps

$$\mathrm{Mor}_{PSh(\mathcal{C})}(S, \mathcal{F}) \to \mathrm{Mor}_{PSh(\mathcal{C})}(S \times_U V, \mathcal{F}).$$

Base change $S \mapsto S \times_U V$ induces an order preserving map $J(U) \to J(V)$. And the restriction maps define a transformation of functors as in Categories, Lemma categories-lemma-functorial-colimit. Hence we get a natural restriction map

$$L\mathcal{F}(U) \longrightarrow L\mathcal{F}(V).$$

00ZH **Lemma 49.1.** *In the situation above.*

(1) *The assignment $U \mapsto L\mathcal{F}(U)$ combined with the restriction mappings defined above is a presheaf.*
(2) *The maps ℓ glue to give a morphism of presheaves $\ell : \mathcal{F} \to L\mathcal{F}$.*
(3) *The rule $\mathcal{F} \mapsto (\mathcal{F} \xrightarrow{\ell} L\mathcal{F})$ is a functor.*
(4) *If $\mathcal{F}$ is a subpresheaf of $\mathcal{G}$, then $L\mathcal{F}$ is a subpresheaf of $L\mathcal{G}$.*
(5) *The map $\ell : \mathcal{F} \to L\mathcal{F}$ has the following property: For every section $s \in L\mathcal{F}(U)$ there exists a covering sieve S on U and an element $\varphi \in \mathrm{Mor}_{PSh(\mathcal{C})}(S, \mathcal{F})$ such that $\ell(\varphi)$ equals the restriction of s to S.*

Proof. Omitted. □

00ZI **Definition 49.2.** Let $\mathcal{C}$ be a category. Let J be a topology on $\mathcal{C}$. We say that a presheaf of sets $\mathcal{F}$ is *separated* if for every object U and every covering sieve S on U the canonical map $\mathcal{F}(U) \to \mathrm{Mor}_{PSh(\mathcal{C})}(S, \mathcal{F})$ is injective.

00ZJ **Theorem 49.3.** *Let $\mathcal{C}$ be a category. Let J be a topology on $\mathcal{C}$. Let $\mathcal{F}$ be a presheaf of sets.*

(1) *The presheaf $L\mathcal{F}$ is separated.*
(2) *If $\mathcal{F}$ is separated, then $L\mathcal{F}$ is a sheaf and the map of presheaves $\mathcal{F} \to L\mathcal{F}$ is injective.*
(3) *If $\mathcal{F}$ is a sheaf, then $\mathcal{F} \to L\mathcal{F}$ is an isomorphism.*
(4) *The presheaf $LL\mathcal{F}$ is always a sheaf.*

Proof. Part (3) is trivial from the definition of L and the definition of a sheaf (Definition 47.10). Part (4) follows formally from the others.

We sketch the proof of (1). Suppose S is a covering sieve of the object U. Suppose that $\varphi_i \in L\mathcal{F}(U)$, $i = 1, 2$ map to the same element in $\mathrm{Mor}_{PSh(\mathcal{C})}(S, L\mathcal{F})$. We may find a single covering sieve S' on U such that both φ_i are represented by elements $\varphi_i \in \mathrm{Mor}_{PSh(\mathcal{C})}(S', \mathcal{F})$. We may assume that $S' = S$ by replacing both S and S' by $S' \cap S$ which is also a covering sieve, see Lemma 47.2. Suppose $V \in \mathrm{Ob}(\mathcal{C})$, and $\alpha : V \to U$ in $S(V)$. Then we have $S \times_U V = h_V$, see Lemma 47.5. Thus the restrictions of φ_i via $V \to U$ correspond to sections $s_{i,V,\alpha}$ of $\mathcal{F}$ over V. The assumption is that there exist a covering sieve $S_{V,\alpha}$ of V such that $s_{i,V,\alpha}$ restrict to the same element of $\mathrm{Mor}_{PSh(\mathcal{C})}(S_{V,\alpha}, \mathcal{F})$. Consider the sieve S'' on U defined by the rule

00ZK
$$\begin{aligned}(f : T \to U) \in S''(T) \quad \Leftrightarrow \quad & \exists\ V,\ \alpha : V \to U,\ \alpha \in S(V), \\ & \exists\ g : T \to V,\ g \in S_{V,\alpha}(T), \\ & f = \alpha \circ g\end{aligned} \tag{49.3.1}$$

By axiom (2) of a topology we see that S'' is a covering sieve on U. By construction we see that φ_1 and φ_2 restrict to the same element of $\mathrm{Mor}_{PSh(\mathcal{C})}(S'', L\mathcal{F})$ as desired.

We sketch the proof of (2). Assume that $\mathcal{F}$ is a separated presheaf of sets on $\mathcal{C}$ with respect to the topology J. Let S be a covering sieve of the object U of $\mathcal{C}$. Suppose

that $\varphi \in \mathrm{Mor}_{\mathcal{C}}(S, L\mathcal{F})$. We have to find an element $s \in L\mathcal{F}(U)$ restricting to φ. Suppose $V \in \mathrm{Ob}(\mathcal{C})$, and $\alpha : V \to U$ in $S(V)$. The value $\varphi(\alpha) \in L\mathcal{F}(V)$ is given by a covering sieve $S_{V,\alpha}$ of V and a morphism of presheaves $\varphi_{V,\alpha} : S_{V,\alpha} \to \mathcal{F}$. As in the proof above, define a covering sieve S'' on U by Equation (49.3.1). We define

$$\varphi'' : S'' \longrightarrow \mathcal{F}$$

by the following simple rule: For every $f : T \to U$, $f \in S''(T)$ choose V, α, g as in Equation (49.3.1). Then set

$$\varphi''(f) = \varphi_{V,\alpha}(g).$$

We claim this is independent of the choice of V, α, g. Consider a second such choiceV', α', g'. The restrictions of $\varphi_{V,\alpha}$ and $\varphi_{V',\alpha'}$ to the intersection of the following covering sieves on T

$$(S_{V,\alpha} \times_{V,g} T) \cap (S_{V',\alpha'} \times_{V',g'} T)$$

agree. Namely, these restrictions both correspond to the restriction of φ to T (via f) and the desired equality follows because $\mathcal{F}$ is separated. Denote the common restriction ψ. The independence of choice follows because $\varphi_{V,\alpha}(g) = \psi(\mathrm{id}_T) = \varphi_{V',\alpha'}(g')$. OK, so now φ'' gives an element $s \in L\mathcal{F}(U)$. We leave it to the reader to check that s restricts to φ. □

00ZL **Definition 49.4.** Let $\mathcal{C}$ be a category endowed with a topology J. Let $\mathcal{F}$ be a presheaf of sets on $\mathcal{C}$. The sheaf $\mathcal{F}^\# := LL\mathcal{F}$ together with the canonical map $\mathcal{F} \to \mathcal{F}^\#$ is called the *sheaf associated to* $\mathcal{F}$.

00ZM **Proposition 49.5.** *Let $\mathcal{C}$ be a category endowed with a topology. Let $\mathcal{F}$ be a presheaf of sets on $\mathcal{C}$. The canonical map $\mathcal{F} \to \mathcal{F}^\#$ has the following universal property: For any map $\mathcal{F} \to \mathcal{G}$, where $\mathcal{G}$ is a sheaf of sets, there is a unique map $\mathcal{F}^\# \to \mathcal{G}$ such that $\mathcal{F} \to \mathcal{F}^\# \to \mathcal{G}$ equals the given map.*

Proof. Same as the proof of Proposition 10.12. □

50. Topologies and sheaves

00ZN

00ZO **Lemma 50.1.** *Let $\mathcal{C}$ be a category endowed with a topology J. Let U be an object of $\mathcal{C}$. Let S be a sieve on U. The following are equivalent*

1. *The sieve S is a covering sieve.*
2. *The sheafification $S^\# \to h_U^\#$ of the map $S \to h_U$ is an isomorphism.*

Proof. First we make a couple of general remarks. We will use that $S^\# = LLS$, and $h_U^\# = LLh_U$. In particular, by Lemma 49.1, we see that $S^\# \to h_U^\#$ is injective. Note that $\mathrm{id}_U \in h_U(U)$. Hence it gives rise to sections of Lh_U and $h_U^\# = LLh_U$ over U which we will also denote id_U.

Suppose S is a covering sieve. It clearly suffices to find a morphism $h_U \to S^\#$ such that the composition $h_U \to h_U^\#$ is the canonical map. To find such a map it suffices to find a section $s \in S^\#(U)$ wich restricts to id_U. But since S is a covering sieve, the element $\mathrm{id}_S \in \mathrm{Mor}_{PSh(\mathcal{C})}(S, S)$ gives rise to a section of LS over U which restricts to id_U in Lh_U. Hence we win.

Suppose that $S^\# \to h_U^\#$ is an isomorphism. Let $1 \in S^\#(U)$ be the element corresponding to id_U in $h_U^\#(U)$. Because $S^\# = LLS$ there exists a covering sieve S' on U such that 1 comes from a

$$\varphi \in \mathrm{Mor}_{PSh(\mathcal{C})}(S', LS).$$

This in turn means that for every $\alpha : V \to U$, $\alpha \in S'(V)$ there exists a covering sieve $S_{V,\alpha}$ on V such that $\varphi(\alpha)$ corresponds to a morphism of presheaves $S_{V,\alpha} \to S$. In other words $S_{V,\alpha}$ is contained in $S \times_U V$. By the second axiom of a topology we see that S is a covering sieve. □

00ZP **Theorem 50.2.** *Let $\mathcal{C}$ be a category. Let J, J' be topologies on $\mathcal{C}$. The following are equivalent*

(1) $J = J'$,
(2) *sheaves for the topology J are the same as sheaves for the topology J'.*

Proof. It is a tautology that if $J = J'$ then the notions of sheaves are the same. Conversely, Lemma 50.1 characterizes covering sieves in terms of the sheafification functor. But the sheafification functor $PSh(\mathcal{C}) \to Sh(\mathcal{C}, J)$ is the left adjoint of the inclusion functor $Sh(\mathcal{C}, J) \to PSh(\mathcal{C})$. Hence if the subcategories $Sh(\mathcal{C}, J)$ and $Sh(\mathcal{C}, J')$ are the same, then the sheafification functors are the same and hence the collections of covering sieves are the same. □

00ZQ **Lemma 50.3.** *Assumption and notation as in Theorem 50.2. Then $J \subset J'$ if and only if every sheaf for the topology J' is a sheaf for the topology J.*

Proof. One direction is clear. For the other direction suppose that $Sh(\mathcal{C}, J') \subset Sh(\mathcal{C}, J)$. By formal nonsense this implies that if $\mathcal{F}$ is a presheaf of sets, and $\mathcal{F} \to \mathcal{F}^\#$, resp. $\mathcal{F} \to \mathcal{F}^{\#,\prime}$ is the sheafification wrt J, resp. J' then there is a canonical map $\mathcal{F}^\# \to \mathcal{F}^{\#,\prime}$ such that $\mathcal{F} \to \mathcal{F}^\# \to \mathcal{F}^{\#,\prime}$ equals the canonical map $\mathcal{F} \to \mathcal{F}^{\#,\prime}$. Of course, $\mathcal{F}^\# \to \mathcal{F}^{\#,\prime}$ identifies the second sheaf as the sheafification of the first with respect to the topology J'. Apply this to the map $S \to h_U$ of Lemma 50.1. We get a commutative diagram

$$\begin{array}{ccccc} S & \longrightarrow & S^\# & \longrightarrow & S^{\#,\prime} \\ \downarrow & & \downarrow & & \downarrow \\ h_U & \longrightarrow & h_U^\# & \longrightarrow & h_U^{\#,\prime} \end{array}$$

And clearly, if S is a covering sieve for the topology J then the middle vertical map is an isomorphism (by the lemma) and we conclude that the right vertical map is an isomorphism as it is the sheafification of the one in the middle wrt J'. By the lemma again we conclude that S is a covering sieve for J' as well. □

51. Topologies and continuous functors

00ZR Explain how a continuous functor gives an adjoint pair of functors on sheaves.

52. Points and topologies

00ZS Recall from Section 32 that given a functor $p = u : \mathcal{C} \to Sets$ we can define a stalk functor

$$PSh(\mathcal{C}) \longrightarrow Sets, \mathcal{F} \longmapsto \mathcal{F}_p.$$

00ZT **Definition 52.1.** Let $\mathcal{C}$ be a category. Let J be a topology on $\mathcal{C}$. A *point* p of the topology is given by a functor $u : \mathcal{C} \to Sets$ such that

(1) For every covering sieve S on U the map $S_p \to (h_U)_p$ is surjective.
(2) The stalk functor $Sh(\mathcal{C}) \to Sets$, $\mathcal{F} \to \mathcal{F}_p$ is exact.

53. Other chapters

References

[AGV71] Michael Artin, Alexander Grothendieck, and Jean-Louis Verdier, *Theorie de topos et cohomologie etale des schemas I, II, III*, Lecture Notes in Mathematics, vol. 269, 270, 305, Springer, 1971.

[Art62] Michael Artin, *Grothendieck topologies: notes on a seminar*, Harvard University, Dept. of Mathematics, 1962.

[KS06] Masaki Kashiwara and Pierre Schapira, *Categories and sheaves*, Grundlehren der Mathematischen Wissenschaften, vol. 332, Springer-Verlag, Berlin, 2006.