PICARD SCHEMES OF CURVES

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Contents

1. Introduction

0B93 In this chapter we do just enough work to construct the Picard scheme of a projective nonsingular curve over an algebraically closed field. See [Kle05] for a more thorough discussion as well as historical background.

Later in the Stacks project we will discuss Hilbert and Quot functors in much greater generality.

2. Hilbert scheme of points

0B94 Let $X \to S$ be a morphism of schemes. Let $d \geq 0$ be an integer. For a scheme T over S we let

$$\mathrm{Hilb}^d_{X/S}(T) = \left\{ \begin{array}{l} Z \subset X_T \text{ closed subscheme such that} \\ Z \to T \text{ is finite locally free of degree } d \end{array} \right\}$$

If $T' \to T$ is a morphism of schemes over S and if $Z \in \mathrm{Hilb}^d_{X/S}(T)$, then the base change $Z_{T'} \subset X_{T'}$ is an element of $\mathrm{Hilb}^d_{X/S}(T')$. In this way we obtain a functor

$$\mathrm{Hilb}^d_{X/S} : (Sch/S)^{opp} \longrightarrow Sets, \quad T \longrightarrow \mathrm{Hilb}^d_{X/S}(T)$$

In general $\mathrm{Hilb}^d_{X/S}$ is an algebraic space (insert future reference here). In this section we will show that $\mathrm{Hilb}^d_{X/S}$ is representable by a scheme if any finite number of points in a fibre of $X \to S$ are contained in an affine open. If $\mathrm{Hilb}^d_{X/S}$ is representable by a scheme, we often denote this scheme by $\underline{\mathrm{Hilb}}^d_{X/S}$.

0B95 **Lemma 2.1.** *Let $X \to S$ be a morphism of schemes. The functor $\mathrm{Hilb}^d_{X/S}$ satisfies the sheaf property for the fpqc topology (Topologies, Definition 9.12).*

Proof. Let $\{T_i \to T\}_{i \in I}$ be an fpqc covering of schemes over S. Set $X_i = X_{T_i} = X \times_S T_i$. Note that $\{X_i \to X_T\}_{i \in I}$ is an fpqc covering of X_T (Topologies, Lemma 9.7) and that $X_{T_i \times_T T_{i'}} = X_i \times_{X_T} X_{i'}$. Suppose that $Z_i \in \mathrm{Hilb}^d_{X/S}(T_i)$ is a collection of elements such that Z_i and $Z_{i'}$ map to the same element of $\mathrm{Hilb}^d_{X/S}(T_i \times_T T_{i'})$. By effective descent for closed immersions (Descent, Lemma 37.2) there is a closed immersion $Z \to X_T$ whose base change by $X_i \to X_T$ is equal to $Z_i \to X_i$. The morphism $Z \to T$ then has the property that its base change to T_i is the morphism $Z_i \to T_i$. Hence $Z \to T$ is finite locally free of degree d by Descent, Lemma 23.30. $\square$

0B96 **Lemma 2.2.** *Let $X \to S$ be a morphism of schemes. If $X \to S$ is of finite presentation, then the functor $\mathrm{Hilb}^d_{X/S}$ is limit preserving (Limits, Remark 6.2).*

Proof. Let $T = \lim T_i$ be a limit of affine schemes over S. We have to show that $\mathrm{Hilb}^d_{X/S}(T) = \mathrm{colim} \, \mathrm{Hilb}^d_{X/S}(T_i)$. Observe that if $Z \to X_T$ is an element of $\mathrm{Hilb}^d_{X/S}(T)$, then $Z \to T$ is of finite presentation. Hence by Limits, Lemma 10.1 there exists an i, a scheme Z_i of finite presentation over T_i, and a morphism $Z_i \to X_{T_i}$ over T_i whose base change to T gives $Z \to X_T$. We apply Limits, Lemma 8.5 to see that we may assume $Z_i \to X_{T_i}$ is a closed immersion after increasing i. We apply Limits, Lemma 8.8 to see that $Z_i \to T_i$ is finite locally free of degree d after possibly increasing i. Then $Z_i \in \mathrm{Hilb}^d_{X/S}(T_i)$ as desired. $\square$

Let S be a scheme. Let $i : X \to Y$ be a closed immersion of schemes over S. Then there is a transformation of functors

$$\mathrm{Hilb}^d_{X/S} \longrightarrow \mathrm{Hilb}^d_{Y/S}$$

which maps an element $Z \in \mathrm{Hilb}^d_{X/S}(T)$ to $i_T(Z) \subset Y_T$ in $\mathrm{Hilb}^d_{Y/S}$. Here $i_T : X_T \to Y_T$ is the base change of i.

0B97 **Lemma 2.3.** *Let S be a scheme. Let $i : X \to Y$ be a closed immersion of schemes. If $\mathrm{Hilb}^d_{Y/S}$ is representable by a scheme, so is $\mathrm{Hilb}^d_{X/S}$ and the corresponding morphism of schemes $\underline{\mathrm{Hilb}}^d_{X/S} \to \underline{\mathrm{Hilb}}^d_{Y/S}$ is a closed immersion.*

Proof. Let T be a scheme over S and let $Z \in \mathrm{Hilb}^d_{Y/S}(T)$. Claim: there is a closed subscheme $T_X \subset T$ such that a morphism of schemes $T' \to T$ factors through T_X if and only if $Z_{T'} \to Y_{T'}$ factors through $X_{T'}$. Applying this to a scheme T_{univ} representing $\mathrm{Hilb}^d_{Y/S}$ and the universal object[1] $Z_{univ} \in \mathrm{Hilb}^d_{Y/S}(T_{univ})$ we get a closed subscheme $T_{univ,X} \subset T_{univ}$ such that $Z_{univ,X} = Z_{univ} \times_{T_{univ}} T_{univ,X}$ is a closed subscheme of $X \times_S T_{univ,X}$ and hence defines an element of $\mathrm{Hilb}^d_{X/S}(T_{univ,X})$. A formal argument then shows that $T_{univ,X}$ is a scheme representing $\mathrm{Hilb}^d_{X/S}$ with universal object $Z_{univ,X}$.

Proof of the claim. Consider $Z' = X_T \times_{Y_T} Z$. Given $T' \to T$ we see that $Z_{T'} \to Y_{T'}$ factors through $X_{T'}$ if and only if $Z'_{T'} \to Z_{T'}$ is an isomorphism. Thus the claim follows from the very general More on Flatness, Lemma 23.4. However, in this special case one can prove the statement directly as follows: first reduce to the case $T = \mathrm{Spec}(A)$ and $Z = \mathrm{Spec}(B)$. After shrinking T further we may assume there is an isomorphism $\varphi : B \to A^{\oplus d}$ as A-modules. Then $Z' = \mathrm{Spec}(B/J)$ for some ideal

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[1]See Categories, Section 3

$J \subset B$. Let $g_\beta \in J$ be a collection of generators and write $\varphi(g_\beta) = (g_\beta^1, \ldots, g_\beta^d)$. Then it is clear that T_X is given by $\mathrm{Spec}(A/(g_\beta^j))$. $\qquad \square$

0B98 **Lemma 2.4.** *Let $X \to S$ be a morphism of schemes. If $X \to S$ is separated and $\mathrm{Hilb}_{X/S}^d$ is representable, then $\underline{\mathrm{Hilb}}_{X/S}^d \to S$ is separated.*

Proof. In this proof all unadorned products are over S. Let $H = \underline{\mathrm{Hilb}}_{X/S}^d$ and let $Z \in \mathrm{Hilb}_{X/S}^d(H)$ be the universal object. Consider the two objects $Z_1, Z_2 \in \mathrm{Hilb}_{X/S}^d(H \times H)$ we get by pulling back Z by the two projections $H \times H \to H$. Then $Z_1 = Z \times H \subset X_{H \times H}$ and $Z_2 = H \times Z \subset X_{H \times H}$. Since H represents the functor $\mathrm{Hilb}_{X/S}^d$, the diagonal morphism $\Delta : H \to H \times H$ has the following universal property: A morphism of schemes $T \to H \times H$ factors through Δ if and only if $Z_{1,T} = Z_{2,T}$ as elements of $\mathrm{Hilb}_{X/S}^d(T)$. Set $Z = Z_1 \times_{X_{H \times H}} Z_2$. Then we see that $T \to H \times H$ factors through Δ if and only if the morphisms $Z_T \to Z_{1,T}$ and $Z_T \to Z_{2,T}$ are isomorphisms. It follows from the very general More on Flatness, Lemma 23.4 that Δ is a closed immersion. In the proof of Lemma 2.3 the reader finds an alternative easier proof of the needed result in our special case. $\qquad \square$

0B99 **Lemma 2.5.** *Let $X \to S$ be a morphism of affine schemes. Let $d \geq 0$. Then $\mathrm{Hilb}_{X/S}^d$ is representable.*

Proof. Say $S = \mathrm{Spec}(R)$. Then we can choose a closed immersion of X into the spectrum of $R[x_i; i \in I]$ for some set I (of sufficiently large cardinality. Hence by Lemma 2.3 we may assume that $X = \mathrm{Spec}(A)$ where $A = R[x_i; i \in I]$. We will use Schemes, Lemma 15.4 to prove the lemma in this case.

Condition (1) of the lemma follows from Lemma 2.1.

For every subset $W \subset A$ of cardinality d we will construct a subfunctor F_W of $\mathrm{Hilb}_{X/S}^d$. (It would be enough to consider the case where W consists of a collection of monomials in the x_i but we do not need this.) Namely, we will say that $Z \in \mathrm{Hilb}_{X/S}^d(T)$ is in $F_W(T)$ if and only if the $\mathcal{O}_T$-linear map

$$\bigoplus_{f \in W} \mathcal{O}_T \longrightarrow (Z \to T)_* \mathcal{O}_Z, \quad (g_f) \longmapsto \sum g_f f|_Z$$

is surjective (equivalently an isomorphism). Here for $f \in A$ and $Z \in \mathrm{Hilb}_{X/S}^d(T)$ we denote $f|_Z$ the pullback of f by the morphism $Z \to X_T \to X$.

Openness, i.e., condition (2)(b) of the lemma. This follows from Algebra, Lemma 79.4.

Covering, i.e., condition (2)(c) of the lemma. Since

$$A \otimes_R \mathcal{O}_T = (X_T \to T)_* \mathcal{O}_{X_T} \to (Z \to T)_* \mathcal{O}_Z$$

is surjective and since $(Z \to T)_* \mathcal{O}_Z$ is finite locally free of rank d, for every point $t \in T$ we can find a finite subset $W \subset A$ of cardinality d whose images form a basis of the d-dimensional $\kappa(t)$-vector space $((Z \to T)_* \mathcal{O}_Z)_t \otimes_{\mathcal{O}_{T,t}} \kappa(t)$. By Nakayama's lemma there is an open neighbourhood $V \subset T$ of t such that $Z_V \in F_W(V)$.

Representable, i.e., condition (2)(a) of the lemma. Let $W \subset A$ have cardinality d. We claim that F_W is representable by an affine scheme over R. We will construct this affine scheme here, but we encourage the reader to think it trough for themselves. Choose a numbering $f_1, \ldots, f_d$ of the elements of W. We will construct a

universal element $Z_{univ} = \mathrm{Spec}(B_{univ})$ of F_W over $T_{univ} = \mathrm{Spec}(R_{univ})$ which will be the spectrum of

$$B_{univ} = R_{univ}[e_1, \ldots, e_d]/(e_k e_l - \sum c_{kl}^m e_m)$$

where the e_l will be the images of the f_l and where the closed immersion $Z_{univ} \to X_{T_{univ}}$ is given by the ring map

$$A \otimes_R R_{univ} \longrightarrow B_{univ}$$

mapping $1 \otimes 1$ to $\sum b^l e_l$ and x_i to $\sum b_i^l e_l$. In fact, we claim that F_W is represented by the spectrum of the ring

$$R_{univ} = R[c_{kl}^m, b^l, b_i^l]/\mathfrak{a}_{univ}$$

where the ideal $\mathfrak{a}_{univ}$ is generated by the following elements:

  (1) multiplication on B_{univ} is commutative, i.e., $c_{lk}^m - c_{kl}^m \in \mathfrak{a}_{univ}$,
  (2) multiplication on B_{univ} is associative, i.e., $c_{lk}^m c_{mn}^p - c_{lq}^p c_{kn}^q \in \mathfrak{a}_{univ}$,
  (3) $\sum b^l e_l$ is a multiplicative 1 in B_{univ}, in other words, we should have $(\sum b^l e_l)e_k = e_k$ for all k, which means $\sum b^l c_{lk}^m - \delta_{km} \in \mathfrak{a}_{univ}$ (Kronecker delta).

After dividing out by the ideal $\mathfrak{a}'_{univ}$ of the elements listed sofar we obtain a well defined ring map

$$\Psi : A \otimes_R R[c_{kl}^m, b^l, b_i^l]/\mathfrak{a}'_{univ} \longrightarrow \left(R[c_{kl}^m, b^l, b_i^l]/\mathfrak{a}'_{univ}\right)[e_1, \ldots, e_d]/(e_k e_l - \sum c_{kl}^m e_m)$$

sending $1 \otimes 1$ to $\sum b^l e_l$ and $x_i \otimes 1$ to $\sum b_i^l e_l$. We need to add some more elements to our ideal because we need

  (5) f_l to map to e_l in B_{univ}. Write $\Psi(f_l) - e_l = \sum h_l^m e_m$ with $h_l^m \in R[c_{kl}^m, b^l, b_i^l]/\mathfrak{a}'_{univ}$ then we need to set h_l^m equal to zero.

Thus setting $\mathfrak{a}_{univ} \subset R[c_{kl}^m, b^l, b_i^l]$ equal to $\mathfrak{a}'_{univ}+$ ideal generated by lifts of h_l^m to $R[c_{kl}^m, b^l, b_i^l]$, then it is clear that F_W is represented by $\mathrm{Spec}(R_{univ})$. $\square$

0B9A   **Proposition 2.6.** *Let $X \to S$ be a morphism of schemes. Let $d \geq 0$. Assume for all $(s, x_1, \ldots, x_d)$ where $s \in S$ and $x_1, \ldots, x_d \in X_s$ there exists an affine open $U \subset X$ with $x_1, \ldots, x_d \in U$. Then $\mathrm{Hilb}_{X/S}^d$ is representable by a scheme.*

Proof. Either using relative glueing (Constructions, Section 2) or using the functorial point of view (Schemes, Lemma 15.4) we reduce to the case where S is affine. Details omitted.

Assume S is affine. For $U \subset X$ affine open, denote $F_U \subset \mathrm{Hilb}_{X/S}^d$ the subfunctor such that for a scheme T/S an element $Z \in \mathrm{Hilb}_{X/S}^d(T)$ is in $F_U(T)$ if and only if $Z \subset U_T$. We will use Schemes, Lemma 15.4 and the subfunctors F_U to conclude.

Condition (1) is Lemma 2.1.

Condition (2)(a) follows from the fact that $F_U = \mathrm{Hilb}_{U/S}^d$ and that this is representable by Lemma 2.5. Namely, if $Z \in F_U(T)$, then Z can be viewed as a closed subscheme of U_T which is finite locally free of degree d over T and hence $Z \in \mathrm{Hilb}_{U/S}^d(T)$. Conversely, if $Z \in \mathrm{Hilb}_{U/S}^d(T)$ then $Z \to U_T \to X_T$ is a closed immersion[2] and we may view Z as an element of $F_U(T)$.

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[2]This is clear if $X \to S$ is separated as in this case Morphisms, Lemma 41.7 tells us that the immersion $\varphi : Z \to X_T$ has closed image and hence is a closed immersion by Schemes, Lemma 10.4. We suggest the reader skip the rest of this footnote as we don't know of any instance where

Let $Z \in \mathrm{Hilb}^d_{X/S}(T)$ for some scheme T over S. Let

$$B = (Z \to T)\left((Z \to X_T \to X)^{-1}(X \setminus U)\right)$$

This is a closed subset of T and it is clear that over the open $T_{Z,U} = T \setminus B$ the restriction $Z_{t'}$ maps into $U_{T'}$. On the other hand, for any $b \in B$ the fibre Z_b does not map into U. Thus we see that given a morphism $T' \to T$ we have $Z_{T'} \in F_U(T')$ $\Leftrightarrow T' \to T$ factors through the open $T_{Z,U}$. This proves condition (2)(b).

Condition (2)(c) follows from our assumption on X/S. All we have to do is show the following: If T is the spectrum of a field and $Z \subset X_T$ is a closed subscheme, finite flat of degree d over T, then $Z \to X_T \to X$ factors through an affine open U of X. This is clear because Z will have at most d points and these will all map into the fibre of X over the image point of $T \to S$. $\square$

0B9B **Remark 2.7.** Let $f : X \to S$ be a morphism of schemes. The assumption of Proposition 2.6 and hence the conclusion holds in each of the following cases:

(1) X is quasi-affine,
(2) f is quasi-affine,
(3) f is quasi-projective,
(4) f is locally projective,
(5) there exists an ample invertible sheaf on X,
(6) there exists an f-ample invertible sheaf on X, and
(7) there exists an f-very ample invertible sheaf on X.

Namely, in each of these cases, every finite set of points of a fibre X_s is contained in a quasi-compact open U of X which comes with an ample invertible sheaf, is isomorphic to an open of an affine scheme, or is isomorphic to an open of Proj of a graded ring (in each case this follows by unwinding the definitions). Thus the existence of suitable affine opens by Properties, Lemma 29.5.

3. Moduli of divisors on smooth curves

0B9C For a smooth morphism $X \to S$ of relative dimension 1 the functor $\mathrm{Hilb}^d_{X/S}$ parametrizes relative effective Cartier divisors as defined in Divisors, Section 18.

0B9D **Lemma 3.1.** *Let $X \to S$ be a smooth morphism of schemes of relative dimension 1. Let $D \subset X$ be a closed subscheme. Consider the following conditions*

(1) $D \to S$ *is finite locally free,*
(2) D *is a relative effective Cartier divisor on X/S,*
(3) $D \to S$ *is locally quasi-finite, flat, and locally of finite presentation, and*
(4) $D \to S$ *is locally quasi-finite and flat.*

We always have the implications

$$(1) \Rightarrow (2) \Leftrightarrow (3) \Rightarrow (4)$$

If S is locally Noetherian, then the last arrow is an if and only if. If $X \to S$ is proper (and S arbitrary), then the first arrow is an if and only if.

the assumptions on $X \to S$ hold but $X \to S$ is not separated. In the general case, let $x \in X_T$ be a point in the closure of $\varphi(Z)$. We have to show that $x \in \varphi(Z)$. Let $t \in T$ be the image of x. By assumption on $X \to S$ we can choose an affine open $W \subset X_T$ containing x and $\varphi(Z_t)$. Then $\varphi^{-1}(W)$ is an open containing the whole fibre Z_t and since $Z \to T$ is closed, we may after replacing T by an open neighbourhood of t assume that $Z = \varphi^{-1}(W)$. Then $\varphi(Z) \subset W$ is closed by the separated case (as $W \to T$ is separated) and we conclude $x \in \varphi(Z)$.

Proof. Equivalence of (2) and (3). This follows from Divisors, Lemma 18.9 if we can show the equivalence of (2) and (3) when S is the spectrum of a field k. Let $x \in X$ be a closed point. As X is smooth of relative dimension 1 over k and we see that $\mathcal{O}_{X,x}$ is a regular local ring of dimension 1 (see Varieties, Lemma 25.3). Thus $\mathcal{O}_{X,x}$ is a discrete valuation ring (Algebra, Lemma 119.7) and hence a PID. It follows that every sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ which is nonvanishing at all the generic points of X is invertible (Divisors, Lemma 15.2). In other words, every closed subscheme of X which does not contain a generic point is an effective Cartier divisor. It follows that (2) and (3) are equivalent.

If S is Noetherian, then any locally quasi-finite morphism $D \to S$ is locally of finite presentation (Morphisms, Lemma 21.9), whence (3) is equivalent to (4).

If $X \to S$ is proper (and S is arbitrary), then $D \to S$ is proper as well. Since a proper locally quasi-finite morphism is finite (More on Morphisms, Lemma 44.1) and a finite, flat, and finitely presented morphism is finite locally free (Morphisms, Lemma 48.2), we see that (1) is equivalent to (2). $\qquad\square$

0B9E **Lemma 3.2.** *Let $X \to S$ be a smooth morphism of schemes of relative dimension 1. Let $D_1, D_2 \subset X$ be closed subschemes finite locally free of degrees d_1, d_2 over S. Then $D_1 + D_2$ is finite locally free of degree $d_1 + d_2$ over S.*

Proof. By Lemma 3.1 we see that D_1 and D_2 are relative effective Cartier divisors on X/S. Thus $D = D_1 + D_2$ is a relative effective Cartier divisor on X/S by Divisors, Lemma 18.3. Hence $D \to S$ is locally quasi-finite, flat, and locally of finite presentation by Lemma 3.1. Applying Morphisms, Lemma 41.11 the surjective integral morphism $D_1 \amalg D_2 \to D$ we find that $D \to S$ is separated. Then Morphisms, Lemma 41.9 implies that $D \to S$ is proper. This implies that $D \to S$ is finite (More on Morphisms, Lemma 44.1) and in turn we see that $D \to S$ is finite locally free (Morphisms, Lemma 48.2). Thus it suffice to show that the degree of $D \to S$ is $d_1 + d_2$. To do this we may base change to a fibre of $X \to S$, hence we may assume that $S = \mathrm{Spec}(k)$ for some field k. In this case, there exists a finite set of closed points $x_1, \dots, x_n \in X$ such that D_1 and D_2 are supported on $\{x_1, \dots, x_n\}$. In fact, there are nonzerodivisors $f_{i,j} \in \mathcal{O}_{X,x_i}$ such that

$$D_1 = \coprod \mathrm{Spec}(\mathcal{O}_{X,x_i}/(f_{i,1})) \quad \text{and} \quad D_2 = \coprod \mathrm{Spec}(\mathcal{O}_{X,x_i}/(f_{i,2}))$$

Then we see that

$$D = \coprod \mathrm{Spec}(\mathcal{O}_{X,x_i}/(f_{i,1}f_{i,2}))$$

From this one sees easily that D has degree $d_1 + d_2$ over k (if need be, use Algebra, Lemma 121.1). $\qquad\square$

0B9F **Lemma 3.3.** *Let $X \to S$ be a smooth morphism of schemes of relative dimension 1. Let $D_1, D_2 \subset X$ be closed subschemes finite locally free of degrees d_1, d_2 over S. If $D_1 \subset D_2$ (as closed subschemes) then there is a closed subscheme $D \subset X$ finite locally free of degree $d_2 - d_1$ over S such that $D_2 = D_1 + D$.*

Proof. This proof is almost exactly the same as the proof of Lemma 3.2. By Lemma 3.1 we see that D_1 and D_2 are relative effective Cartier divisors on X/S. By Divisors, Lemma 18.4 there is a relative effective Cartier divisor $D \subset X$ such that $D_2 = D_1 + D$. Hence $D \to S$ is locally quasi-finite, flat, and locally of finite presentation by Lemma 3.1. Since D is a closed subscheme of D_2, we see that

$D \to S$ is finite. It follows that $D \to S$ is finite locally free (Morphisms, Lemma 48.2). Thus it suffice to show that the degree of $D \to S$ is $d_2 - d_1$. This follows from Lemma 3.2. $\square$

Let $X \to S$ be a smooth morphism of schemes of relative dimension 1. By Lemma 3.1 for a scheme T over S and $D \in \mathrm{Hilb}^d_{X/S}(T)$, we can view D as a relative effective Cartier divisor on X_T/T such that $D \to T$ is finite locally free of degree d. Hence, by Lemma 3.2 we obtain a transformation of functors

$$\mathrm{Hilb}^{d_1}_{X/S} \times \mathrm{Hilb}^{d_2}_{X/S} \longrightarrow \mathrm{Hilb}^{d_1+d_2}_{X/S}, \quad (D_1, D_2) \longmapsto D_1 + D_2$$

If $\mathrm{Hilb}^d_{X/S}$ is representable for all degrees d, then this transformation of functors corresponds to a morphism of schemes

$$\underline{\mathrm{Hilb}}^{d_1}_{X/S} \times_S \underline{\mathrm{Hilb}}^{d_2}_{X/S} \longrightarrow \underline{\mathrm{Hilb}}^{d_1+d_2}_{X/S}$$

over S. Observe that $\underline{\mathrm{Hilb}}^0_{X/S} = S$ and $\underline{\mathrm{Hilb}}^1_{X/S} = X$. A special case of the morphism above is the morphism

$$\underline{\mathrm{Hilb}}^d_{X/S} \times_S X \longrightarrow \underline{\mathrm{Hilb}}^{d+1}_{X/S}, \quad (D, x) \longmapsto D + x$$

0B9G **Lemma 3.4.** *Let $X \to S$ be a smooth morphism of schemes of relative dimension* 1 *such that the functors* $\mathrm{Hilb}^d_{X/S}$ *are representable. The morphism* $\underline{\mathrm{Hilb}}^d_{X/S} \times_S X \to \underline{\mathrm{Hilb}}^{d+1}_{X/S}$ *is finite locally free of degree* $d + 1$.

Proof. Let $D_{univ} \subset X \times_S \underline{\mathrm{Hilb}}^{d+1}_{X/S}$ be the universal object. There is a commutative diagram

$$\underline{\mathrm{Hilb}}^d_{X/S} \times_S X \longrightarrow D_{univ} \hookrightarrow \underline{\mathrm{Hilb}}^{d+1}_{X/S} \times_S X$$
$$\underline{\mathrm{Hilb}}^{d+1}_{X/S}$$

where the top horizontal arrow maps (D', x) to $(D' + x, x)$. We claim this morphism is an isomorphism which certainly proves the lemma. Namely, given a scheme T over S, a T-valued point ξ of D_{univ} is given by a pair $\xi = (D, x)$ where $D \subset X_T$ is a closed subscheme finite locally free of degree $d + 1$ over T and $x : T \to X$ is a morphism whose graph $x : T \to X_T$ factors through D. Then by Lemma 3.3 we can write $D = D' + x$ for some $D' \subset X_T$ finite locally free of degree d over T. Sending $\xi = (D, x)$ to the pair (D', x) is the desired inverse. $\square$

0B9H **Lemma 3.5.** *Let $X \to S$ be a smooth morphism of schemes of relative dimension* 1 *such that the functors* $\mathrm{Hilb}^d_{X/S}$ *are representable. The schemes* $\underline{\mathrm{Hilb}}^d_{X/S}$ *are smooth over S of relative dimension d.*

Proof. We have $\underline{\mathrm{Hilb}}^0_{X/S} = S$ and $\underline{\mathrm{Hilb}}^1_{X/S} = X$ thus the result is true for $d = 0, 1$. Assuming the result for d, we see that $\underline{\mathrm{Hilb}}^d_{X/S} \times_S X$ is smooth over S (Morphisms, Lemma 34.5 and 34.4). Since $\underline{\mathrm{Hilb}}^d_{X/S} \times_S X \to \underline{\mathrm{Hilb}}^{d+1}_{X/S}$ is finite locally free of degree $d + 1$ by Lemma 3.4 the result follows from Descent, Lemma 14.5. We omit the verification that the relative dimension is as claimed (you can do this by looking at fibres, or by keeping track of the dimensions in the argument above). $\square$

We collect all the information obtained sofar in the case of a proper smooth curve over a field.

0B9I **Proposition 3.6.** *Let X be a geometrically irreducible smooth proper curve over a field k.*

(1) *The functors $\mathrm{Hilb}^d_{X/k}$ are representable by smooth proper varieties $\underline{\mathrm{Hilb}}^d_{X/k}$ of dimension d over k.*

(2) *For a field extension k'/k the k'-rational points of $\underline{\mathrm{Hilb}}^d_{X/k}$ are in 1-to-1 bijection with effective Cartier divisors of degree d on $X_{k'}$.*

(3) *For $d_1, d_2 \geq 0$ there is a morphism*

$$\underline{\mathrm{Hilb}}^{d_1}_{X/k} \times_k \underline{\mathrm{Hilb}}^{d_2}_{X/k} \longrightarrow \underline{\mathrm{Hilb}}^{d_1+d_2}_{X/k}$$

which is finite locally free of degree $\binom{d_1+d_2}{d_1}$.

Proof. The functors $\mathrm{Hilb}^d_{X/k}$ are representable by Proposition 2.6 (see also Remark 2.7) and the fact that X is projective (Varieties, Lemma 43.4). The schemes $\underline{\mathrm{Hilb}}^d_{X/k}$ are separated over k by Lemma 2.4. The schemes $\underline{\mathrm{Hilb}}^d_{X/k}$ are smooth over k by Lemma 3.5. Starting with $X = \underline{\mathrm{Hilb}}^1_{X/k}$, the morphisms of Lemma 3.4, and induction we find a morphism

$$X^d = X \times_k X \times_k \ldots \times_k X \longrightarrow \underline{\mathrm{Hilb}}^d_{X/k}, \quad (x_1, \ldots, x_d) \longrightarrow x_1 + \ldots + x_d$$

which is finite locally free of degree $d!$. Since X is proper over k, so is X^d, hence $\underline{\mathrm{Hilb}}^d_{X/k}$ is proper over k by Morphisms, Lemma 41.9. Since X is geometrically irreducible over k, the product X^d is irreducible (Varieties, Lemma 8.4) hence the image is irreducible (in fact geometrically irreducible). This proves (1). Part (2) follows from the definitions. Part (3) follows from the commutative diagram

$$
\begin{array}{ccc}
X^{d_1} \times_k X^{d_2} & =\!\!=\!\!=\!\!=\!\!=\!\!=\!\!= & X^{d_1+d_2} \\
\downarrow & & \downarrow \\
\underline{\mathrm{Hilb}}^{d_1}_{X/k} \times_k \underline{\mathrm{Hilb}}^{d_2}_{X/k} & \longrightarrow & \underline{\mathrm{Hilb}}^{d_1+d_2}_{X/k}
\end{array}
$$

and multiplicativity of degrees of finite locally free morphisms. $\qquad\square$

0B9J **Remark 3.7.** Let X be a geometrically irreducible smooth proper curve over a field k as in Proposition 3.6. Let $d \geq 0$. The universal closed object is a relatively effective divisor

$$D_{univ} \subset \underline{\mathrm{Hilb}}^{d+1}_{X/k} \times_k X$$

over $\underline{\mathrm{Hilb}}^{d+1}_{X/k}$ by Lemma 3.1. In fact, D_{univ} is isomorphic as a scheme to $\underline{\mathrm{Hilb}}^d_{X/k} \times_k X$, see proof of Lemma 3.4. In particular, D_{univ} is an effective Cartier divisor and we obtain an invertible module $\mathcal{O}(D_{univ})$. If $[D] \in \underline{\mathrm{Hilb}}^{d+1}_{X/k}$ denotes the k-rational point corresponding to the effective Cartier divisor $D \subset X$ of degree $d + 1$, then the restriction of $\mathcal{O}(D_{univ})$ to the fibre $[D] \times X$ is $\mathcal{O}_X(D)$.

4. The Picard functor

0B9K Given any scheme X we denote $\mathrm{Pic}(X)$ the set of isomorphism classes of invertible $\mathcal{O}_X$-modules. See Modules, Definition 25.9. Given a morphism $f : X \to Y$ of

schemes, pullback defines a group homomorphism $\text{Pic}(Y) \to \text{Pic}(X)$. The assignment $X \rightsquigarrow \text{Pic}(X)$ is a contravariant functor from the category of schemes to the category of abelian groups. This functor is not representable, but it turns out that a relative variant of this construction sometimes is representable.

Let us define the Picard functor for a morphism of schemes $f : X \to S$. The idea behind our construction is that we'll take it to be the sheaf $R^1 f_* \mathbf{G}_m$ where we use the fppf topology to compute the higher direct image. Unwinding the definitions this leads to the following more direct definition.

0B9L **Definition 4.1.** Let Sch_{fppf} be a big site as in Topologies, Definition 7.8. Let $f : X \to S$ be a morphism of this site. The *Picard functor* $\text{Pic}_{X/S}$ is the fppf sheafification of the functor

$$(Sch/S)_{fppf} \longrightarrow Sets, \quad T \longmapsto \text{Pic}(X_T)$$

If this functor is representable, then we denote $\underline{\text{Pic}}_{X/S}$ a scheme representing it.

An often used remark is that if $T \in \text{Ob}((Sch/S)_{fppf})$, then $\text{Pic}_{X_T/T}$ is the restriction of $\text{Pic}_{X/S}$ to $(Sch/T)_{fppf}$. It turns out to be nontrivial to see what the value of $\text{Pic}_{X/S}$ is on schemes T over S. Here is a lemma that helps with this task.

0B9M **Lemma 4.2.** *Let $f : X \to S$ be as in Definition 4.1. If $\mathcal{O}_T \to f_{T,*}\mathcal{O}_{X_T}$ is an isomorphism for all $T \in \text{Ob}((Sch/S)_{fppf})$, then*

$$0 \to \text{Pic}(T) \to \text{Pic}(X_T) \to \text{Pic}_{X/S}(T)$$

is an exact sequence for all T.

Proof. We may replace S by T and X by X_T and assume that $S = T$ to simplify the notation. Let $\mathcal{N}$ be an invertible $\mathcal{O}_S$-module. If $f^* \mathcal{N} \cong \mathcal{O}_X$, then we see that $f_* f^* \mathcal{N} \cong f_* \mathcal{O}_X \cong \mathcal{O}_S$ by assumption. Since $\mathcal{N}$ is locally trivial, we see that the canonical map $\mathcal{N} \to f_* f^* \mathcal{N}$ is locally an isomorphism (because $\mathcal{O}_S \to f_* f^* \mathcal{O}_S$ is an isomorphism by assumption). Hence we conclude that $\mathcal{N} \to f_* f^* \mathcal{N} \to \mathcal{O}_S$ is an isomorphism and we see that $\mathcal{N}$ is trivial. This proves the first arrow is injective.

Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module which is in the kernel of $\text{Pic}(X) \to \text{Pic}_{X/S}(S)$. Then there exists an fppf covering $\{S_i \to S\}$ such that $\mathcal{L}$ pulls back to the trivial invertible sheaf on X_{S_i}. Choose a trivializing section s_i. Then $\text{pr}_0^* s_i$ and $\text{pr}_1^* s_j$ are both trivialising sections of $\mathcal{L}$ over $X_{S_i \times_S S_j}$ and hence differ by a multiplicative unit

$$f_{ij} \in \Gamma(X_{S_i \times_S S_j}, \mathcal{O}^*_{X_{S_i \times_S S_j}}) = \Gamma(S_i \times_S S_j, \mathcal{O}^*_{S_i \times_S S_j})$$

(equality by our assumption on pushforward of structure sheaves). Of course these elements satisfy the cocycle condition on $S_i \times_S S_j \times_S S_k$, hence they define a descent datum on invertible sheaves for the fppf covering $\{S_i \to S\}$. By Descent, Proposition 5.2 there is an invertible $\mathcal{O}_S$-module $\mathcal{N}$ with trivializations over S_i whose associated descent datum is $\{f_{ij}\}$. Then $f^* \mathcal{N} \cong \mathcal{L}$ as the functor from descent data to modules is fully faithful (see proposition cited above). □

0B9N **Lemma 4.3.** *Let $f : X \to S$ be as in Definition 4.1. Assume f has a section σ and that $\mathcal{O}_T \to f_{T,*}\mathcal{O}_{X_T}$ is an isomorphism for all $T \in \text{Ob}((Sch/S)_{fppf})$. Then*

$$0 \to \text{Pic}(T) \to \text{Pic}(X_T) \to \text{Pic}_{X/S}(T) \to 0$$

is a split exact sequence with splitting given by $\sigma_T^* : \text{Pic}(X_T) \to \text{Pic}(T)$.

Proof. Denote $K(T) = \operatorname{Ker}(\sigma_T^* : \operatorname{Pic}(X_T) \to \operatorname{Pic}(T))$. Since σ is a section of f we see that $\operatorname{Pic}(X_T)$ is the direct sum of $\operatorname{Pic}(T)$ and $K(T)$. Thus by Lemma 4.2 we see that $K(T) \subset \operatorname{Pic}_{X/S}(T)$ for all T. Moreover, it is clear from the construction that $\operatorname{Pic}_{X/S}$ is the sheafification of the presheaf K. To finish the proof it suffices to show that K satisfies the sheaf condition for fppf coverings which we do in the next paragraph.

Let $\{T_i \to T\}$ be an fppf covering. Let $\mathcal{L}_i$ be elements of $K(T_i)$ which map to the same elements of $K(T_i \times_T T_j)$ for all i and j. Choose an isomorphism $\alpha_i : \mathcal{O}_{T_i} \to \sigma_{T_i}^* \mathcal{L}_i$ for all i. Choose an isomorphism

$$\varphi_{ij} : \mathcal{L}_i|_{X_{T_i \times_T T_j}} \longrightarrow \mathcal{L}_j|_{X_{T_i \times_T T_j}}$$

If the map

$$\alpha_j|_{T_i \times_T T_j} \circ \sigma_{T_i \times_T T_j}^* \varphi_{ij} \circ \alpha_i|_{T_i \times_T T_j} : \mathcal{O}_{T_i \times_T T_j} \to \mathcal{O}_{T_i \times_T T_j}$$

is not equal to multiplication by 1 but some u_{ij}, then we can scale φ_{ij} by u_{ij}^{-1} to correct this. Having done this, consider the self map

$$\varphi_{ki}|_{X_{T_i \times_T T_j \times_T T_k}} \circ \varphi_{jk}|_{X_{T_i \times_T T_j \times_T T_k}} \circ \varphi_{ij}|_{X_{T_i \times_T T_j \times_T T_k}} \quad \text{on} \quad \mathcal{L}_i|_{X_{T_i \times_T T_j \times_T T_k}}$$

which is given by multiplication by some regular function f_{ijk} on the scheme $X_{T_i \times_T T_j \times_T T_k}$. By our choice of φ_{ij} we see that the pullback of this map by σ is equal to multiplication by 1. By our assumption on functions on X, we see that $f_{ijk} = 1$. Thus we obtain a descent datum for the fppf covering $\{X_{T_i} \to X\}$. By Descent, Proposition 5.2 there is an invertible $\mathcal{O}_{X_T}$-module $\mathcal{L}$ and an isomorphism $\alpha : \mathcal{O}_T \to \sigma_T^* \mathcal{L}$ whose pullback to X_{T_i} recovers $(\mathcal{L}_i, \alpha_i)$ (small detail omitted). Thus $\mathcal{L}$ defines an object of $K(T)$ as desired. $\square$

5. A representability criterion

0B9P To prove the Picard functor is representable we will use the following criterion.

0B9Q **Lemma 5.1.** *Let k be a field. Let $G : (Sch/k)^{opp} \to Groups$ be a functor. With terminology as in Schemes, Definition 15.3, assume that*

    (1) *G satisfies the sheaf property for the Zariski topology,*
    (2) *there exists a subfunctor $F \subset G$ such that*
        (a) *F is representable,*
        (b) *$F \subset G$ is representable by open immersion,*
        (c) *for every field extension K of k and $g \in G(K)$ there exists a $g' \in G(k)$ such that $g'g \in F(K)$.*

Then G is representable by a group scheme over k.

Proof. This follows from Schemes, Lemma 15.4. Namely, take $I = G(k)$ and for $i = g' \in I$ take $F_i \subset G$ the subfunctor which associates to T over k the set of elements $g \in G(T)$ with $g'g \in F(T)$. Then $F_i \cong F$ by multiplication by g'. The map $F_i \to G$ is isomorphic to the map $F \to G$ by multiplication by g', hence is representable by open immersions. Finally, the collection $(F_i)_{i \in I}$ covers G by assumption $(2)(c)$. Thus the lemma mentioned above applies and the proof is complete. $\square$

6. The Picard scheme of a curve

0B9R   In this section we will apply Lemma 5.1 to show that $\mathrm{Pic}_{X/k}$ is representable, when k is an algebraically closed field and X is a smooth projective curve over k. To make this work we use a bit of cohomology and base change developed in the chapter on derived categories of schemes.

0B9U   **Lemma 6.1.** *Let k be a field. Let X be a smooth projective curve over k which has a k-rational point. Then the hypotheses of Lemma 4.3 are satisfied.*

Proof. The meaning of the phrase "has a k-rational point" is exactly that the structure morphism $f : X \to \mathrm{Spec}(k)$ has a section, which verifies the first condition. By Varieties, Lemma 26.2 we see that $k' = H^0(X, \mathcal{O}_X)$ is a field extension of k. Since X has a k-rational point there is a k-algebra homomorphism $k' \to k$ and we conclude $k' = k$. Since k is a field, any morphism $T \to \mathrm{Spec}(k)$ is flat. Hence we see by cohomology and base change (Cohomology of Schemes, Lemma 5.2) that $\mathcal{O}_T \to f_{T,*}\mathcal{O}_{X_T}$ is an isomorphism. This finishes the proof. $\square$

Let X be a smooth projective curve over a field k with a k-rational point σ. Then the functor

$$\mathrm{Pic}_{X/k,\sigma} : (Sch/k)^{opp} \longrightarrow Ab, \quad T \longmapsto \mathrm{Ker}(\mathrm{Pic}(X_T) \xrightarrow{\sigma_T^*} \mathrm{Pic}(T))$$

is isomorphic to $\mathrm{Pic}_{X/k}$ on $(Sch/k)_{fppf}$ by Lemmas 6.1 and 4.3. Hence it will suffice to prove that $\mathrm{Pic}_{X/k,\sigma}$ is representable. We will use the notation "$\mathcal{L} \in \mathrm{Pic}_{X/k,\sigma}(T)$" to signify that T is a scheme over k and $\mathcal{L}$ is an invertible $\mathcal{O}_{X_T}$-module whose restriction to T via σ_T is isomorphic to $\mathcal{O}_T$.

0B9V   **Lemma 6.2.** *Let k be a field. Let X be a smooth projective curve over k with a k-rational point σ. For a scheme T over k, consider the subset $F(T) \subset \mathrm{Pic}_{X/k,\sigma}(T)$ consisting of $\mathcal{L}$ such that $Rf_{T,*}\mathcal{L}$ is isomorphic to an invertible $\mathcal{O}_T$-module placed in degree 0. Then $F \subset \mathrm{Pic}_{X/k,\sigma}$ is a subfunctor and the inclusion is representable by open immersions.*

Proof. Immediate from Derived Categories of Schemes, Lemma 32.3 applied with $i = 0$ and $r = 1$ and Schemes, Definition 15.3. $\square$

To continue it is convenient to make the following definition.

0B9W   **Definition 6.3.** Let k be a field. Let X be a smooth projective geometrically irreducible curve over k. The *genus* of X is $g = \dim_k H^1(X, \mathcal{O}_X)$.

0B9X   **Lemma 6.4.** *Let k be a field. Let X be a smooth projective curve of genus g over k with a k-rational point σ. The open subfunctor F defined in Lemma 6.2 is representable by an open subscheme of $\underline{\mathrm{Hilb}}^g_{X/k}$.*

Proof. In this proof unadorned products are over $\mathrm{Spec}(k)$. By Proposition 3.6 the scheme $H = \underline{\mathrm{Hilb}}^g_{X/k}$ exists. Consider the universal divisor $D_{univ} \subset H \times X$ and the associated invertible sheaf $\mathcal{O}(D_{univ})$, see Remark 3.7. We adjust by tensoring with the pullback via $\sigma_H : H \to H \times X$ to get

$$\mathcal{L}_H = \mathcal{O}(D_{univ}) \otimes_{\mathcal{O}_{H \times X}} \mathrm{pr}_H^* \sigma_H^* \mathcal{O}(D_{univ})^{\otimes -1} \in \mathrm{Pic}_{X/k,\sigma}(H)$$

By the Yoneda lemma (Categories, Lemma 3.5) the invertible sheaf $\mathcal{L}_H$ defines a natural transformation

$$h_H \longrightarrow \mathrm{Pic}_{X/k,\sigma}$$

Because F is an open subfuctor, there exists a maximal open $W \subset H$ such that $\mathcal{L}_H|_{W \times X}$ is in $F(W)$. Of course, this open is nothing else than the open subscheme constructed in Derived Categories of Schemes, Lemma 32.3 with $i = 0$ and $r = 1$ for the morphism $H \times X \to H$ and the sheaf $\mathcal{F} = \mathcal{O}(D_{univ})$. Applying the Yoneda lemma again we obtain a commutative diagram

$$
\begin{array}{ccc}
h_W & \longrightarrow & F \\
\downarrow & & \downarrow \\
h_H & \longrightarrow & \mathrm{Pic}_{X/k,\sigma}
\end{array}
$$

To finish the proof we will show that the top horizontal arrow is an isomorphism.

Let $\mathcal{L} \in F(T) \subset \mathrm{Pic}_{X/k,\sigma}(T)$. Let $\mathcal{N}$ be the invertible $\mathcal{O}_T$-module such that $Rf_{T,*}\mathcal{L} \cong \mathcal{N}[0]$. The adjunction map

$$f_T^*\mathcal{N} \longrightarrow \mathcal{L} \quad \text{corresponds to a section } s \text{ of} \quad \mathcal{L} \otimes f_T^*\mathcal{N}^{\otimes -1}$$

on X_T. Claim: The zero scheme of s is a relative effective Cartier divisor D on $(T \times X)/T$ finite locally free of degree g over T.

Let us finish the proof of the lemma admitting the claim. Namely, D defines a morphism $m : T \to H$ such that D is the pullback of D_{univ}. Then

$$(m \times \mathrm{id}_X)^*\mathcal{O}(D_{univ}) \cong \mathcal{O}_{T \times X}(D)$$

Hence $(m \times \mathrm{id}_X)^*\mathcal{L}_H$ and $\mathcal{O}(D)$ differ by the pullback of an invertible sheaf on H. This in particular shows that $m : T \to H$ factors through the open $W \subset H$ above. Moreover, it follows that these invertible modules define, after adjusting by pullback via σ_T as above, the same element of $\mathrm{Pic}_{X/k,\sigma}(T)$. Chasing diagrams using Yoneda's lemma we see that $m \in h_W(T)$ maps to $\mathcal{L} \in F(T)$. We omit the verification that the rule $F(T) \to h_W(T)$, $\mathcal{L} \mapsto m$ defines an inverse of the transformation of functors above.

Proof of the claim. Since D is a locally principal closed subscheme of $T \times X$, it suffices to show that the fibres of D over T are effective Cartier divisors, see Lemma 3.1 and Divisors, Lemma 18.9. Because taking cohomology of $\mathcal{L}$ commutes with base change (Derived Categories of Schemes, Lemma 30.4) we reduce to $T = \mathrm{Spec}(K)$ where K/k is a field extension. Then $\mathcal{L}$ is an invertible sheaf on X_K with $H^0(X_K, \mathcal{L}) = K$ and $H^1(X_K, \mathcal{L}) = 0$. Thus

$$\deg(\mathcal{L}) = \chi(X_K, \mathcal{L}) - \chi(X_K, \mathcal{O}_{X_K}) = 1 - (1 - g) = g$$

See Varieties, Definition 44.1. To finish the proof we have to show a nonzero section of $\mathcal{L}$ defines an effective Cartier divisor on X_K. This is clear. $\qquad\square$

0B9Y **Lemma 6.5.** *Let k be a separably closed field. Let X be a smooth projective curve of genus g over k. Let K/k be a field extension and let $\mathcal{L}$ be an invertible sheaf on X_K. Then there exists an invertible sheaf $\mathcal{L}_0$ on X such that $\dim_K H^0(X_K, \mathcal{L} \otimes_{\mathcal{O}_{X_K}} \mathcal{L}_0|_{X_K}) = 1$ and $\dim_K H^1(X_K, \mathcal{L} \otimes_{\mathcal{O}_{X_K}} \mathcal{L}_0|_{X_K}) = 0$.*

Proof. This proof is a variant of the proof of Varieties, Lemma 44.16. We encourage the reader to read that proof first.

First we pick an ample invertible sheaf $\mathcal{L}_0$ and we replace $\mathcal{L}$ by $\mathcal{L} \otimes_{\mathcal{O}_{X_K}} \mathcal{L}_0^{\otimes n}|_{X_K}$ for some $n \gg 0$. The result will be that we may assume that $H^0(X_K, \mathcal{L}) \neq 0$ and

$H^1(X_K, \mathcal{L}) = 0$. Namely, we will get the vanishing by Cohomology of Schemes, Lemma 17.1 and the nonvanishing because the degree of the tensor product is $\gg 0$. We will finish the proof by descending induction on $t = \dim_K H^0(X_K, \mathcal{L})$. The base case $t = 1$ is trivial. Assume $t > 1$.

Observe that for a k-rational point x of X, the inverse image x_K is a K-rational point of X_K. Moreover, there are infinitely many k-rational points by Varieties, Lemma 25.6. Therefore the points x_K form a Zariski dense collection of points of X_K.

Let $s \in H^0(X_K, \mathcal{L})$ be nonzero. From the previous paragraph we deduce there exists a k-rational point x such that s does not vanish in x_K. Let $\mathcal{I}$ be the ideal sheaf of $i : x_K \to X_K$ as in Varieties, Lemma 43.8. Look at the short exact sequence

$$0 \to \mathcal{I} \otimes_{\mathcal{O}_{X_K}} \mathcal{L} \to \mathcal{L} \to i_* i^* \mathcal{L} \to 0$$

Observe that $H^0(X_K, i_* i^* \mathcal{L}) = H^0(x_K, i^* \mathcal{L})$ has dimension 1 over K. Since s does not vanish at x we conclude that

$$H^0(X_K, \mathcal{L}) \longrightarrow H^0(X, i_* i^* \mathcal{L})$$

is surjective. Hence $\dim_K H^0(X_K, \mathcal{I} \otimes_{\mathcal{O}_{X_K}} \mathcal{L}) = t - 1$. Finally, the long exact sequence of cohomology also shows that $H^1(X_K, \mathcal{I} \otimes_{\mathcal{O}_{X_K}} \mathcal{L}) = 0$ thereby finishing the proof of the induction step. $\square$

0B9Z **Proposition 6.6.** *Let k be a separably closed field. Let X be a smooth projective curve over k. The Picard functor $\mathrm{Pic}_{X/k}$ is representable.*

Proof. Since k is separably closed there exists a k-rational point σ of X, see Varieties, Lemma 25.6. As discussed above, it suffices to show that the functor $\mathrm{Pic}_{X/k,\sigma}$ classifying invertible modules trivial along σ is representable. To do this we will check conditions (1), (2)(a), (2)(b), and (2)(c) of Lemma 5.1.

The functor $\mathrm{Pic}_{X/k,\sigma}$ satisfies the sheaf condition for the fppf topology because it is isomorphic to $\mathrm{Pic}_{X/k}$. It would be more correct to say that we've shown the sheaf condition for $\mathrm{Pic}_{X/k,\sigma}$ in the proof of Lemma 4.3 which applies by Lemma 6.1. This proves condition (1)

As our subfunctor we use F as defined in Lemma 6.2. Condition (2)(b) follows. Condition (2)(a) is Lemma 6.4. Condition (2)(c) is Lemma 6.5. $\square$

In fact, the proof given above produces more information which we collect here.

0BA0 **Lemma 6.7.** *Let k be a separably closed field. Let X be a smooth projective curve of genus g over k.*

(1) $\underline{\mathrm{Pic}}_{X/k}$ *is a disjoint union of g-dimensional smooth proper varieties $\underline{\mathrm{Pic}}^d_{X/k}$,*

(2) *k-points of $\underline{\mathrm{Pic}}^d_{X/k}$ correspond to invertible $\mathcal{O}_X$-modules of degree d,*

(3) $\underline{\mathrm{Pic}}^0_{X/k}$ *is an open and closed subgroup scheme,*

(4) *for $d \geq 0$ there is a canonical morphism $\gamma_d : \underline{\mathrm{Hilb}}^d_{X/k} \to \underline{\mathrm{Pic}}^d_{X/k}$*

(5) *the morphisms γ_d are surjective for $d \geq g$ and smooth for $d \geq 2g - 1$,*

(6) *the morphism $\underline{\mathrm{Hilb}}^g_{X/k} \to \underline{\mathrm{Pic}}^g_{X/k}$ is birational.*

Proof. Pick a k-rational point σ of X. Recall that $\mathrm{Pic}_{X/k}$ is isomorphic to the functor $\mathrm{Pic}_{X/k,\sigma}$. By Derived Categories of Schemes, Lemma 32.2 for every $d \in \mathbf{Z}$ there is an open subfunctor

$$\mathrm{Pic}^d_{X/k,\sigma} \subset \mathrm{Pic}_{X/k,\sigma}$$

whose value on a scheme T over k consists of those $\mathcal{L} \in \mathrm{Pic}_{X/k,\sigma}(T)$ such that $\chi(X_t, \mathcal{L}_t) = d + 1 - g$ and moreover we have

$$\mathrm{Pic}_{X/k,\sigma} = \coprod\nolimits_{d \in \mathbf{Z}} \mathrm{Pic}^d_{X/k,\sigma}$$

as fppf sheaves. It follows that the scheme $\underline{\mathrm{Pic}}_{X/k}$ (which exists by Proposition 6.6) has a corresponding decomposition

$$\underline{\mathrm{Pic}}_{X/k,\sigma} = \coprod\nolimits_{d \in \mathbf{Z}} \underline{\mathrm{Pic}}^d_{X/k,\sigma}$$

where the points of $\underline{\mathrm{Pic}}^d_{X/k,\sigma}$ correspond to isomorphism classes of invertible modules of degree d on X.

Fix $d \geq 0$. There is a morphism

$$\gamma_d : \underline{\mathrm{Hilb}}^d_{X/k} \longrightarrow \underline{\mathrm{Pic}}^d_{X/k}$$

coming from the invertible sheaf $\mathcal{O}(D_{univ})$ on $\underline{\mathrm{Hilb}}^d_{X/k} \times_k X$ (Remark 3.7) by the Yoneda lemma (Categories, Lemma 3.5). Our proof of the representability of the Picard functor of X/k in Proposition 6.6 and Lemma 6.4 shows that γ_g induces an open immersion on a nonempty open of $\underline{\mathrm{Hilb}}^g_{X/k}$. Moreover, the proof shows that the translates of this open by k-rational points of the group scheme $\underline{\mathrm{Pic}}_{X/k}$ define an open covering. Since $\underline{\mathrm{Hilb}}^g_{X/K}$ is smooth of dimension g (Proposition 3.6) over k, we conclude that the group scheme $\underline{\mathrm{Pic}}_{X/k}$ is smooth of dimension g over k.

By Groupoids, Lemma 7.3 we see that $\underline{\mathrm{Pic}}_{X/k}$ is separated. Hence, for every $d \geq 0$, the image of γ_d is a proper variety over k (Morphisms, Lemma 41.10).

Let $d \geq g$. Then for any field extension K/k and any invertible $\mathcal{O}_{X_K}$-module $\mathcal{L}$ of degree d, we see that $\chi(X_K, \mathcal{L}) = d + 1 - g > 0$. Hence $\mathcal{L}$ has a nonzero section and we conclude that $\mathcal{L} = \mathcal{O}_{X_K}(D)$ for some divisor $D \subset X_K$ of degree d. It follows that γ_d is surjective.

Combining the facts mentioned above we see that $\underline{\mathrm{Pic}}^d_{X/k}$ is proper for $d \geq g$. This finishes the proof of (2) because now we see that $\underline{\mathrm{Pic}}^d_{X/k}$ is proper for $d \geq g$ but then all $\underline{\mathrm{Pic}}^d_{X/k}$ are proper by translation.

It remains to prove that γ_d is smooth for $d \geq 2g - 1$. Consider an invertible $\mathcal{O}_X$-module $\mathcal{L}$ of degree d. Then the fibre of the point corresponding to $\mathcal{L}$ is

$$Z = \{D \subset X \mid \mathcal{O}_X(D) \cong \mathcal{L}\} \subset \underline{\mathrm{Hilb}}^d_{X/k}$$

with its natural scheme structure. Since any isomorphism $\mathcal{O}_X(D) \to \mathcal{L}$ is well defined up to multiplying by a nonzero scalar, we see that the canonical section $1 \in \mathcal{O}_X(D)$ is mapped to a section $s \in \Gamma(X, \mathcal{L})$ well defined up to multiplication by a nonzero scalar. In this way we obtain a morphism

$$Z \longrightarrow \mathrm{Proj}(\mathrm{Sym}(\Gamma(X, \mathcal{L})^*))$$

(dual because of our conventions). This morphism is an isomorphism, because given an section of $\mathcal{L}$ we can take the associated effective Cartier divisor, in other

words we can construct an inverse of the displayed morphism; we omit the precise formulation and proof. Since $\dim H^0(X, \mathcal{L}) = d + 1 - g$ for every $\mathcal{L}$ of degree $d \geq 2g - 1$ by Varieties, Lemma 44.17 we see that $\mathrm{Proj}(\mathrm{Sym}(\Gamma(X, \mathcal{L})^*)) \cong \mathbf{P}_k^{d-g}$. We conclude that $\dim(Z) = \dim(\mathbf{P}_k^{d-g}) = d - g$. We conclude that the fibres of the morphism γ_d all have dimension equal to the difference of the dimensions of $\underline{\mathrm{Hilb}}_{X/k}^d$ and $\underline{\mathrm{Pic}}_{X/k}^d$. It follows that γ_d is flat, see Algebra, Lemma 128.1. As moreover the fibres are smooth, we conclude that γ_d is smooth by Morphisms, Lemma 34.3. $\square$

7. Some remarks on Picard groups

0CDS This section continues the discussion in Varieties, Section 30 and will be continued in Algebraic Curves, Section 17.

0CDT **Lemma 7.1.** *Let k be a field. Let X be a quasi-compact and quasi-separated scheme over k with $H^0(X, \mathcal{O}_X) = k$. If X has a k-rational point, then for any Galois extension k'/k we have*

$$\mathrm{Pic}(X) = \mathrm{Pic}(X_{k'})^{Gal(k'/k)}$$

Moreover the action of $\mathrm{Gal}(k'/k)$ on $\mathrm{Pic}(X_{k'})$ is continuous.

Proof. Since $\mathrm{Gal}(k'/k) = \mathrm{Aut}(k'/k)$ it acts (from the right) on $\mathrm{Spec}(k')$, hence it acts (from the right) on $X_{k'} = X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k')$, and since $\mathrm{Pic}(-)$ is a contravariant functor, it acts (from the left) on $\mathrm{Pic}(X_{k'})$. If k'/k is an infinite Galois extension, then we write $k' = \mathrm{colim}\, k'_\lambda$ as a filtered colimit of finite Galois extensions, see Fields, Lemma 22.3. Then $X_{k'} = \lim X_{k_\lambda}$ (as in Limits, Section 2) and we obtain

$$\mathrm{Pic}(X_{k'}) = \mathrm{colim}\, \mathrm{Pic}(X_{k_\lambda})$$

by Limits, Lemma 10.3. Moreover, the transition maps in this system of abelian groups are injective by Varieties, Lemma 30.3. It follows that every element of $\mathrm{Pic}(X_{k'})$ is fixed by one of the open subgroups $\mathrm{Gal}(k'/k_\lambda)$, which exactly means that the action is continuous. Injectivity of the transition maps implies that it suffices to prove the statement on fixed points in the case that k'/k is finite Galois.

Assume k'/k is finite Galois with Galois group $G = \mathrm{Gal}(k'/k)$. Let $\mathcal{L}$ be an element of $\mathrm{Pic}(X_{k'})$ fixed by G. We will use Galois descent (Descent, Lemma 6.1) to prove that $\mathcal{L}$ is the pullback of an invertible sheaf on X. Recall that $f_\sigma = \mathrm{id}_X \times \mathrm{Spec}(\sigma) : X_{k'} \to X_{k'}$ and that σ acts on $\mathrm{Pic}(X_{k'})$ by pulling back by f_σ. Hence for each $\sigma \in G$ we can choose an isomorphism $\varphi_\sigma : \mathcal{L} \to f_\sigma^* \mathcal{L}$ because $\mathcal{L}$ is a fixed by the G-action. The trouble is that we don't know if we can choose φ_σ such that the cocycle condition $\varphi_{\sigma\tau} = f_\sigma^* \varphi_\tau \circ \varphi_\sigma$ holds. To see that this is possible we use that X has a k-rational point $x \in X(k)$. Of course, x similarly determines a k'-rational point $x' \in X_{k'}$ which is fixed by f_σ for all σ. Pick a nonzero element s in the fibre of $\mathcal{L}$ at x'; the fibre is the 1-dimensional $k' = \kappa(x')$-vector space

$$\mathcal{L}_{x'} \otimes_{\mathcal{O}_{X_{k'}, x'}} \kappa(x').$$

Then $f_\sigma^* s$ is a nonzero element of the fibre of $f_\sigma^* \mathcal{L}$ at x'. Since we can multiply φ_σ by an element of $(k')^*$ we may assume that φ_σ sends s to $f_\sigma^* s$. Then we see that both $\varphi_{\sigma\tau}$ and $f_\sigma^* \varphi_\tau \circ \varphi_\sigma$ send s to $f_{\sigma\tau}^* s = f_\tau^* f_\sigma^* s$. Since $H^0(X_{k'}, \mathcal{O}_{X_{k'}}) = k'$ these two isomorphisms have to be the same (as one is a global unit times the other and they agree in x') and the proof is complete. $\square$

0CD5 **Lemma 7.2.** *Let k be a field of characteristic $p > 0$. Let X be a quasi-compact and quasi-separated scheme over k with $H^0(X, \mathcal{O}_X) = k$. Let n be an integer prime to p. Then the map*

$$\operatorname{Pic}(X)[n] \longrightarrow \operatorname{Pic}(X_{k'})[n]$$

is bijective for any purely inseparable extension k'/k.

Proof. First we observe that the map $\operatorname{Pic}(X) \to \operatorname{Pic}(X_{k'})$ is injective by Varieties, Lemma 30.3. Hence we have to show the map in the lemma is surjective. Let $\mathcal{L}$ be an invertible $\mathcal{O}_{X_{k'}}$-module which has order dividing n in $\operatorname{Pic}(X_{k'})$. Choose an isomorphism $\alpha : \mathcal{L}^{\otimes n} \to \mathcal{O}_{X_{k'}}$ of invertible modules. We will prove that we can descend the pair $(\mathcal{L}, \alpha)$ to X.

Set $A = k' \otimes_k k'$. Since k'/k is purely inseparable, the kernel of the multiplication map $A \to k'$ is a locally nilpotent ideal I of A. Observe that

$$X_A = X \times_{\operatorname{Spec}(k)} \operatorname{Spec}(A) = X_{k'} \times_X X_{k'}$$

comes with two projections $\operatorname{pr}_i : X_A \to X_{k'}$, $i = 0, 1$ which agree over A/I. Hence the invertible modules $\mathcal{L}_i = \operatorname{pr}_i^* \mathcal{L}$ agree over the closed subscheme $X_{A/I} = X_{k'}$. Since $X_{A/I} \to X_A$ is a thickening and since $\mathcal{L}_i$ are n-torsion, we see that there exists an isomorphism $\varphi : \mathcal{L}_0 \to \mathcal{L}_1$ by More on Morphisms, Lemma 4.2. We may pick φ to reduce to the identity modulo I. Namely, $H^0(X, \mathcal{O}_X) = k$ implies $H^0(X_{k'}, \mathcal{O}_{X_{k'}}) = k'$ by Cohomology of Schemes, Lemma 5.2 and $A \to k'$ is surjective hence we can adjust φ by multiplying by a suitable element of A. Consider the map

$$\lambda : \mathcal{O}_{X_A} \xrightarrow{\operatorname{pr}_0^* \alpha^{-1}} \mathcal{L}_0^{\otimes n} \xrightarrow{\varphi^{\otimes n}} \mathcal{L}_1^{\otimes n} \xrightarrow{\operatorname{pr}_0^* \alpha} \mathcal{O}_{X_A}$$

We can view λ as an element of A because $H^0(X_A, \mathcal{O}_{X_A}) = A$ (same reference as above). Since φ reduces to the identity modulo I we see that $\lambda = 1 \bmod I$. Then there is a unique nth root of λ in $1 + I$ (Algebra, Lemma 32.8) and after multiplying φ by its inverse we get $\lambda = 1$. We claim that $(\mathcal{L}, \varphi)$ is a descent datum for the fpqc covering $\{X_{k'} \to X\}$ (Descent, Definition 2.1). If true, then $\mathcal{L}$ is the pullback of an invertible $\mathcal{O}_X$-module $\mathcal{N}$ by Descent, Proposition 5.2. Injectivity of the map on Picard groups shows that $\mathcal{N}$ is a torsion element of $\operatorname{Pic}(X)$ of the same order as $\mathcal{L}$.

Proof of the claim. To see this we have to verify that

$$\operatorname{pr}_{12}^* \varphi \circ \operatorname{pr}_{01}^* \varphi = \operatorname{pr}_{02}^* \varphi \quad \text{on} \quad X_{k'} \times_X X_{k'} \times_X X_{k'} = X_{k' \otimes_k k' \otimes_k k'}$$

As before the diagonal morphism $\Delta : X_{k'} \to X_{k' \otimes_k k' \otimes_k k'}$ is a thickening. The left and right hand sides of the equality signs are maps $a, b : p_0^* \mathcal{L} \to p_2^* \mathcal{L}$ compatible with $p_0^* \alpha$ and $p_2^* \alpha$ where $p_i : X_{k' \otimes_k k' \otimes_k k'} \to X_{k'}$ are the projection morphisms. Finally, a, b pull back to the same map under Δ. Affine locally (in local trivializations) this means that a, b are given by multiplication by invertible functions which reduce to the same function modulo a locally nilpotent ideal and which have the same nth powers. Then it follows from Algebra, Lemma 32.8 that these functions are the same. $\square$

8. Other chapters

Preliminaries

References

[Kle05] Steven L. Kleiman, *The Picard scheme*, Fundamental algebraic geometry, Math. Surveys Monogr., vol. 123, Amer. Math. Soc., Providence, RI, 2005, pp. 235–321.