FORMAL DEFORMATION THEORY

06G7

Contents

1. Introduction

06G8 This chapter develops formal deformation theory in a form applicable later in the Stacks project, closely following Rim [GRR72, Exposee VI] and Schlessinger [Sch68]. We strongly encourage the reader new to this topic to read the paper by Schlessinger

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first, as it is sufficiently general for most applications, and Schlessinger's results are indeed used in most papers that use this kind of formal deformation theory.

Let Λ be a complete Noetherian local ring with residue field k, and let $\mathcal{C}_\Lambda$ denote the category of Artinian local Λ-algebras with residue field k. Given a functor $F : \mathcal{C}_\Lambda \to \textit{Sets}$ such that $F(k)$ is a one element set, Schlessinger's paper introduced conditions (H1)-(H4) such that:

(1) F has a "hull" if and only if (H1)-(H3) hold.
(2) F is prorepresentable if and only if (H1)-(H4) hold.

The purpose of this chapter is to generalize these results in two ways exactly as is done in Rim's paper:

(A) The functor F is replaced by a category $\mathcal{F}$ cofibered in groupoids over $\mathcal{C}_\Lambda$, see Section 3.
(B) We let Λ be a Noetherian ring and $\Lambda \to k$ a finite ring map to a field. The category $\mathcal{C}_\Lambda$ is the category of Artinian local Λ-algebras A endowed with a given identification $A/\mathfrak{m}_A = k$.

The analogue of the condition that $F(k)$ is a one element set is that $\mathcal{F}(k)$ is the trivial groupoid. If $\mathcal{F}$ satisfies this condition then we say it is a *predeformation category*, but in general we do not make this assumption. Rim's paper [GRR72, Exposee VI] is the original source for the results in this document. We also mention the useful paper [TV13], which discusses deformation theory with groupoids but in less generality than we do here.

An important role is played by the "completion" $\widehat{\mathcal{C}}_\Lambda$ of the category $\mathcal{C}_\Lambda$. An object of $\widehat{\mathcal{C}}_\Lambda$ is a Noetherian complete local Λ-algebra R whose residue field is identified with k, see Section 4. On the one hand $\mathcal{C}_\Lambda \subset \widehat{\mathcal{C}}_\Lambda$ is a strictly full subcategory and on the other hand $\widehat{\mathcal{C}}_\Lambda$ is a full subcategory of the category of pro-objects of $\mathcal{C}_\Lambda$. A functor $\mathcal{C}_\Lambda \to \textit{Sets}$ is *prorepresentable* if it is isomorphic to the restriction of a representable functor $\underline{R} = \mathrm{Mor}_{\widehat{\mathcal{C}}_\Lambda}(R, -)$ to $\mathcal{C}_\Lambda$ where $R \in \mathrm{Ob}(\widehat{\mathcal{C}}_\Lambda)$.

Categories cofibred in groupoids are dual to categories fibred in groupoids; we introduce them in Section 5. A *smooth* morphism of categories cofibred in groupoids over $\mathcal{C}_\Lambda$ is one that satisfies the infinitesimal lifting criterion for objects, see Section 8. This is analogous to the definition of a formally smooth ring map, see Algebra, Definition 138.1 and is exactly dual to the notion in Criteria for Representability, Section 6. This is an important notion as we eventually want to prove that certain kinds of categories cofibred in groupoids have a smooth prorepresentable presentation, much like the characterization of algebraic stacks in Algebraic Stacks, Sections 16 and 17. A *versal formal object* of a category $\mathcal{F}$ cofibred in groupoids over $\mathcal{C}_\Lambda$ is an object $\xi \in \widehat{\mathcal{F}}(R)$ of the completion such that the associated morphism $\underline{\xi} : \underline{R}|_{\mathcal{C}_\Lambda} \to \mathcal{F}$ is smooth.

In Section 10, we define conditions (S1) and (S2) on $\mathcal{F}$ generalizing Schlessinger's (H1) and (H2). The analogue of Schlessinger's (H3)—the condition that $\mathcal{F}$ has finite dimensional tangent space—is not given a name. A key step in the development of the theory is the existence of versal formal objects for predeformation categories satisfying (S1), (S2) and (H3), see Lemma 13.4. Schlessinger's notion of a *hull* for a functor $F : \mathcal{C}_\Lambda \to \textit{Sets}$ is, in our terminology, a versal formal object $\xi \in \widehat{F}(R)$ such that the induced map of tangent spaces $d\underline{\xi} : T\underline{R}|_{\mathcal{C}_\Lambda} \to TF$ is an isomorphism. In the literature a hull is often called a "miniversal" object. We do not do so, and here is why. It can happen that a functor has a versal formal object without having a hull. Moreover, we show in Section 14 that if a predeformation category has a versal formal object, then it always has a *minimal* one (as defined in Definition 14.4) which is unique up to isomorphism, see Lemma 14.5. But it can happen that the minimal versal formal object does not induce an isomorphism on tangent spaces! (See Examples 15.3 and 15.8.)

Keeping in mind the differences pointed out above, Theorem 15.5 is the direct generalization of (1) above: it recovers Schlessinger's result in the case that $\mathcal{F}$ is a functor and it characterizes minimal versal formal objects, in the presence of conditions (S1) and (S2), in terms of the map $d\underline{\xi} : T\underline{R}|_{\mathcal{C}_\Lambda} \to TF$ on tangent spaces.

In Section 16, we define Rim's condition (RS) on $\mathcal{F}$ generalizing Schlessinger's (H4). A *deformation category* is defined as a predeformation category satisfying (RS). The analogue to prorepresentable functors are the categories cofibred in groupoids over $\mathcal{C}_\Lambda$ which have a *presentation by a smooth prorepresentable groupoid in functors* on $\mathcal{C}_\Lambda$, see Definitions 21.1, 22.1, and 23.1. This notion of a presentation takes into account the groupoid structure of the fibers of $\mathcal{F}$. In Theorem 26.4 we prove that $\mathcal{F}$ has a presentation by a smooth prorepresentable groupoid in functors if and only if $\mathcal{F}$ has a finite dimensional tangent space and finite dimensional infinitesimal automorphism space. This is the generalization of (2) above: it reduces to Schlessinger's result in the case that $\mathcal{F}$ is a functor. There is a final Section 27 where we discuss how to use minimal versal formal objects to produce a (unique up to isomorphism) minimal presentation by a smooth prorepresentable groupoid in functors.

We also find the following conceptual explanation for Schlessinger's conditions. If a predeformation category $\mathcal{F}$ satisfies (RS), then the associated functor of isomorphism classes $\overline{\mathcal{F}} : \mathcal{C}_\Lambda \to \textit{Sets}$ satisfies (H1) and (H2) (Lemmas 16.6 and 10.5). Conversely, if a functor $F : \mathcal{C}_\Lambda \to \textit{Sets}$ arises naturally as the functor of isomorphism classes of a category $\mathcal{F}$ cofibered in groupoids, then it seems to happen in practice that an argument showing F satisfies (H1) and (H2) will also show $\mathcal{F}$ satisfies (RS). Examples are discussed in Deformation Problems, Section 1. Moreover, if $\mathcal{F}$ satisfies (RS), then condition (H4) for $\overline{\mathcal{F}}$ has a simple interpretation in terms of extending automorphisms of objects of $\mathcal{F}$ (Lemma 16.7). These observations suggest that (RS) should be regarded as the fundamental deformation theoretic glueing condition.

2. Notation and Conventions

06G9 A ring is commutative with 1. The maximal ideal of a local ring A is denoted by $\mathfrak{m}_A$. The set of positive integers is denoted by $\mathbf{N} = \{1, 2, 3, \ldots\}$. If U is an object of a category $\mathcal{C}$, we denote by $\underline{U}$ the functor $\mathrm{Mor}_\mathcal{C}(U, -) : \mathcal{C} \to \textit{Sets}$, see Remarks 5.2 (12). Warning: this may conflict with the notation in other chapters where we sometimes use $\underline{U}$ to denote $h_U(-) = \mathrm{Mor}_\mathcal{C}(-, U)$.

Throughout this chapter Λ is a Noetherian ring and $\Lambda \to k$ is a finite ring map from Λ to a field. The kernel of this map is denoted $\mathfrak{m}_\Lambda$ and the image $k' \subset k$. It turns out that $\mathfrak{m}_\Lambda$ is a maximal ideal, $k' = \Lambda/\mathfrak{m}_\Lambda$ is a field, and the extension k/k' is finite. See discussion surrounding (3.3.1).

3. The base category

06GB Motivation. An important application of formal deformation theory is to criteria for representability by algebraic spaces. Suppose given a locally Noetherian base S and a functor $F : (Sch/S)_{fppf}^{opp} \to Sets$. Let k be a finite type field over S, i.e., we are given a finite type morphism $\mathrm{Spec}(k) \to S$. One of Artin's criteria is that for any element $x \in F(\mathrm{Spec}(k))$ the predeformation functor associated to the triple (S, k, x) should be prorepresentable. By Morphisms, Lemma 16.1 the condition that k is of finite type over S means that there exists an affine open $\mathrm{Spec}(\Lambda) \subset S$ such that k is a finite Λ-algebra. This motivates why we work throughout this chapter with a base category as follows.

06GC **Definition 3.1.** Let Λ be a Noetherian ring and let $\Lambda \to k$ be a finite ring map where k is a field. We define $\mathcal{C}_\Lambda$ to be the category with

(1) objects are pairs (A, φ) where A is an Artinian local Λ-algebra and where $\varphi : A/\mathfrak{m}_A \to k$ is a Λ-algebra isomorphism, and
(2) morphisms $f : (B, \psi) \to (A, \varphi)$ are local Λ-algebra homomorphisms such that $\varphi \circ (f \bmod \mathfrak{m}) = \psi$.

We say we are in the *classical case* if Λ is a Noetherian complete local ring and k is its residue field.

Note that if $\Lambda \to k$ is surjective and if A is an Artinian local Λ-algebra, then the identification φ, if it exists, is unique. Moreover, in this case any Λ-algebra map $A \to B$ is going to be compatible with the identifications. Hence in this case $\mathcal{C}_\Lambda$ is just the category of local Artinian Λ-algebras whose residue field "is" k. By abuse of notation we also denote objects of $\mathcal{C}_\Lambda$ simply A in the general case. Moreover, we will often write $A/\mathfrak{m} = k$, i.e., we will pretend all rings in $\mathcal{C}_\Lambda$ have residue field k (since all ring maps in $\mathcal{C}_\Lambda$ are compatible with the given identifications this should never cause any problems). Throughout the rest of this chapter the base ring Λ and the field k are fixed. The category $\mathcal{C}_\Lambda$ will be the base category for the cofibered categories considered below.

06GD **Definition 3.2.** Let $f : B \to A$ be a ring map in $\mathcal{C}_\Lambda$. We say f is a *small extension* if it is surjective and $\mathrm{Ker}(f)$ is a nonzero principal ideal which is annihilated by $\mathfrak{m}_B$.

By the following lemma we can often reduce arguments involving surjective ring maps in $\mathcal{C}_\Lambda$ to the case of small extensions.

06GE **Lemma 3.3.** *Let $f : B \to A$ be a surjective ring map in $\mathcal{C}_\Lambda$. Then f can be factored as a composition of small extensions.*

Proof. Let I be the kernel of f. The maximal ideal $\mathfrak{m}_B$ is nilpotent since B is Artinian, say $\mathfrak{m}_B^n = 0$. Hence we get a factorization

$$B = B/I\mathfrak{m}_B^{n-1} \to B/I\mathfrak{m}_B^{n-2} \to \ldots \to B/I \cong A$$

of f into a composition of surjective maps whose kernels are annihilated by the maximal ideal. Thus it suffices to prove the lemma when f itself is such a map, i.e. when I is annihilated by $\mathfrak{m}_B$. In this case I is a k-vector space, which has finite dimension, see Algebra, Lemma 53.6. Take a basis $x_1, \ldots, x_n$ of I as a k-vector space to get a factorization

$$B \to B/(x_1) \to \ldots \to B/(x_1, \ldots, x_n) \cong A$$

of f into a composition of small extensions. $\square$

The next lemma says that we can compute the length of a module over a local Λ-algebra with residue field k in terms of the length over Λ. To explain the notation in the statement, let $k' \subset k$ be the image of our fixed finite ring map $\Lambda \to k$. Note that $k' \subset k$ is a finite extension of rings. Hence k' is a field and k/k' is a finite extension of fields, see Algebra, Lemma 36.18. Moreover, as $\Lambda \to k'$ is surjective we see that its kernel is a maximal ideal $\mathfrak{m}_\Lambda$. Thus

06S2 (3.3.1)
$$[k : k'] = [k : \Lambda/\mathfrak{m}_\Lambda] < \infty$$

and in the classical case we have $k = k'$. The notation $k' = \Lambda/\mathfrak{m}_\Lambda$ will be fixed throughout this chapter.

06GG **Lemma 3.4.** *Let A be a local Λ-algebra with residue field k. Let M be an A-module. Then $[k : k']length_A(M) = length_\Lambda(M)$. In the classical case we have $length_A(M) = length_\Lambda(M)$.*

Proof. If M is a simple A-module then $M \cong k$ as an A-module, see Algebra, Lemma 52.10. In this case $\text{length}_A(M) = 1$ and $\text{length}_\Lambda(M) = [k' : k]$, see Algebra, Lemma 52.6. If $\text{length}_A(M)$ is finite, then the result follows on choosing a filtration of M by A-submodules with simple quotients using additivity, see Algebra, Lemma 52.3. If $\text{length}_A(M)$ is infinite, the result follows from the obvious inequality $\text{length}_A(M) \leq \text{length}_\Lambda(M)$. $\square$

06S3 **Lemma 3.5.** *Let $A \to B$ be a ring map in $\mathcal{C}_\Lambda$. The following are equivalent*

(1) *f is surjective,*
(2) *$\mathfrak{m}_A/\mathfrak{m}_A^2 \to \mathfrak{m}_B/\mathfrak{m}_B^2$ is surjective, and*
(3) *$\mathfrak{m}_A/(\mathfrak{m}_\Lambda A + \mathfrak{m}_A^2) \to \mathfrak{m}_B/(\mathfrak{m}_\Lambda B + \mathfrak{m}_B^2)$ is surjective.*

Proof. For any ring map $f : A \to B$ in $\mathcal{C}_\Lambda$ we have $f(\mathfrak{m}_A) \subset \mathfrak{m}_B$ for example because $\mathfrak{m}_A$, $\mathfrak{m}_B$ is the set of nilpotent elements of A, B. Suppose f is surjective. Let $y \in \mathfrak{m}_B$. Choose $x \in A$ with $f(x) = y$. Since f induces an isomorphism $A/\mathfrak{m}_A \to B/\mathfrak{m}_B$ we see that $x \in \mathfrak{m}_A$. Hence the induced map $\mathfrak{m}_A/\mathfrak{m}_A^2 \to \mathfrak{m}_B/\mathfrak{m}_B^2$ is surjective. In this way we see that (1) implies (2).

It is clear that (2) implies (3). The map $A \to B$ gives rise to a canonical commutative diagram

$$\begin{tikzcd}
\mathfrak{m}_\Lambda/\mathfrak{m}_\Lambda^2 \otimes_{k'} k \arrow[r] \arrow[d] & \mathfrak{m}_A/\mathfrak{m}_A^2 \arrow[r] \arrow[d] & \mathfrak{m}_A/(\mathfrak{m}_\Lambda A + \mathfrak{m}_A^2) \arrow[r] \arrow[d] & 0 \\
\mathfrak{m}_\Lambda/\mathfrak{m}_\Lambda^2 \otimes_{k'} k \arrow[r] & \mathfrak{m}_B/\mathfrak{m}_B^2 \arrow[r] & \mathfrak{m}_B/(\mathfrak{m}_\Lambda B + \mathfrak{m}_B^2) \arrow[r] & 0
\end{tikzcd}$$

with exact rows. Hence if (3) holds, then so does (2).

Assume (2). To show that $A \to B$ is surjective it suffices by Nakayama's lemma (Algebra, Lemma 20.1) to show that $A/\mathfrak{m}_A \to B/\mathfrak{m}_A B$ is surjective. (Note that $\mathfrak{m}_A$ is a nilpotent ideal.) As $k = A/\mathfrak{m}_A = B/\mathfrak{m}_B$ it suffices to show that $\mathfrak{m}_A B \to \mathfrak{m}_B$ is surjective. Applying Nakayama's lemma once more we see that it suffices to see that $\mathfrak{m}_A B/\mathfrak{m}_A\mathfrak{m}_B \to \mathfrak{m}_B/\mathfrak{m}_B^2$ is surjective which is what we assumed. $\square$

If $A \to B$ is a ring map in $\mathcal{C}_\Lambda$, then the map $\mathfrak{m}_A/(\mathfrak{m}_\Lambda A + \mathfrak{m}_A^2) \to \mathfrak{m}_B/(\mathfrak{m}_\Lambda B + \mathfrak{m}_B^2)$ is the map on relative cotangent spaces. Here is a formal definition.

06GY **Definition 3.6.** Let $R \to S$ be a local homomorphism of local rings. The *relative cotangent space*[1] of R over S is the $S/\mathfrak{m}_S$-vector space $\mathfrak{m}_S/(\mathfrak{m}_R S + \mathfrak{m}_S^2)$.

If $f_1 : A_1 \to A$ and $f_2 : A_2 \to A$ are two ring maps, then the fiber product $A_1 \times_A A_2$ is the subring of $A_1 \times A_2$ consisting of elements whose two projections to A are equal. Throughout this chapter we will be considering conditions involving such a fiber product when f_1 and f_2 are in $\mathcal{C}_\Lambda$. It isn't always the case that the fibre product is an object of $\mathcal{C}_\Lambda$.

06S4 **Example 3.7.** Let p be a prime number and let $n \in \mathbf{N}$. Let $\Lambda = \mathbf{F}_p(t_1, t_2, \ldots, t_n)$ and let $k = \mathbf{F}_p(x_1, \ldots, x_n)$ with map $\Lambda \to k$ given by $t_i \mapsto x_i^p$. Let $A = k[\epsilon] = k[x]/(x^2)$. Then A is an object of $\mathcal{C}_\Lambda$. Suppose that $D : k \to k$ is a derivation of k over Λ, for example $D = \partial/\partial x_i$. Then the map

$$f_D : k \longrightarrow k[\epsilon], \quad a \mapsto a + D(a)\epsilon$$

is a morphism of $\mathcal{C}_\Lambda$. Set $A_1 = A_2 = k$ and set $f_1 = f_{\partial/\partial x_1}$ and $f_2(a) = a$. Then $A_1 \times_A A_2 = \{a \in k \mid \partial/\partial x_1(a) = 0\}$ which does not surject onto k. Hence the fibre product isn't an object of $\mathcal{C}_\Lambda$.

It turns out that this problem can only occur if the residue field extension k/k' (3.3.1) is inseparable and neither f_1 nor f_2 is surjective.

06GH **Lemma 3.8.** *Let $f_1 : A_1 \to A$ and $f_2 : A_2 \to A$ be ring maps in $\mathcal{C}_\Lambda$. Then:*
(1) *If f_1 or f_2 is surjective, then $A_1 \times_A A_2$ is in $\mathcal{C}_\Lambda$.*
(2) *If f_2 is a small extension, then so is $A_1 \times_A A_2 \to A_1$.*
(3) *If the field extension k/k' is separable, then $A_1 \times_A A_2$ is in $\mathcal{C}_\Lambda$.*

Proof. The ring $A_1 \times_A A_2$ is a Λ-algebra via the map $\Lambda \to A_1 \times_A A_2$ induced by the maps $\Lambda \to A_1$ and $\Lambda \to A_2$. It is a local ring with unique maximal ideal

$$\mathfrak{m}_{A_1} \times_{\mathfrak{m}_A} \mathfrak{m}_{A_2} = \mathrm{Ker}(A_1 \times_A A_2 \longrightarrow k)$$

A ring is Artinian if and only if it has finite length as a module over itself, see Algebra, Lemma 53.6. Since A_1 and A_2 are Artinian, Lemma 3.4 implies $\text{length}_\Lambda(A_1)$ and $\text{length}_\Lambda(A_2)$, and hence $\text{length}_\Lambda(A_1 \times A_2)$, are all finite. As $A_1 \times_A A_2 \subset A_1 \times A_2$ is a Λ-submodule, this implies $\text{length}_{A_1 \times_A A_2}(A_1 \times_A A_2) \leq \text{length}_\Lambda(A_1 \times_A A_2)$ is finite. So $A_1 \times_A A_2$ is Artinian. Thus the only thing that is keeping $A_1 \times_A A_2$ from being an object of $\mathcal{C}_\Lambda$ is the possibility that its residue field maps to a proper subfield of k via the map $A_1 \times_A A_2 \to A \to A/\mathfrak{m}_A = k$ above.

Proof of (1). If f_2 is surjective, then the projection $A_1 \times_A A_2 \to A_1$ is surjective. Hence the composition $A_1 \times_A A_2 \to A_1 \to A_1/\mathfrak{m}_{A_1} = k$ is surjective and we conclude that $A_1 \times_A A_2$ is an object of $\mathcal{C}_\Lambda$.

Proof of (2). If f_2 is a small extension then $A_2 \to A$ and $A_1 \times_A A_2 \to A_1$ are both surjective with the same kernel. Hence the kernel of $A_1 \times_A A_2 \to A_1$ is a 1-dimensional k-vector space and we see that $A_1 \times_A A_2 \to A_1$ is a small extension.

Proof of (3). Choose $\overline{x} \in k$ such that $k = k'(\overline{x})$ (see Fields, Lemma 19.1). Let $P'(T) \in k'[T]$ be the minimal polynomial of $\overline{x}$ over k'. Since k/k' is separable we see that $\mathrm{d}P/\mathrm{d}T(\overline{x}) \neq 0$. Choose a monic $P \in \Lambda[T]$ which maps to P' under the

[1]Caution: We will see later that in our general setting the tangent space of an object $A \in \mathcal{C}_\Lambda$ over Λ should not be defined simply as the k-linear dual of the relative cotangent space. In fact, the correct definition of the relative cotangent space is $\Omega_{S/R} \otimes_S S/\mathfrak{m}_S$.

surjective map $\Lambda[T] \to k'[T]$. Because A, A_1, A_2 are henselian, see Algebra, Lemma 153.10, we can find $x, x_1, x_2 \in A, A_1, A_2$ with $P(x) = 0, P(x_1) = 0, P(x_2) = 0$ and such that the image of x, x_1, x_2 in k is $\overline{x}$. Then $(x_1, x_2) \in A_1 \times_A A_2$ because x_1, x_2 map to $x \in A$ by uniqueness, see Algebra, Lemma 153.2. Hence the residue field of $A_1 \times_A A_2$ contains a generator of k over k' and we win. $\square$

Next we define essential surjections in $\mathcal{C}_\Lambda$. A necessary and sufficient condition for a surjection in $\mathcal{C}_\Lambda$ to be essential is given in Lemma 3.12.

06GF **Definition 3.9.** Let $f : B \to A$ be a ring map in $\mathcal{C}_\Lambda$. We say f is an *essential surjection* if it has the following properties:

(1) f is surjective.
(2) If $g : C \to B$ is a ring map in $\mathcal{C}_\Lambda$ such that $f \circ g$ is surjective, then g is surjective.

Using Lemma 3.5, we can characterize essential surjections in $\mathcal{C}_\Lambda$ as follows.

06S5 **Lemma 3.10.** *Let $f : B \to A$ be a ring map in $\mathcal{C}_\Lambda$. The following are equivalent*

(1) *f is an essential surjection,*
(2) *the map $B/\mathfrak{m}_B^2 \to A/\mathfrak{m}_A^2$ is an essential surjection, and*
(3) *the map $B/(\mathfrak{m}_\Lambda B + \mathfrak{m}_B^2) \to A/(\mathfrak{m}_\Lambda A + \mathfrak{m}_A^2)$ is an essential surjection.*

Proof. Assume (3). Let $C \to B$ be a ring map in $\mathcal{C}_\Lambda$ such that $C \to A$ is surjective. Then $C \to A/(\mathfrak{m}_\Lambda A + \mathfrak{m}_A^2)$ is surjective too. We conclude that $C \to B/(\mathfrak{m}_\Lambda B + \mathfrak{m}_B^2)$ is surjective by our assumption. Hence $C \to B$ is surjective by applying Lemma 3.5 (2 times).

Assume (1). Let $C \to B/(\mathfrak{m}_\Lambda B + \mathfrak{m}_B^2)$ be a morphism of $\mathcal{C}_\Lambda$ such that $C \to A/(\mathfrak{m}_\Lambda A + \mathfrak{m}_A^2)$ is surjective. Set $C' = C \times_{B/(\mathfrak{m}_\Lambda B + \mathfrak{m}_B^2)} B$ which is an object of $\mathcal{C}_\Lambda$ by Lemma 3.8. Note that $C' \to A/(\mathfrak{m}_\Lambda A + \mathfrak{m}_A^2)$ is still surjective, hence $C' \to A$ is surjective by Lemma 3.5. Thus $C' \to B$ is surjective by our assumption. This implies that $C' \to B/(\mathfrak{m}_\Lambda B + \mathfrak{m}_B^2)$ is surjective, which implies by the construction of C' that $C \to B/(\mathfrak{m}_\Lambda B + \mathfrak{m}_B^2)$ is surjective.

In the first paragraph we proved (3) $\Rightarrow$ (1) and in the second paragraph we proved (1) $\Rightarrow$ (3). The equivalence of (2) and (3) is a special case of the equivalence of (1) and (3), hence we are done. $\square$

To analyze essential surjections in $\mathcal{C}_\Lambda$ a bit more we introduce some notation. Suppose that A is an object of $\mathcal{C}_\Lambda$ or more generally any Λ-algebra equipped with a Λ-algebra surjection $A \to k$. There is a canonical exact sequence

06S6 (3.10.1) $$\mathfrak{m}_A/\mathfrak{m}_A^2 \xrightarrow{\mathrm{d}_A} \Omega_{A/\Lambda} \otimes_A k \to \Omega_{k/\Lambda} \to 0$$

see Algebra, Lemma 131.9. Note that $\Omega_{k/\Lambda} = \Omega_{k/k'}$ with k' as in (3.3.1). Let $H_1(L_{k/\Lambda})$ be the first homology module of the naive cotangent complex of k over Λ, see Algebra, Definition 134.1. Then we can extend (3.10.1) to the exact sequence

06S7 (3.10.2) $$H_1(L_{k/\Lambda}) \to \mathfrak{m}_A/\mathfrak{m}_A^2 \xrightarrow{\mathrm{d}_A} \Omega_{A/\Lambda} \otimes_A k \to \Omega_{k/\Lambda} \to 0,$$

see Algebra, Lemma 134.4. If $B \to A$ is a ring map in $\mathcal{C}_\Lambda$ or more generally a map of Λ-algebras equipped with Λ-algebra surjections onto k, then we obtain a commutative diagram

06S8 (3.10.3)
$$\begin{tikzcd} H_1(L_{k/\Lambda}) \arrow[r] \arrow[d, equal] & \mathfrak{m}_B/\mathfrak{m}_B^2 \arrow[r, "\mathrm{d}_B"'] \arrow[d] & \Omega_{B/\Lambda} \otimes_B k \arrow[r] \arrow[d] & \Omega_{k/\Lambda} \arrow[r] \arrow[d, equal] & 0 \\ H_1(L_{k/\Lambda}) \arrow[r] & \mathfrak{m}_A/\mathfrak{m}_A^2 \arrow[r, "\mathrm{d}_A"] & \Omega_{A/\Lambda} \otimes_A k \arrow[r] & \Omega_{k/\Lambda} \arrow[r] & 0 \end{tikzcd}$$

with exact rows.

06S9 **Lemma 3.11.** *There is a canonical map*

$$\mathfrak{m}_\Lambda/\mathfrak{m}_\Lambda^2 \longrightarrow H_1(L_{k/\Lambda}).$$

If $k' \subset k$ is separable (for example if the characteristic of k is zero), then this map induces an isomorphism $\mathfrak{m}_\Lambda/\mathfrak{m}_\Lambda^2 \otimes_{k'} k = H_1(L_{k/\Lambda})$. If $k = k'$ (for example in the classical case), then $\mathfrak{m}_\Lambda/\mathfrak{m}_\Lambda^2 = H_1(L_{k/\Lambda})$. The composition

$$\mathfrak{m}_\Lambda/\mathfrak{m}_\Lambda^2 \longrightarrow H_1(L_{k/\Lambda}) \longrightarrow \mathfrak{m}_A/\mathfrak{m}_A^2$$

comes from the canonical map $\mathfrak{m}_\Lambda \to \mathfrak{m}_A$.

Proof. Note that $H_1(L_{k'/\Lambda}) = \mathfrak{m}_\Lambda/\mathfrak{m}_\Lambda^2$ as $\Lambda \to k'$ is surjective with kernel $\mathfrak{m}_\Lambda$. The map arises from functoriality of the naive cotangent complex. If $k' \subset k$ is separable, then $k' \to k$ is an étale ring map, see Algebra, Lemma 143.4. Thus its naive cotangent complex has trivial homology groups, see Algebra, Definition 143.1. Then Algebra, Lemma 134.4 applied to the ring maps $\Lambda \to k' \to k$ implies that $\mathfrak{m}_\Lambda/\mathfrak{m}_\Lambda^2 \otimes_{k'} k = H_1(L_{k/\Lambda})$. We omit the proof of the final statement. ☐

06H0 **Lemma 3.12.** *Let $f : B \to A$ be a ring map in $\mathcal{C}_\Lambda$. Notation as in (3.10.3).*

1. *The equivalent conditions of Lemma 3.5 characterizing when f is surjective are also equivalent to*
   (a) $\mathrm{Im}(d_B) \to \mathrm{Im}(d_A)$ *is surjective, and*
   (b) *the map $\Omega_{B/\Lambda} \otimes_B k \to \Omega_{A/\Lambda} \otimes_A k$ is surjective.*
2. *The following are equivalent*
   (a) *f is an essential surjection (see Lemma 3.10),*
   (b) *the map $\mathrm{Im}(d_B) \to \mathrm{Im}(d_A)$ is an isomorphism, and*
   (c) *the map $\Omega_{B/\Lambda} \otimes_B k \to \Omega_{A/\Lambda} \otimes_A k$ is an isomorphism.*
3. *If k/k' is separable, then f is an essential surjection if and only if the map $\mathfrak{m}_B/(\mathfrak{m}_\Lambda B + \mathfrak{m}_B^2) \to \mathfrak{m}_A/(\mathfrak{m}_\Lambda A + \mathfrak{m}_A^2)$ is an isomorphism.*
4. *If f is a small extension, then f is not essential if and only if f has a section $s : A \to B$ in $\mathcal{C}_\Lambda$ with $f \circ s = id_A$.*

Proof. Proof of (1). It follows from (3.10.3) that (1)(a) and (1)(b) are equivalent. Also, if $A \to B$ is surjective, then (1)(a) and (1)(b) hold. Assume (1)(a). Since the kernel of d_A is the image of $H_1(L_{k/\Lambda})$ which also maps to $\mathfrak{m}_B/\mathfrak{m}_B^2$ we conclude that $\mathfrak{m}_B/\mathfrak{m}_B^2 \to \mathfrak{m}_A/\mathfrak{m}_A^2$ is surjective. Hence $B \to A$ is surjective by Lemma 3.5. This finishes the proof of (1).

Proof of (2). The equivalence of (2)(b) and (2)(c) is immediate from (3.10.3).

Assume (2)(b). Let $g : C \to B$ be a ring map in $\mathcal{C}_\Lambda$ such that $f \circ g$ is surjective. We conclude that $\mathfrak{m}_C/\mathfrak{m}_C^2 \to \mathfrak{m}_A/\mathfrak{m}_A^2$ is surjective by Lemma 3.5. Hence $\mathrm{Im}(\mathrm{d}_C) \to \mathrm{Im}(\mathrm{d}_A)$ is surjective and by the assumption we see that $\mathrm{Im}(\mathrm{d}_C) \to \mathrm{Im}(\mathrm{d}_B)$ is surjective. It follows that $C \to B$ is surjective by (1).

Assume (2)(a). Then f is surjective and we see that $\Omega_{B/\Lambda} \otimes_B k \to \Omega_{A/\Lambda} \otimes_A k$ is surjective. Let K be the kernel. Note that $K = \mathrm{d}_B(\mathrm{Ker}(\mathfrak{m}_B/\mathfrak{m}_B^2 \to \mathfrak{m}_A/\mathfrak{m}_A^2))$ by (3.10.3). Choose a splitting

$$\Omega_{B/\Lambda} \otimes_B k = \Omega_{A/\Lambda} \otimes_A k \oplus K$$

of k-vector space. The map $\mathrm{d} : B \to \Omega_{B/\Lambda}$ induces via the projection onto K a map $D : B \to K$. Set $C = \{b \in B \mid D(b) = 0\}$. The Leibniz rule shows that this is a Λ-subalgebra of B. Let $\overline{x} \in k$. Choose $x \in B$ mapping to $\overline{x}$. If $D(x) \neq 0$, then we can find an element $y \in \mathfrak{m}_B$ such that $D(y) = D(x)$. Hence $x - y \in C$ is an element which maps to $\overline{x}$. Thus $C \to k$ is surjective and C is an object of $\mathcal{C}_\Lambda$. Similarly, pick $\omega \in \mathrm{Im}(\mathrm{d}_A)$. We can find $x \in \mathfrak{m}_B$ such that $\mathrm{d}_B(x)$ maps to ω by (1). If $D(x) \neq 0$, then we can find an element $y \in \mathfrak{m}_B$ which maps to zero in $\mathfrak{m}_A/\mathfrak{m}_A^2$ such that $D(y) = D(x)$. Hence $z = x - y$ is an element of $\mathfrak{m}_C$ whose image $\mathrm{d}_C(z) \in \Omega_{C/k} \otimes_C k$ maps to ω. Hence $\mathrm{Im}(\mathrm{d}_C) \to \mathrm{Im}(\mathrm{d}_A)$ is surjective. We conclude that $C \to A$ is surjective by (1). Hence $C \to B$ is surjective by assumption. Hence $D = 0$, i.e., $K = 0$, i.e., (2)(c) holds. This finishes the proof of (2).

Proof of (3). If k'/k is separable, then $H_1(L_{k/\Lambda}) = \mathfrak{m}_\Lambda/\mathfrak{m}_\Lambda^2 \otimes_{k'} k$, see Lemma 3.11. Hence $\mathrm{Im}(\mathrm{d}_A) = \mathfrak{m}_A/(\mathfrak{m}_\Lambda A + \mathfrak{m}_A^2)$ and similarly for B. Thus (3) follows from (2).

Proof of (4). A section s of f is not surjective (by definition a small extension has nontrivial kernel), hence f is not essentially surjective. Conversely, assume f is a small extension but not an essential surjection. Choose a ring map $C \to B$ in $\mathcal{C}_\Lambda$ which is not surjective, such that $C \to A$ is surjective. Let $C' \subset B$ be the image of $C \to B$. Then $C' \neq B$ but C' surjects onto A. Since $f : B \to A$ is a small extension, $\mathrm{length}_C(B) = \mathrm{length}_C(A) + 1$. Thus $\mathrm{length}_C(C') \leq \mathrm{length}_C(A)$ since C' is a proper subring of B. But $C' \to A$ is surjective, so in fact we must have $\mathrm{length}_C(C') = \mathrm{length}_C(A)$ and $C' \to A$ is an isomorphism which gives us our section. □

06SA **Example 3.13.** Let $\Lambda = k[[x]]$ be the power series ring in 1 variable over k. Set $A = k$ and $B = \Lambda/(x^2)$. Then $B \to A$ is an essential surjection by Lemma 3.12 because it is a small extension and the map $B \to A$ does not have a right inverse (in the category $\mathcal{C}_\Lambda$). But the map

$$k \cong \mathfrak{m}_B/\mathfrak{m}_B^2 \longrightarrow \mathfrak{m}_A/\mathfrak{m}_A^2 = 0$$

is not an isomorphism. Thus in Lemma 3.12 (3) it is necessary to consider the map of relative cotangent spaces $\mathfrak{m}_B/(\mathfrak{m}_\Lambda B + \mathfrak{m}_B^2) \to \mathfrak{m}_A/(\mathfrak{m}_\Lambda A + \mathfrak{m}_A^2)$.

4. The completed base category

06GV The following "completion" of the category $\mathcal{C}_\Lambda$ will serve as the base category of the completion of a category cofibered in groupoids over $\mathcal{C}_\Lambda$ (Section 7).

06GW **Definition 4.1.** Let Λ be a Noetherian ring and let $\Lambda \to k$ be a finite ring map where k is a field. We define $\widehat{\mathcal{C}}_\Lambda$ to be the category with

(1) objects are pairs (R, φ) where R is a Noetherian complete local Λ-algebra and where $\varphi : R/\mathfrak{m}_R \to k$ is a Λ-algebra isomorphism, and
(2) morphisms $f : (S, \psi) \to (R, \varphi)$ are local Λ-algebra homomorphisms such that $\varphi \circ (f \bmod \mathfrak{m}) = \psi$.

As in the discussion following Definition 3.1 we will usually denote an object of $\widehat{\mathcal{C}}_\Lambda$ simply R, with the identification $R/\mathfrak m_R = k$ understood. In this section we discuss some basic properties of objects and morphisms of the category $\widehat{\mathcal{C}}_\Lambda$ paralleling our discussion of the category $\mathcal{C}_\Lambda$ in the previous section.

Our first observation is that any object $A \in \mathcal{C}_\Lambda$ is an object of $\widehat{\mathcal{C}}_\Lambda$ as an Artinian local ring is always Noetherian and complete with respect to its maximal ideal (which is after all a nilpotent ideal). Moreover, it is clear from the definitions that $\mathcal{C}_\Lambda \subset \widehat{\mathcal{C}}_\Lambda$ is the strictly full subcategory consisting of all Artinian rings. As it turns out, conversely every object of $\widehat{\mathcal{C}}_\Lambda$ is a limit of objects of $\mathcal{C}_\Lambda$.

Suppose that R is an object of $\widehat{\mathcal{C}}_\Lambda$. Consider the rings $R_n = R/\mathfrak m_R^n$ for $n \in \mathbf{N}$. These are Noetherian local rings with a unique nilpotent prime ideal, hence Artinian, see Algebra, Proposition 60.7. The ring maps

$$\ldots \to R_{n+1} \to R_n \to \ldots \to R_2 \to R_1 = k$$

are all surjective. Completeness of R by definition means that $R = \lim R_n$. If $f : R \to S$ is a ring map in $\widehat{\mathcal{C}}_\Lambda$ then we obtain a system of ring maps $f_n : R_n \to S_n$ whose limit is the given map.

06GZ **Lemma 4.2.** *Let $f : R \to S$ be a ring map in $\widehat{\mathcal{C}}_\Lambda$. The following are equivalent*

(1) *f is surjective,*
(2) *the map $\mathfrak m_R/\mathfrak m_R^2 \to \mathfrak m_S/\mathfrak m_S^2$ is surjective, and*
(3) *the map $\mathfrak m_R/(\mathfrak m_\Lambda R + \mathfrak m_R^2) \to \mathfrak m_S/(\mathfrak m_\Lambda S + \mathfrak m_S^2)$ is surjective.*

Proof. Note that for $n \geq 2$ we have the equality of relative cotangent spaces

$$\mathfrak m_R/(\mathfrak m_\Lambda R + \mathfrak m_R^2) = \mathfrak m_{R_n}/(\mathfrak m_\Lambda R_n + \mathfrak m_{R_n}^2)$$

and similarly for S. Hence by Lemma 3.5 we see that $R_n \to S_n$ is surjective for all n. Now let K_n be the kernel of $R_n \to S_n$. Then the sequences

$$0 \to K_n \to R_n \to S_n \to 0$$

form an exact sequence of directed inverse systems. The system (K_n) is Mittag-Leffler since each K_n is Artinian. Hence by Algebra, Lemma 86.4 taking limits preserves exactness. So $\lim R_n \to \lim S_n$ is surjective, i.e., f is surjective. □

06SB **Lemma 4.3.** *The category $\widehat{\mathcal{C}}_\Lambda$ admits pushouts.*

Proof. Let $R \to S_1$ and $R \to S_2$ be morphisms of $\widehat{\mathcal{C}}_\Lambda$. Consider the ring $C = S_1 \otimes_R S_2$. This ring has a finitely generated maximal ideal $\mathfrak m = \mathfrak m_{S_1} \otimes S_2 + S_1 \otimes \mathfrak m_{S_2}$ with residue field k. Set $C^\wedge$ equal to the completion of C with respect to $\mathfrak m$. Then $C^\wedge$ is a Noetherian ring complete with respect to the maximal ideal $\mathfrak m^\wedge = \mathfrak m C^\wedge$ whose residue field is identified with k, see Algebra, Lemma 97.5. Hence $C^\wedge$ is an object of $\widehat{\mathcal{C}}_\Lambda$. Then $S_1 \to C^\wedge$ and $S_2 \to C^\wedge$ turn $C^\wedge$ into a pushout over R in $\widehat{\mathcal{C}}_\Lambda$ (details omitted). □

We will not need the following lemma.

06H1 **Lemma 4.4.** *The category $\widehat{\mathcal{C}}_\Lambda$ admits coproducts of pairs of objects.*

Proof. Let R and S be objects of $\widehat{\mathcal{C}}_\Lambda$. Consider the ring $C = R \otimes_\Lambda S$. There is a canonical surjective map $C \to R \otimes_\Lambda S \to k \otimes_\Lambda k \to k$ where the last map is the multiplication map. The kernel of $C \to k$ is a maximal ideal $\mathfrak m$. Note that $\mathfrak m$ is generated by $\mathfrak{m}_R C$, $\mathfrak{m}_S C$ and finitely many elements of C which map to generators of the kernel of $k \otimes_\Lambda k \to k$. Hence $\mathfrak{m}$ is a finitely generated ideal. Set $C^\wedge$ equal to the completion of C with respect to $\mathfrak{m}$. Then $C^\wedge$ is a Noetherian ring complete with respect to the maximal ideal $\mathfrak{m}^\wedge = \mathfrak{m}C^\wedge$ with residue field k, see Algebra, Lemma 97.5. Hence $C^\wedge$ is an object of $\widehat{\mathcal{C}}_\Lambda$. Then $R \to C^\wedge$ and $S \to C^\wedge$ turn $C^\wedge$ into a coproduct in $\widehat{\mathcal{C}}_\Lambda$ (details omitted). □

An empty coproduct in a category is an initial object of the category. In the classical case $\widehat{\mathcal{C}}_\Lambda$ has an initial object, namely Λ itself. More generally, if $k' = k$, then the completion $\Lambda^\wedge$ of Λ with respect to $\mathfrak{m}_\Lambda$ is an initial object. More generally still, if $k' \subset k$ is separable, then $\widehat{\mathcal{C}}_\Lambda$ has an initial object too. Namely, choose a monic polynomial $P \in \Lambda[T]$ such that $k \cong k'[T]/(P')$ where $p' \in k'[T]$ is the image of P. Then $R = \Lambda^\wedge[T]/(P)$ is an initial object, see proof of Lemma 3.8.

If R is an initial object as above, then we have $\mathcal{C}_\Lambda = \mathcal{C}_R$ and $\widehat{\mathcal{C}}_\Lambda = \widehat{\mathcal{C}}_R$ which effectively brings the whole discussion in this chapter back to the classical case. But, if $k' \subset k$ is inseparable, then an initial object does not exist.

06SC **Lemma 4.5.** *Let S be an object of $\widehat{\mathcal{C}}_\Lambda$. Then $\dim_k Der_\Lambda(S, k) < \infty$.*

Proof. Let $x_1, \ldots, x_n \in \mathfrak{m}_S$ map to a k-basis for the relative cotangent space $\mathfrak{m}_S/(\mathfrak{m}_\Lambda S + \mathfrak{m}_S^2)$. Choose $y_1, \ldots, y_m \in S$ whose images in k generate k over k'. We claim that $\dim_k \text{Der}_\Lambda(S, k) \leq n+m$. To see this it suffices to prove that if $D(x_i) = 0$ and $D(y_j) = 0$, then $D = 0$. Let $a \in S$. We can find a polynomial $P = \sum \lambda_J y^J$ with $\lambda_J \in \Lambda$ whose image in k is the same as the image of a in k. Then we see that $D(a - P) = D(a) - D(P) = D(a)$ by our assumption that $D(y_j) = 0$ for all j. Thus we may assume $a \in \mathfrak{m}_S$. Write $a = \sum a_i x_i$ with $a_i \in S$. By the Leibniz rule

$$D(a) = \sum x_i D(a_i) + \sum a_i D(x_i) = \sum x_i D(a_i)$$

as we assumed $D(x_i) = 0$. We have $\sum x_i D(a_i) = 0$ as multiplication by x_i is zero on k. □

06SD **Lemma 4.6.** *Let $f : R \to S$ be a morphism of $\widehat{\mathcal{C}}_\Lambda$. If $Der_\Lambda(S, k) \to Der_\Lambda(R, k)$ is injective, then f is surjective.*

Proof. If f is not surjective, then $\mathfrak{m}_S/(\mathfrak{m}_R S + \mathfrak{m}_S^2)$ is nonzero by Lemma 4.2. Then also $Q = S/(f(R) + \mathfrak{m}_R S + \mathfrak{m}_S^2)$ is nonzero. Note that Q is a $k = R/\mathfrak{m}_R$-vector space via f. We turn Q into an S-module via $S \to k$. The quotient map $D : S \to Q$ is an R-derivation: if $a_1, a_2 \in S$, we can write $a_1 = f(b_1) + a_1'$ and $a_2 = f(b_2) + a_2'$ for some $b_1, b_2 \in R$ and $a_1', a_2' \in \mathfrak{m}_S$. Then b_i and a_i have the same image in k for $i = 1, 2$ and

$$\begin{aligned} a_1 a_2 &= (f(b_1) + a_1')(f(b_2) + a_2') \\ &= f(b_1)a_2' + f(b_2)a_1' \\ &= f(b_1)(f(b_2) + a_2') + f(b_2)(f(b_1) + a_1') \\ &= f(b_1)a_2 + f(b_2)a_1 \end{aligned}$$

in Q which proves the Leibniz rule. Hence $D : S \to Q$ is a Λ-derivation which is zero on composing with $R \to S$. Since $Q \neq 0$ there also exist derivations $D : S \to k$ which are zero on composing with $R \to S$, i.e., $\text{Der}_\Lambda(S, k) \to \text{Der}_\Lambda(R, k)$ is not injective. □

06SE **Lemma 4.7.** *Let R be an object of $\widehat{\mathcal{C}}_\Lambda$. Let (J_n) be a decreasing sequence of ideals such that $\mathfrak{m}_R^n \subset J_n$. Set $J = \bigcap J_n$. Then the sequence (J_n/J) defines the $\mathfrak{m}_{R/J}$-adic topology on R/J.*

Proof. It is clear that $\mathfrak{m}_{R/J}^n \subset J_n/J$. Thus it suffices to show that for every n there exists an N such that $J_N/J \subset \mathfrak{m}_{R/J}^n$. This is equivalent to $J_N \subset \mathfrak{m}_R^n + J$. For each n the ring $R/\mathfrak{m}_R^n$ is Artinian, hence there exists a N_n such that

$$J_{N_n} + \mathfrak{m}_R^n = J_{N_n+1} + \mathfrak{m}_R^n = \ldots$$

Set $E_n = (J_{N_n} + \mathfrak{m}_R^n)/\mathfrak{m}_R^n$. Set $E = \lim E_n \subset \lim R/\mathfrak{m}_R^n = R$. Note that $E \subset J$ as for any $f \in E$ and any m we have $f \in J_m + \mathfrak{m}_R^n$ for all $n \gg 0$, so $f \in J_m$ by Krull's intersection theorem, see Algebra, Lemma 51.4. Since the transition maps $E_n \to E_{n-1}$ are all surjective, we see that J surjects onto E_n. Hence for $N = N_n$ works. □

06SF **Lemma 4.8.** *Let $\ldots \to A_3 \to A_2 \to A_1$ be a sequence of surjective ring maps in $\mathcal{C}_\Lambda$. If $\dim_k(\mathfrak{m}_{A_n}/\mathfrak{m}_{A_n}^2)$ is bounded, then $S = \lim A_n$ is an object in $\widehat{\mathcal{C}}_\Lambda$ and the ideals $I_n = \mathrm{Ker}(S \to A_n)$ define the $\mathfrak{m}_S$-adic topology on S.*

Proof. We will use freely that the maps $S \to A_n$ are surjective for all n. Note that the maps $\mathfrak{m}_{A_{n+1}}/\mathfrak{m}_{A_{n+1}}^2 \to \mathfrak{m}_{A_n}/\mathfrak{m}_{A_n}^2$ are surjective, see Lemma 4.2. Hence for n sufficiently large the dimension $\dim_k(\mathfrak{m}_{A_n}/\mathfrak{m}_{A_n}^2)$ stabilizes to an integer, say r. Thus we can find $x_1, \ldots, x_r \in \mathfrak{m}_S$ whose images in A_n generate $\mathfrak{m}_{A_n}$. Moreover, pick $y_1, \ldots, y_t \in S$ whose images in k generate k over Λ. Then we get a ring map $P = \Lambda[z_1, \ldots, z_{r+t}] \to S$, $z_i \mapsto x_i$ and $z_{r+j} \mapsto y_j$ such that the composition $P \to S \to A_n$ is surjective for all n. Let $\mathfrak{m} \subset P$ be the kernel of $P \to k$. Let $R = P^\wedge$ be the $\mathfrak{m}$-adic completion of P; this is an object of $\widehat{\mathcal{C}}_\Lambda$. Since we still have the compatible system of (surjective) maps $R \to A_n$ we get a map $R \to S$. Set $J_n = \mathrm{Ker}(R \to A_n)$. Set $J = \bigcap J_n$. By Lemma 4.7 we see that $R/J = \lim R/J_n = \lim A_n = S$ and that the ideals $J_n/J = I_n$ define the $\mathfrak{m}$-adic topology. (Note that for each n we have $\mathfrak{m}_R^{N_n} \subset J_n$ for some N_n and not necessarily $N_n = n$, so a renumbering of the ideals J_n may be necessary before applying the lemma.) □

06SG **Lemma 4.9.** *Let $R', R \in \mathrm{Ob}(\widehat{\mathcal{C}}_\Lambda)$. Suppose that $R = R' \oplus I$ for some ideal I of R. Let $x_1, \ldots, x_r \in I$ map to a basis of $I/\mathfrak{m}_R I$. Set $S = R'[[X_1, \ldots, X_r]]$ and consider the R'-algebra map $S \to R$ mapping X_i to x_i. Assume that for every $n \gg 0$ the map $S/\mathfrak{m}_S^n \to R/\mathfrak{m}_R^n$ has a left inverse in $\mathcal{C}_\Lambda$. Then $S \to R$ is an isomorphism.*

Proof. As $R = R' \oplus I$ we have

$$\mathfrak{m}_R/\mathfrak{m}_R^2 = \mathfrak{m}_{R'}/\mathfrak{m}_{R'}^2 \oplus I/\mathfrak{m}_R I$$

and similarly

$$\mathfrak{m}_S/\mathfrak{m}_S^2 = \mathfrak{m}_{R'}/\mathfrak{m}_{R'}^2 \oplus \bigoplus kX_i$$

Hence for $n > 1$ the map $S/\mathfrak{m}_S^n \to R/\mathfrak{m}_R^n$ induces an isomorphism on cotangent spaces. Thus a left inverse $h_n : R/\mathfrak{m}_R^n \to S/\mathfrak{m}_S^n$ is surjective by Lemma 4.2. Since h_n is injective as a left inverse it is an isomorphism. Thus the canonical surjections $S/\mathfrak{m}_S^n \to R/\mathfrak{m}_R^n$ are all isomorphisms and we win. □

5. Categories cofibered in groupoids

06GA In developing the theory we work with categories *cofibered* in groupoids. We assume as known the definition and basic properties of categories *fibered* in groupoids, see Categories, Section 35.

06GJ **Definition 5.1.** Let $\mathcal{C}$ be a category. A *category cofibered in groupoids over* $\mathcal{C}$ is a category $\mathcal{F}$ equipped with a functor $p : \mathcal{F} \to \mathcal{C}$ such that $\mathcal{F}^{opp}$ is a category fibered in groupoids over $\mathcal{C}^{opp}$ via $p^{opp} : \mathcal{F}^{opp} \to \mathcal{C}^{opp}$.

Explicitly, $p : \mathcal{F} \to \mathcal{C}$ is cofibered in groupoids if the following two conditions hold:

(1) For every morphism $f : U \to V$ in $\mathcal{C}$ and every object x lying over U, there is a morphism $x \to y$ of $\mathcal{F}$ lying over f.
(2) For every pair of morphisms $a : x \to y$ and $b : x \to z$ of $\mathcal{F}$ and any morphism $f : p(y) \to p(z)$ such that $p(b) = f \circ p(a)$, there exists a unique morphism $c : y \to z$ of $\mathcal{F}$ lying over f such that $b = c \circ a$.

06GK **Remarks 5.2.** Everything about categories fibered in groupoids translates directly to the cofibered setting. The following remarks are meant to fix notation. Let $\mathcal{C}$ be a category.

(1) We often omit the functor $p : \mathcal{F} \to \mathcal{C}$ from the notation.
(2) The fiber category over an object U in $\mathcal{C}$ is denoted by $\mathcal{F}(U)$. Its objects are those of $\mathcal{F}$ lying over U and its morphisms are those of $\mathcal{F}$ lying over id_U. If x, y are objects of $\mathcal{F}(U)$, we sometimes write $\text{Mor}_U(x, y)$ for $\text{Mor}_{\mathcal{F}(U)}(x, y)$.
(3) The fibre categories $\mathcal{F}(U)$ are groupoids, see Categories, Lemma 35.2. Hence the morphisms in $\mathcal{F}(U)$ are all isomorphisms. We sometimes write $\text{Aut}_U(x)$ for $\text{Mor}_{\mathcal{F}(U)}(x, x)$.

06SH (4) Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}$, let $f : U \to V$ be a morphism in $\mathcal{C}$, and let $x \in \text{Ob}(\mathcal{F}(U))$. A *pushforward* of x along f is a morphism $x \to y$ of $\mathcal{F}$ lying over f. A pushforward is unique up to unique isomorphism (see the discussion following Categories, Definition 33.1). We sometimes write $x \to f_*x$ for "the" pushforward of x along f.
(5) A *choice of pushforwards for* $\mathcal{F}$ is the choice of a pushforward of x along f for every pair (x, f) as above. We can make such a choice of pushforwards for $\mathcal{F}$ by the axiom of choice.
(6) Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}$. Given a choice of pushforwards for $\mathcal{F}$, there is an associated pseudo-functor $\mathcal{C} \to$ *Groupoids*. We will never use this construction so we give no details.

06GL (7) A morphism of categories cofibered in groupoids over $\mathcal{C}$ is a functor commuting with the projections to $\mathcal{C}$. If $\mathcal{F}$ and $\mathcal{F}'$ are categories cofibered in groupoids over $\mathcal{C}$, we denote the morphisms from $\mathcal{F}$ to $\mathcal{F}'$ by $\text{Mor}_{\mathcal{C}}(\mathcal{F}, \mathcal{F}')$.

06GM (8) Categories cofibered in groupoids form a $(2,1)$-category $\text{Cof}(\mathcal{C})$. Its 1-morphisms are the morphisms described in (7). If $p : \mathcal{F} \to C$ and $p' : \mathcal{F}' \to \mathcal{C}$ are categories cofibered in groupoids and $\varphi, \psi : \mathcal{F} \to \mathcal{F}'$ are 1-morphisms, then a 2-morphism $t : \varphi \to \psi$ is a morphism of functors such that $p'(t_x) = \text{id}_{p(x)}$ for all $x \in \text{Ob}(\mathcal{F})$.

06GN (9) Let $F : \mathcal{C} \to$ *Groupoids* be a functor. There is a category cofibered in groupoids $\mathcal{F} \to \mathcal{C}$ associated to F as follows. An object of $\mathcal{F}$ is a pair (U, x) where $U \in \text{Ob}(\mathcal{C})$ and $x \in \text{Ob}(F(U))$. A morphism $(U, x) \to (V, y)$ is a pair

(f, a) where $f \in \mathrm{Mor}_{\mathcal{C}}(U, V)$ and $a \in \mathrm{Mor}_{F(V)}(F(f)(x), y)$. The functor $\mathcal{F} \to \mathcal{C}$ sends (U, x) to U. See Categories, Section 37.

07W5 (10) Let $\mathcal{F}$ be cofibered in groupoids over $\mathcal{C}$. For $U \in \mathrm{Ob}(\mathcal{C})$ set $\overline{\mathcal{F}}(U)$ equal to the set of isomorphisms classes of the category $\mathcal{F}(U)$. If $f : U \to V$ is a morphism of $\mathcal{C}$, then we obtain a map of sets $\overline{\mathcal{F}}(U) \to \overline{\mathcal{F}}(V)$ by mapping the isomorphism class of x to the isomorphism class of a pushforward f_*x of x see (4). Then $\overline{\mathcal{F}} : \mathcal{C} \to \mathit{Sets}$ is a functor. Similarly, if $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of cofibered categories, we denote by $\overline{\varphi} : \overline{\mathcal{F}} \to \overline{\mathcal{G}}$ the associated morphism of functors.

06GP (11) Let $F : \mathcal{C} \to \mathit{Sets}$ be a functor. We can think of a set as a discrete category, i.e., as a groupoid with only identity morphisms. Then the construction (9) associates to F a category cofibered in sets. This defines a fully faithful embedding of the category of functors $\mathcal{C} \to \mathit{Sets}$ to the category of categories cofibered in groupoids over $\mathcal{C}$. We identify the category of functors with its image under this embedding. Hence if $F : \mathcal{C} \to \mathit{Sets}$ is a functor, we denote the associated category cofibered in sets also by F; and if $\varphi : F \to G$ is a morphism of functors, we denote still by φ the corresponding morphism of categories cofibered in sets, and vice-versa. See Categories, Section 38.

06GQ (12) Let U be an object of $\mathcal{C}$. We write $\underline{U}$ for the functor $\mathrm{Mor}_{\mathcal{C}}(U, -) : \mathcal{C} \to \mathit{Sets}$. This defines a fully faithful embedding of $\mathcal{C}^{opp}$ into the category of functors $\mathcal{C} \to \mathit{Sets}$. Hence, if $f : U \to V$ is a morphism, we are justified in denoting still by f the induced morphism $\underline{V} \to \underline{U}$, and vice-versa.

06SI (13) Fiber products of categories cofibered in groupoids: If $\mathcal{F} \to \mathcal{H}$ and $\mathcal{G} \to \mathcal{H}$ are morphisms of categories cofibered in groupoids over $\mathcal{C}_\Lambda$, then a construction of their 2-fiber product is given by the construction for their 2-fiber product as categories over $\mathcal{C}_\Lambda$, as described in Categories, Lemma 32.3.

0DZJ (14) Products of categories cofibered in groupoids: If $\mathcal{F}$ and $\mathcal{G}$ are categories cofibered in groupoids over $\mathcal{C}_\Lambda$ then their product is defined to be the 2-fiber product $\mathcal{F} \times_{\mathcal{C}_\Lambda} \mathcal{G}$ as described in Categories, Lemma 32.3.

06GR (15) Restricting the base category: Let $p : \mathcal{F} \to \mathcal{C}$ be a category cofibered in groupoids, and let $\mathcal{C}'$ be a full subcategory of $\mathcal{C}$. The restriction $\mathcal{F}|_{\mathcal{C}'}$ is the full subcategory of $\mathcal{F}$ whose objects lie over objects of $\mathcal{C}'$. It is a category cofibered in groupoids via the functor $p|_{\mathcal{C}'} : \mathcal{F}|_{\mathcal{C}'} \to \mathcal{C}'$.

6. Prorepresentable functors and predeformation categories

06GI Our basic goal is to understand categories cofibered in groupoids over $\mathcal{C}_\Lambda$ and $\widehat{\mathcal{C}}_\Lambda$. Since $\mathcal{C}_\Lambda$ is a full subcategory of $\widehat{\mathcal{C}}_\Lambda$ we can restrict categories cofibred in groupoids over $\widehat{\mathcal{C}}_\Lambda$ to $\mathcal{C}_\Lambda$, see Remarks 5.2 (15). In particular we can do this with functors, in particular with representable functors. The functors on $\mathcal{C}_\Lambda$ one obtains in this way are called prorepresentable functors.

06GX **Definition 6.1.** Let $F : \mathcal{C}_\Lambda \to \mathit{Sets}$ be a functor. We say F is *prorepresentable* if there exists an isomorphism $F \cong \underline{R}|_{\mathcal{C}_\Lambda}$ of functors for some $R \in \mathrm{Ob}(\widehat{\mathcal{C}}_\Lambda)$.

Note that if $F : \mathcal{C}_\Lambda \to \mathit{Sets}$ is prorepresentable by $R \in \mathrm{Ob}(\widehat{\mathcal{C}}_\Lambda)$, then

$$F(k) = \mathrm{Mor}_{\widehat{\mathcal{C}}_\Lambda}(R, k) = \{*\}$$

is a singleton. The categories cofibered in groupoids over $\mathcal{C}_\Lambda$ that are arise in deformation theory will often satisfy an analogous condition.

06GS **Definition 6.2.** A *predeformation category* $\mathcal{F}$ is a category cofibered in groupoids over $\mathcal{C}_\Lambda$ such that $\mathcal{F}(k)$ is equivalent to a category with a single object and a single morphism, i.e., $\mathcal{F}(k)$ contains at least one object and there is a unique morphism between any two objects. A *morphism of predeformation categories* is a morphism of categories cofibered in groupoids over $\mathcal{C}_\Lambda$.

A feature of a predeformation category is the following. Let $x_0 \in \mathrm{Ob}(\mathcal{F}(k))$. Then every object of $\mathcal{F}$ comes equipped with a unique morphism to x_0. Namely, if x is an object of $\mathcal{F}$ over A, then we can choose a pushforward $x \to q_*x$ where $q : A \to k$ is the quotient map. There is a unique isomorphism $q_*x \to x_0$ and the composition $x \to q_*x \to x_0$ is the desired morphism.

06GT **Remark 6.3.** We say that a functor $F : \mathcal{C}_\Lambda \to \textit{Sets}$ is a *predeformation functor* if the associated cofibered set is a predeformation category, i.e. if $F(k)$ is a one element set. Thus if $\mathcal{F}$ is a predeformation category, then $\overline{\mathcal{F}}$ is a predeformation functor.

06GU **Remark 6.4.** Let $p : \mathcal{F} \to \mathcal{C}_\Lambda$ be a category cofibered in groupoids, and let $x \in \mathrm{Ob}(\mathcal{F}(k))$. We denote by $\mathcal{F}_x$ the category of objects over x. An object of $\mathcal{F}_x$ is an arrow $y \to x$. A morphism $(y \to x) \to (z \to x)$ in $\mathcal{F}_x$ is a commutative diagram

$$\begin{tikzcd} y \arrow[rr] \arrow[dr] & & z \arrow[dl] \\ & x & \end{tikzcd}$$

There is a forgetful functor $\mathcal{F}_x \to \mathcal{F}$. We define the functor $p_x : \mathcal{F}_x \to \mathcal{C}_\Lambda$ as the composition $\mathcal{F}_x \to \mathcal{F} \xrightarrow{p} \mathcal{C}_\Lambda$. Then $p_x : \mathcal{F}_x \to \mathcal{C}_\Lambda$ is a predeformation category (proof omitted). In this way we can pass from an arbitrary category cofibered in groupoids over $\mathcal{C}_\Lambda$ to a predeformation category at any $x \in \mathrm{Ob}(\mathcal{F}(k))$.

7. Formal objects and completion categories

06H2 In this section we discuss how to go between categories cofibred in groupoids over $\mathcal{C}_\Lambda$ to categories cofibred in groupoids over $\widehat{\mathcal{C}}_\Lambda$ and vice versa.

06H3 **Definition 7.1.** Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. The *category* $\widehat{\mathcal{F}}$ *of formal objects of* $\mathcal{F}$ is the category with the following objects and morphisms.

(1) A *formal object* $\xi = (R, \xi_n, f_n)$ *of* $\mathcal{F}$ consists of an object R of $\widehat{\mathcal{C}}_\Lambda$, and a collection indexed by $n \in \mathbf{N}$ of objects ξ_n of $\mathcal{F}(R/\mathfrak{m}_R^n)$ and morphisms $f_n : \xi_{n+1} \to \xi_n$ lying over the projection $R/\mathfrak{m}_R^{n+1} \to R/\mathfrak{m}_R^n$.
(2) Let $\xi = (R, \xi_n, f_n)$ and $\eta = (S, \eta_n, g_n)$ be formal objects of $\mathcal{F}$. A *morphism* $a : \xi \to \eta$ *of formal objects* consists of a map $a_0 : R \to S$ in $\widehat{\mathcal{C}}_\Lambda$ and a collection $a_n : \xi_n \to \eta_n$ of morphisms of $\mathcal{F}$ lying over $R/\mathfrak{m}_R^n \to S/\mathfrak{m}_S^n$, such that for every n the diagram

$$\begin{tikzcd} \xi_{n+1} \arrow[r, "f_n"'] \arrow[d, "a_{n+1}"'] & \xi_n \arrow[d, "a_n"] \\ \eta_{n+1} \arrow[r, "g_n"] & \eta_n \end{tikzcd}$$

commutes.

The category of formal objects comes with a functor $\widehat{p} : \widehat{\mathcal{F}} \to \widehat{\mathcal{C}}_\Lambda$ which sends an object (R, ξ_n, f_n) to R and a morphism $(R, \xi_n, f_n) \to (S, \eta_n, g_n)$ to the map $R \to S$.

06H4 **Lemma 7.2.** *Let $p : \mathcal{F} \to \mathcal{C}_\Lambda$ be a category cofibered in groupoids. Then $\widehat{p} : \widehat{\mathcal{F}} \to \widehat{\mathcal{C}}_\Lambda$ is a category cofibered in groupoids.*

Proof. Let $R \to S$ be a ring map in $\widehat{\mathcal{C}}_\Lambda$. Let (R, ξ_n, f_n) be an object of $\widehat{\mathcal{F}}$. For each n choose a pushforward $\xi_n \to \eta_n$ of ξ_n along $R/\mathfrak{m}_R^n \to S/\mathfrak{m}_S^n$. For each n there exists a unique morphism $g_n : \eta_{n+1} \to \eta_n$ in $\mathcal{F}$ lying over $S/\mathfrak{m}_S^{n+1} \to S/\mathfrak{m}_S^n$ such that

$$\begin{tikzcd} \xi_{n+1} \arrow[r, "f_n"'] \arrow[d] & \xi_n \arrow[d] \\ \eta_{n+1} \arrow[r, "g_n"] & \eta_n \end{tikzcd}$$

commutes (by the first axiom of a category cofibred in groupoids). Hence we obtain a morphism $(R, \xi_n, f_n) \to (S, \eta_n, g_n)$ lying over $R \to S$, i.e., the first axiom of a category cofibred in groupoids holds for $\widehat{\mathcal{F}}$. To see the second axiom suppose that we have morphisms $a : (R, \xi_n, f_n) \to (S, \eta_n, g_n)$ and $b : (R, \xi_n, f_n) \to (T, \theta_n, h_n)$ in $\widehat{\mathcal{F}}$ and a morphism $c_0 : S \to T$ in $\widehat{\mathcal{C}}_\Lambda$ such that $c_0 \circ a_0 = b_0$. By the second axiom of a category cofibred in groupoids for $\mathcal{F}$ we obtain unique maps $c_n : \eta_n \to \theta_n$ lying over $S/\mathfrak{m}_S^n \to T/\mathfrak{m}_T^n$ such that $c_n \circ a_n = b_n$. Setting $c = (c_n)_{n \geq 0}$ gives the desired morphism $c : (S, \eta_n, g_n) \to (T, \theta_n, h_n)$ in $\widehat{\mathcal{F}}$ (we omit the verification that $h_n \circ c_{n+1} = c_n \circ g_n$). $\square$

06H5 **Definition 7.3.** Let $p : \mathcal{F} \to \mathcal{C}_\Lambda$ be a category cofibered in groupoids. The category cofibered in groupoids $\widehat{p} : \widehat{\mathcal{F}} \to \widehat{\mathcal{C}}_\Lambda$ is called the *completion of $\mathcal{F}$*.

If $\mathcal{F}$ is a category cofibered in groupoids over $\mathcal{C}_\Lambda$, we have defined $\widehat{\mathcal{F}}(R)$ for $R \in \mathrm{Ob}(\widehat{\mathcal{C}}_\Lambda)$ in terms of the filtration of R by powers of its maximal ideal. But suppose $\mathcal{I} = (I_n)$ is a filtration of R by ideals inducing the $\mathfrak{m}_R$-adic topology. We define $\widehat{\mathcal{F}}_\mathcal{I}(R)$ to be the category with the following objects and morphisms:

(1) An object is a collection $(\xi_n, f_n)_{n \in \mathbf{N}}$ of objects ξ_n of $\mathcal{F}(R/I_n)$ and morphisms $f_n : \xi_{n+1} \to \xi_n$ lying over the projections $R/I_{n+1} \to R/I_n$.
(2) A morphism $a : (\xi_n, f_n) \to (\eta_n, g_n)$ consists of a collection $a_n : \xi_n \to \eta_n$ of morphisms in $\mathcal{F}(R/I_n)$, such that for every n the diagram

$$\begin{tikzcd} \xi_{n+1} \arrow[r, "f_n"] \arrow[d, "a_{n+1}"'] & \xi_n \arrow[d, "a_n"] \\ \eta_{n+1} \arrow[r, "g_n"] & \eta_n \end{tikzcd}$$

commutes.

06H6 **Lemma 7.4.** *In the situation above, $\widehat{\mathcal{F}}_\mathcal{I}(R)$ is equivalent to the category $\widehat{\mathcal{F}}(R)$.*

Proof. An equivalence $\widehat{\mathcal{F}}_\mathcal{I}(R) \to \widehat{\mathcal{F}}(R)$ can be defined as follows. For each n, let $m(n)$ be the least m that $I_m \subset \mathfrak{m}_R^n$. Given an object (ξ_n, f_n) of $\widehat{\mathcal{F}}_\mathcal{I}(R)$, let η_n be the pushforward of $\xi_{m(n)}$ along $R/I_{m(n)} \to R/\mathfrak{m}_R^n$. Let $g_n : \eta_{n+1} \to \eta_n$ be the unique morphism of $\mathcal{F}$ lying over $R/\mathfrak{m}_R^{n+1} \to R/\mathfrak{m}_R^n$ such that

$$\begin{tikzcd} \xi_{m(n+1)} \arrow[r, "f_{m(n)} \circ \ldots \circ f_{m(n+1)-1}"'] \arrow[d] & \xi_{m(n)} \arrow[d] \\ \eta_{n+1} \arrow[r, "g_n"] & \eta_n \end{tikzcd}$$

commutes (existence and uniqueness is guaranteed by the axioms of a cofibred category). The functor $\widehat{\mathcal{F}}_{\mathcal{I}}(R) \to \widehat{\mathcal{F}}(R)$ sends (ξ_n, f_n) to (R, η_n, g_n). We omit the verification that this is indeed an equivalence of categories. $\square$

06H7 **Remark 7.5.** Let $p : \mathcal{F} \to \mathcal{C}_\Lambda$ be a category cofibered in groupoids. Suppose that for each $R \in \mathrm{Ob}(\widehat{\mathcal{C}}_\Lambda)$ we are given a filtration $\mathcal{I}_R$ of R by ideals. If $\mathcal{I}_R$ induces the $\mathfrak{m}_R$-adic topology on R for all R, then one can define a category $\widehat{\mathcal{F}}_{\mathcal{I}}$ by mimicking the definition of $\widehat{\mathcal{F}}$. This category comes equipped with a morphism $\widehat{p}_{\mathcal{I}} : \widehat{\mathcal{F}}_{\mathcal{I}} \to \widehat{\mathcal{C}}_\Lambda$ making it into a category cofibered in groupoids such that $\widehat{\mathcal{F}}_{\mathcal{I}}(R)$ is isomorphic to $\widehat{\mathcal{F}}_{\mathcal{I}_R}(R)$ as defined above. The categories cofibered in groupoids $\widehat{\mathcal{F}}_{\mathcal{I}}$ and $\widehat{\mathcal{F}}$ are equivalent, by using over an object $R \in \mathrm{Ob}(\widehat{\mathcal{C}}_\Lambda)$ the equivalence of Lemma 7.4.

06H8 **Remark 7.6.** Let $F : \mathcal{C}_\Lambda \to \mathit{Sets}$ be a functor. Identifying functors with cofibered sets, the completion of F is the functor $\widehat{F} : \widehat{\mathcal{C}}_\Lambda \to \mathit{Sets}$ given by $\widehat{F}(S) = \lim F(S/\mathfrak{m}_S^n)$. This agrees with the definition in Schlessinger's paper [Sch68].

06SJ **Remark 7.7.** Let $\mathcal{F}$ be a category cofibred in groupoids over $\mathcal{C}_\Lambda$. We claim that there is a canonical equivalence

$$can : \widehat{\mathcal{F}}|_{\mathcal{C}_\Lambda} \longrightarrow \mathcal{F}.$$

Namely, let $A \in \mathrm{Ob}(\mathcal{C}_\Lambda)$ and let (A, ξ_n, f_n) be an object of $\widehat{\mathcal{F}}|_{\mathcal{C}_\Lambda}(A)$. Since A is Artinian there is a minimal $m \in \mathbf{N}$ such that $\mathfrak{m}_A^m = 0$. Then can sends (A, ξ_n, f_n) to ξ_m. This functor is an equivalence of categories cofibered in groupoids by Categories, Lemma 35.9 because it is an equivalence on all fibre categories by Lemma 7.4 and the fact that the $\mathfrak{m}_A$-adic topology on a local Artinian ring A comes from the zero ideal. We will frequently identify $\mathcal{F}$ with a full subcategory of $\widehat{\mathcal{F}}$ via a quasi-inverse to the functor can.

06H9 **Remark 7.8.** Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of categories cofibered in groupoids over $\mathcal{C}_\Lambda$. Then there is an induced morphism $\widehat{\varphi} : \widehat{\mathcal{F}} \to \widehat{\mathcal{G}}$ of categories cofibered in groupoids over $\widehat{\mathcal{C}}_\Lambda$. It sends an object $\xi = (R, \xi_n, f_n)$ of $\widehat{\mathcal{F}}$ to $(R, \varphi(\xi_n), \varphi(f_n))$, and it sends a morphism $(a_0 : R \to S, a_n : \xi_n \to \eta_n)$ between objects ξ and η of $\widehat{\mathcal{F}}$ to $(a_0 : R \to S, \varphi(a_n) : \varphi(\xi_n) \to \varphi(\eta_n))$. Finally, if $t : \varphi \to \varphi'$ is a 2-morphism between 1-morphisms $\varphi, \varphi' : \mathcal{F} \to \mathcal{G}$ of categories cofibred in groupoids, then we obtain a 2-morphism $\widehat{t} : \widehat{\varphi} \to \widehat{\varphi}'$. Namely, for $\xi = (R, \xi_n, f_n)$ as above we set $\widehat{t}_\xi = (t_{\varphi(\xi_n)})$. Hence completion defines a functor between 2-categories

$$\widehat{\ } : \mathrm{Cof}(\mathcal{C}_\Lambda) \longrightarrow \mathrm{Cof}(\widehat{\mathcal{C}}_\Lambda)$$

from the 2-category of categories cofibred in groupoids over $\mathcal{C}_\Lambda$ to the 2-category of categories cofibred in groupoids over $\widehat{\mathcal{C}}_\Lambda$.

06HA **Remark 7.9.** We claim the completion functor of Remark 7.8 and the restriction functor $|_{\mathcal{C}_\Lambda} : \mathrm{Cof}(\widehat{\mathcal{C}}_\Lambda) \to \mathrm{Cof}(\mathcal{C}_\Lambda)$ of Remarks 5.2 (15) are "2-adjoint" in the following precise sense. Let $\mathcal{F} \in \mathrm{Ob}(\mathrm{Cof}(\mathcal{C}_\Lambda))$ and let $\mathcal{G} \in \mathrm{Ob}(\mathrm{Cof}(\widehat{\mathcal{C}}_\Lambda))$. Then there is an equivalence of categories

$$\Phi : \mathrm{Mor}_{\mathcal{C}_\Lambda}(\mathcal{G}|_{\mathcal{C}_\Lambda}, \mathcal{F}) \longrightarrow \mathrm{Mor}_{\widehat{\mathcal{C}}_\Lambda}(\mathcal{G}, \widehat{\mathcal{F}})$$

To describe this equivalence, we define canonical morphisms $\mathcal{G} \to \widehat{\mathcal{G}|_{\mathcal{C}_\Lambda}}$ and $\widehat{\mathcal{F}}|_{\mathcal{C}_\Lambda} \to \mathcal{F}$ as follows

(1) Let $R \in \mathrm{Ob}(\widehat{\mathcal{C}}_\Lambda))$ and let ξ be an object of the fiber category $\mathcal{G}(R)$. Choose a pushforward $\xi \to \xi_n$ of ξ to $R/\mathfrak{m}_R^n$ for each $n \in \mathbf{N}$, and let $f_n : \xi_{n+1} \to \xi_n$ be the induced morphism. Then $\mathcal{G} \to \widehat{\mathcal{G}|_{\mathcal{C}_\Lambda}}$ sends ξ to (R, ξ_n, f_n).
(2) This is the equivalence $can : \widehat{\mathcal{F}}|_{\mathcal{C}_\Lambda} \to \mathcal{F}$ of Remark 7.7.

Having said this, the equivalence $\Phi : \mathrm{Mor}_{\mathcal{C}_\Lambda}(\mathcal{G}|_{\mathcal{C}_\Lambda}, \mathcal{F}) \to \mathrm{Mor}_{\widehat{\mathcal{C}}_\Lambda}(\mathcal{G}, \widehat{\mathcal{F}})$ sends a morphism $\varphi : \mathcal{G}|_{\mathcal{C}_\Lambda} \to \mathcal{F}$ to

$$\mathcal{G} \to \widehat{\mathcal{G}|_{\mathcal{C}_\Lambda}} \xrightarrow{\widehat{\varphi}} \widehat{\mathcal{F}}$$

There is a quasi-inverse $\Psi : \mathrm{Mor}_{\widehat{\mathcal{C}}_\Lambda}(\mathcal{G}, \widehat{\mathcal{F}}) \to \mathrm{Mor}_{\mathcal{C}_\Lambda}(\mathcal{G}|_{\mathcal{C}_\Lambda}, \mathcal{F})$ to Φ which sends $\psi : \mathcal{G} \to \widehat{\mathcal{F}}$ to

$$\mathcal{G}|_{\mathcal{C}_\Lambda} \xrightarrow{\psi|_{\mathcal{C}_\Lambda}} \widehat{\mathcal{F}}|_{\mathcal{C}_\Lambda} \to \mathcal{F}.$$

We omit the verification that Φ and Ψ are quasi-inverse. We also do not address functoriality of Φ (because it would lead into 3-category territory which we want to avoid at all cost).

06HB **Remark 7.10.** For a category $\mathcal{C}$ we denote by $\mathrm{CofSet}(\mathcal{C})$ the category of cofibered sets over $\mathcal{C}$. It is a 1-category isomorphic the category of functors $\mathcal{C} \to \textit{Sets}$. See Remarks 5.2 (11). The completion and restriction functors restrict to functors $\widehat{\ } : \mathrm{CofSet}(\mathcal{C}_\Lambda) \to \mathrm{CofSet}(\widehat{\mathcal{C}}_\Lambda)$ and $|_{\mathcal{C}_\Lambda} : \mathrm{CofSet}(\widehat{\mathcal{C}}_\Lambda) \to \mathrm{CofSet}(\mathcal{C}_\Lambda)$ which we denote by the same symbols. As functors on the categories of cofibered sets, completion and restriction are adjoints in the usual 1-categorical sense: the same construction as in Remark 7.9 defines a functorial bijection

$$\mathrm{Mor}_{\mathcal{C}_\Lambda}(G|_{\mathcal{C}_\Lambda}, F) \longrightarrow \mathrm{Mor}_{\widehat{\mathcal{C}}_\Lambda}(G, \widehat{F})$$

for $F \in \mathrm{Ob}(\mathrm{CofSet}(\mathcal{C}_\Lambda))$ and $G \in \mathrm{Ob}(\mathrm{CofSet}(\widehat{\mathcal{C}}_\Lambda))$. Again the map $\widehat{F}|_{\mathcal{C}_\Lambda} \to F$ is an isomorphism.

06HE **Remark 7.11.** Let $G : \widehat{\mathcal{C}}_\Lambda \to \textit{Sets}$ be a functor that commutes with limits. Then the map $G \to \widehat{G|_{\mathcal{C}_\Lambda}}$ described in Remark 7.9 is an isomorphism. Indeed, if S is an object of $\widehat{\mathcal{C}}_\Lambda$, then we have canonical bijections

$$\widehat{G|_{\mathcal{C}_\Lambda}}(S) = \lim_n G(S/\mathfrak{m}_S^n) = G(\lim_n S/\mathfrak{m}_S^n) = G(S).$$

In particular, if R is an object of $\widehat{\mathcal{C}}_\Lambda$ then $\underline{R} = \widehat{\underline{R}|_{\mathcal{C}_\Lambda}}$ because the representable functor $\underline{R}$ commutes with limits by definition of limits.

06HC **Remark 7.12.** Let R be an object of $\widehat{\mathcal{C}}_\Lambda$. It defines a functor $\underline{R} : \widehat{\mathcal{C}}_\Lambda \to \textit{Sets}$ as described in Remarks 5.2 (12). As usual we identify this functor with the associated cofibered set. If $\mathcal{F}$ is a cofibered category over $\mathcal{C}_\Lambda$, then there is an equivalence of categories

06SK (7.12.1)
$$\mathrm{Mor}_{\mathcal{C}_\Lambda}(\underline{R}|_{\mathcal{C}_\Lambda}, \mathcal{F}) \longrightarrow \widehat{\mathcal{F}}(R).$$

It is given by the composition

$$\mathrm{Mor}_{\mathcal{C}_\Lambda}(\underline{R}|_{\mathcal{C}_\Lambda}, \mathcal{F}) \xrightarrow{\Phi} \mathrm{Mor}_{\widehat{\mathcal{C}}_\Lambda}(\underline{R}, \widehat{\mathcal{F}}) \xrightarrow{\sim} \widehat{\mathcal{F}}(R)$$

where Φ is as in Remark 7.9 and the second equivalence comes from the 2-Yoneda lemma (the cofibered analogue of Categories, Lemma 41.2). Explicitly, the equivalence sends a morphism $\varphi : \underline{R}|_{\mathcal{C}_\Lambda} \to \mathcal{F}$ to the formal object $(R, \varphi(R \to R/\mathfrak{m}_R^n), \varphi(f_n))$ in $\widehat{\mathcal{F}}(R)$, where $f_n : R/\mathfrak{m}_R^{n+1} \to R/\mathfrak{m}_R^n$ is the projection.

Assume a choice of pushforwards for $\mathcal{F}$ has been made. Given any $\xi \in \mathrm{Ob}(\widehat{\mathcal{F}}(R))$ we construct an explicit $\underline{\xi} : \underline{R}|_{\mathcal{C}_\Lambda} \to \mathcal{F}$ which maps to ξ under (7.12.1). Namely, say $\xi = (R, \xi_n, f_n)$. An object α in $\underline{R}|_{\mathcal{C}_\Lambda}$ is the same thing as a morphism $\alpha : R \to A$ of $\widehat{\mathcal{C}}_\Lambda$ with A Artinian. Let $m \in \mathbf{N}$ be minimal such that $\mathfrak{m}_A^m = 0$. Then α factors through a unique $\alpha_m : R/\mathfrak{m}_R^m \to A$ and we can set $\underline{\xi}(\alpha) = \alpha_{m,*}\xi_m$. We omit the description of $\underline{\xi}$ on morphisms and we omit the proof that $\underline{\xi}$ maps to ξ via (7.12.1).

Assume a choice of pushforwards for $\widehat{\mathcal{F}}$ has been made. In this case the proof of Categories, Lemma 41.2 gives an explicit quasi-inverse

$$\iota : \widehat{\mathcal{F}}(R) \longrightarrow \mathrm{Mor}_{\widehat{\mathcal{C}}_\Lambda}(\underline{R}, \widehat{\mathcal{F}})$$

to the 2-Yoneda equivalence which takes ξ to the morphism $\iota(\xi) : \underline{R} \to \widehat{\mathcal{F}}$ sending $f \in \underline{R}(S) = \mathrm{Mor}_{\mathcal{C}_\Lambda}(R, S)$ to $f_*\xi$. A quasi-inverse to (7.12.1) is then

$$\widehat{\mathcal{F}}(R) \xrightarrow{\iota} \mathrm{Mor}_{\widehat{\mathcal{C}}_\Lambda}(\underline{R}, \widehat{\mathcal{F}}) \xrightarrow{\Psi} \mathrm{Mor}_{\mathcal{C}_\Lambda}(\underline{R}|_{\mathcal{C}_\Lambda}, \mathcal{F})$$

where Ψ is as in Remark 7.9. Given $\xi \in \mathrm{Ob}(\widehat{\mathcal{F}}(R))$ we have $\Psi(\iota(\xi)) \cong \underline{\xi}$ where $\underline{\xi}$ is as in the previous paragraph, because both are mapped to ξ under the equivalence of categories (7.12.1). Using $\underline{R} = \widehat{\underline{R}|_{\mathcal{C}_\Lambda}}$ (see Remark 7.11) and unwinding the definitions of Φ and Ψ we conclude that $\iota(\xi)$ is isomorphic to the completion of $\underline{\xi}$.

06SL **Remark 7.13.** Let $\mathcal{F}$ be a category cofibred in groupoids over $\mathcal{C}_\Lambda$. Let $\xi = (R, \xi_n, f_n)$ and $\eta = (S, \eta_n, g_n)$ be formal objects of $\mathcal{F}$. Let $a = (a_n) : \xi \to \eta$ be a morphism of formal objects, i.e., a morphism of $\widehat{\mathcal{F}}$. Let $f = \widehat{p}(a) = a_0 : R \to S$ be the projection of a in $\widehat{\mathcal{C}}_\Lambda$. Then we obtain a 2-commutative diagram

$$\begin{tikzcd}
\underline{R}|_{\mathcal{C}_\Lambda} \arrow[dr, "\underline{\xi}"'] & & \underline{S}|_{\mathcal{C}_\Lambda} \arrow[ll, "f"'] \arrow[dl, "\underline{\eta}"] \\
& \mathcal{F} &
\end{tikzcd}$$

where $\underline{\xi}$ and $\underline{\eta}$ are the morphisms constructed in Remark 7.12. To see this let $\alpha : S \to A$ be an object of $\underline{S}|_{\mathcal{C}_\Lambda}$ (see loc. cit.). Let $m \in \mathbf{N}$ be minimal such that $\mathfrak{m}_A^m = 0$. We get a commutative diagram

$$\begin{tikzcd}
R \arrow[r] \arrow[d, "f"] & R/\mathfrak{m}_R^m \arrow[d, "f_m"'] \arrow[dr, "\beta_m"] & \\
S \arrow[r] & S/\mathfrak{m}_S^m \arrow[r, "\alpha_m"] & A
\end{tikzcd}$$

such that the bottom arrows compose to give α. Then $\underline{\eta}(\alpha) = \alpha_{m,*}\eta_m$ and $\underline{\xi}(\alpha \circ f) = \beta_{m,*}\xi_m$. The morphism $a_m : \xi_m \to \eta_m$ lies over f_m hence we obtain a canonical morphism

$$\underline{\xi}(\alpha \circ f) = \beta_{m,*}\xi_m \longrightarrow \underline{\eta}(\alpha) = \alpha_{m,*}\eta_m$$

lying over id_A such that

$$\begin{tikzcd} \xi_m \arrow[r] \arrow[d, "a_m"] & \beta_{m,*}\xi_m \arrow[d] \\ \eta_m \arrow[r] & \alpha_{m,*}\eta_m \end{tikzcd}$$

commutes by the axioms of a category cofibred in groupoids. This defines a transformation of functors $\underline{\xi} \circ f \to \underline{\eta}$ which witnesses the 2-commutativity of the first diagram of this remark.

06HD **Remark 7.14.** According to Remark 7.12, giving a formal object ξ of $\mathcal{F}$ is equivalent to giving a prorepresentable functor $U : \mathcal{C}_\Lambda \to \textit{Sets}$ and a morphism $U \to \mathcal{F}$.

8. Smooth morphisms

06HF In this section we discuss smooth morphisms of categories cofibered in groupoids over $\mathcal{C}_\Lambda$.

06HG **Definition 8.1.** Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of categories cofibered in groupoids over $\mathcal{C}_\Lambda$. We say φ is *smooth* if it satisfies the following condition: Let $B \to A$ be a surjective ring map in $\mathcal{C}_\Lambda$. Let $y \in \mathrm{Ob}(\mathcal{G}(B)), x \in \mathrm{Ob}(\mathcal{F}(A))$, and $y \to \varphi(x)$ be a morphism lying over $B \to A$. Then there exists $x' \in \mathrm{Ob}(\mathcal{F}(B))$, a morphism $x' \to x$ lying over $B \to A$, and a morphism $\varphi(x') \to y$ lying over $\mathrm{id} : B \to B$, such that the diagram

$$\begin{tikzcd} \varphi(x') \arrow[r] \arrow[dr] & y \arrow[d] \\ & \varphi(x) \end{tikzcd}$$

commutes.

06HH **Lemma 8.2.** *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of categories cofibered in groupoids over $\mathcal{C}_\Lambda$. Then φ is smooth if the condition in Definition 8.1 is assumed to hold only for small extensions $B \to A$.*

Proof. Let $B \to A$ be a surjective ring map in $\mathcal{C}_\Lambda$. Let $y \in \mathrm{Ob}(\mathcal{G}(B))$, $x \in \mathrm{Ob}(\mathcal{F}(A))$, and $y \to \varphi(x)$ be a morphism lying over $B \to A$. By Lemma 3.3 we can factor $B \to A$ into small extensions $B = B_n \to B_{n-1} \to \ldots \to B_0 = A$. We argue by induction on n. If $n = 1$ the result is true by assumption. If $n > 1$, then denote $f : B = B_n \to B_{n-1}$ and denote $g : B_{n-1} \to B_0 = A$. Choose a pushforward $y \to f_*y$ of y along f, so that the morphism $y \to \varphi(x)$ factors as $y \to f_*y \to \varphi(x)$. By the induction hypothesis we can find $x_{n-1} \to x$ lying over $g : B_{n-1} \to A$ and $a : \varphi(x_{n-1}) \to f_*y$ lying over $\mathrm{id} : B_{n-1} \to B_{n-1}$ such that

$$\begin{tikzcd} \varphi(x_{n-1}) \arrow[r, "a"'] \arrow[dr] & f_*y \arrow[d] \\ & \varphi(x) \end{tikzcd}$$

commutes. We can apply the assumption to the composition $y \to \varphi(x_{n-1})$ of $y \to f_*y$ with $a^{-1} : f_*y \to \varphi(x_{n-1})$. We obtain $x_n \to x_{n-1}$ lying over $B_n \to B_{n-1}$ and $\varphi(x_n) \to y$ lying over $\mathrm{id} : B_n \to B_n$ so that the diagram

$$\begin{tikzcd} \varphi(x_n) \arrow[r] \arrow[d] & y \arrow[d] \\ \varphi(x_{n-1}) \arrow[r, "a"] \arrow[dr] & f_*y \arrow[d] \\ & \varphi(x) \end{tikzcd}$$

commutes. Then the composition $x_n \to x_{n-1} \to x$ and $\varphi(x_n) \to y$ are the morphisms required by the definition of smoothness. $\square$

06HI **Remark 8.3.** Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of categories cofibered in groupoids over $\mathcal{C}_\Lambda$. Let $B \to A$ be a ring map in $\mathcal{C}_\Lambda$. Choices of pushforwards along $B \to A$ for objects in the fiber categories $\mathcal{F}(B)$ and $\mathcal{G}(B)$ determine functors $\mathcal{F}(B) \to \mathcal{F}(A)$ and $\mathcal{G}(B) \to \mathcal{G}(A)$ fitting into a 2-commutative diagram

$$\begin{tikzcd} \mathcal{F}(B) \arrow[r, "\varphi"] \arrow[d] & \mathcal{G}(B) \arrow[d] \\ \mathcal{F}(A) \arrow[r, "\varphi"] & \mathcal{G}(A). \end{tikzcd}$$

Hence there is an induced functor $\mathcal{F}(B) \to \mathcal{F}(A) \times_{\mathcal{G}(A)} \mathcal{G}(B)$. Unwinding the definitions shows that $\varphi : \mathcal{F} \to \mathcal{G}$ is smooth if and only if this induced functor is essentially surjective whenever $B \to A$ is surjective (or equivalently, by Lemma 8.2, whenever $B \to A$ is a small extension).

06HJ **Remark 8.4.** The characterization of smooth morphisms in Remark 8.3 is analogous to Schlessinger's notion of a smooth morphism of functors, cf. [Sch68, Definition 2.2.]. In fact, when $\mathcal{F}$ and $\mathcal{G}$ are cofibered in sets then our notion is equivalent to Schlessinger's. Namely, in this case let $F, G : \mathcal{C}_\Lambda \to Sets$ be the corresponding functors, see Remarks 5.2 (11). Then $F \to G$ is smooth if and only if for every surjection of rings $B \to A$ in $\mathcal{C}_\Lambda$ the map $F(B) \to F(A) \times_{G(A)} G(B)$ is surjective.

06HK **Remark 8.5.** Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. Then the morphism $\mathcal{F} \to \overline{\mathcal{F}}$ is smooth. Namely, suppose that $f : B \to A$ is a ring map in $\mathcal{C}_\Lambda$. Let $x \in \mathrm{Ob}(\mathcal{F}(A))$ and let $\overline{y} \in \overline{\mathcal{F}}(B)$ be the isomorphism class of $y \in \mathrm{Ob}(\mathcal{F}(B))$ such that $\overline{f_*y} = \overline{x}$. Then we simply take $x' = y$, the implied morphism $x' = y \to x$ over $B \to A$, and the equality $\overline{x'} = \overline{y}$ as the solution to the problem posed in Definition 8.1.

If $R \to S$ is a ring map $\widehat{\mathcal{C}}_\Lambda$, then there is an induced morphism $\underline{S} \to \underline{R}$ between the functors $\underline{S}, \underline{R} : \widehat{\mathcal{C}}_\Lambda \to Sets$. In this situation, smoothness of the restriction $\underline{S}|_{\mathcal{C}_\Lambda} \to \underline{R}|_{\mathcal{C}_\Lambda}$ is a familiar notion:

06HL **Lemma 8.6.** *Let $R \to S$ be a ring map in $\widehat{\mathcal{C}}_\Lambda$. Then the induced morphism $\underline{S}|_{\mathcal{C}_\Lambda} \to \underline{R}|_{\mathcal{C}_\Lambda}$ is smooth if and only if S is a power series ring over R.*

Proof. Assume S is a power series ring over R. Say $S = R[[x_1, \ldots, x_n]]$. Smoothness of $\underline{S}|_{\mathcal{C}_\Lambda} \to \underline{R}|_{\mathcal{C}_\Lambda}$ means the following (see Remark 8.4): Given a surjective ring map $B \to A$ in $\mathcal{C}_\Lambda$, a ring map $R \to B$, a ring map $S \to A$ such that the solid diagram

$$\begin{tikzcd} S \arrow[r] \arrow[rd, dotted] & A \\ R \arrow[u] \arrow[r] & B \arrow[u] \end{tikzcd}$$

is commutative then a dotted arrow exists making the diagram commute. (Note the similarity with Algebra, Definition 138.1.) To construct the dotted arrow choose elements $b_i \in B$ whose images in A are equal to the images of x_i in A. Note that $b_i \in \mathfrak{m}_B$ as x_i maps to an element of $\mathfrak{m}_A$. Hence there is a unique R-algebra map $R[[x_1, \ldots, x_n]] \to B$ which maps x_i to b_i and which can serve as our dotted arrow.

Conversely, assume $\underline{S}|_{\mathcal{C}_\Lambda} \to \underline{R}|_{\mathcal{C}_\Lambda}$ is smooth. Let $x_1, \ldots, x_n \in S$ be elements whose images form a basis in the relative cotangent space $\mathfrak{m}_S/(\mathfrak{m}_R S + \mathfrak{m}_S^2)$ of S over R. Set $T = R[[X_1, \ldots, X_n]]$. Note that both

$$S/(\mathfrak{m}_R S + \mathfrak{m}_S^2) \cong R/\mathfrak{m}_R[x_1, \ldots, x_n]/(x_i x_j)$$

and

$$T/(\mathfrak{m}_R T + \mathfrak{m}_T^2) \cong R/\mathfrak{m}_R[X_1, \ldots, X_n]/(X_i X_j).$$

Let $S/(\mathfrak{m}_R S + \mathfrak{m}_S^2) \to T/(\mathfrak{m}_R T + \mathfrak{m}_T^2)$ be the local R-algebra isomorphism given by mapping the class of x_i to the class of X_i. Let $f_1 : S \to T/(\mathfrak{m}_R T + \mathfrak{m}_T^2)$ be the composition $S \to S/(\mathfrak{m}_R S + \mathfrak{m}_S^2) \to T/(\mathfrak{m}_R T + \mathfrak{m}_T^2)$. The assumption that $\underline{S}|_{\mathcal{C}_\Lambda} \to \underline{R}|_{\mathcal{C}_\Lambda}$ is smooth means we can lift f_1 to a map $f_2 : S \to T/\mathfrak{m}_T^2$, then to a map $f_3 : S \to T/\mathfrak{m}_T^3$, and so on, for all $n \geq 1$. Thus we get an induced map $f : S \to T = \lim T/\mathfrak{m}_T^n$ of local R-algebras. By our choice of f_1, the map f induces an isomorphism $\mathfrak{m}_S/(\mathfrak{m}_R S + \mathfrak{m}_S^2) \to \mathfrak{m}_T/(\mathfrak{m}_R T + \mathfrak{m}_T^2)$ of relative cotangent spaces. Hence f is surjective by Lemma 4.2 (where we think of f as a map in $\widehat{\mathcal{C}}_R$). Choose preimages $y_i \in S$ of $X_i \in T$ under f. As T is a power series ring over R there exists a local R-algebra homomorphism $s : T \to S$ mapping X_i to y_i. By construction $f \circ s = \text{id}$. Then s is injective. But s induces an isomorphism on relative cotangent spaces since f does, so it is also surjective by Lemma 4.2 again. Hence s and f are isomorphisms. □

Smooth morphisms satisfy the following functorial properties.

06HM **Lemma 8.7.** *Let $\varphi : \mathcal{F} \to \mathcal{G}$ and $\psi : \mathcal{G} \to \mathcal{H}$ be morphisms of categories cofibered in groupoids over $\mathcal{C}_\Lambda$.*

1. *If φ and ψ are smooth, then $\psi \circ \varphi$ is smooth.*
2. *If φ is essentially surjective and $\psi \circ \varphi$ is smooth, then ψ is smooth.*
3. *If $\mathcal{G}' \to \mathcal{G}$ is a morphism of categories cofibered in groupoids and φ is smooth, then $\mathcal{F} \times_\mathcal{G} \mathcal{G}' \to \mathcal{G}'$ is smooth.*

Proof. Statements (1) and (2) follow immediately from the definitions. Proof of (3) omitted. Hints: use the formulation of smoothness given in Remark 8.3 and use that $\mathcal{F} \times_\mathcal{G} \mathcal{G}'$ is the 2-fibre product, see Remarks 5.2 (13). □

06HN **Lemma 8.8.** *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a smooth morphism of categories cofibered in groupoids over $\mathcal{C}_\Lambda$. Assume $\varphi : \mathcal{F}(k) \to \mathcal{G}(k)$ is essentially surjective. Then $\varphi : \mathcal{F} \to \mathcal{G}$ and $\widehat{\varphi} : \widehat{\mathcal{F}} \to \widehat{\mathcal{G}}$ are essentially surjective.*

Proof. Let y be an object of $\mathcal{G}$ lying over $A \in \mathrm{Ob}(\mathcal{C}_\Lambda)$. Let $y \to y_0$ be a pushforward of y along $A \to k$. By the assumption on essential surjectivity of $\varphi : \mathcal{F}(k) \to \mathcal{G}(k)$ there exist an object x_0 of $\mathcal{F}$ lying over k and an isomorphism $y_0 \to \varphi(x_0)$. Smoothness of φ implies there exists an object x of $\mathcal{F}$ over A whose image $\varphi(x)$ is isomorphic to y. Thus $\varphi : \mathcal{F} \to \mathcal{G}$ is essentially surjective.

Let $\eta = (R, \eta_n, g_n)$ be an object of $\widehat{\mathcal{G}}$. We construct an object ξ of $\widehat{\mathcal{F}}$ with an isomorphism $\eta \to \varphi(\xi)$. By the assumption on essential surjectivity of $\varphi : \mathcal{F}(k) \to \mathcal{G}(k)$, there exists a morphism $\eta_1 \to \varphi(\xi_1)$ in $\mathcal{G}(k)$ for some $\xi_1 \in \mathrm{Ob}(\mathcal{F}(k))$. The morphism $\eta_2 \xrightarrow{g_1} \eta_1 \to \varphi(\xi_1)$ lies over the surjective ring map $R/\mathfrak{m}_R^2 \to k$, hence by smoothness of φ there exists $\xi_2 \in \mathrm{Ob}(\mathcal{F}(R/\mathfrak{m}_R^2))$, a morphism $f_1 : \xi_2 \to \xi_1$ lying over $R/\mathfrak{m}_R^2 \to k$, and a morphism $\eta_2 \to \varphi(\xi_2)$ such that

$$\begin{tikzcd} \varphi(\xi_2) \arrow[r, "\varphi(f_1)"] & \varphi(\xi_1) \\ \eta_2 \arrow[u] \arrow[r, "g_1"] & \eta_1 \arrow[u] \end{tikzcd}$$

commutes. Continuing in this way we construct an object $\xi = (R, \xi_n, f_n)$ of $\widehat{\mathcal{F}}$ and a morphism $\eta \to \varphi(\xi) = (R, \varphi(\xi_n), \varphi(f_n))$ in $\widehat{\mathcal{G}}(R)$. $\square$

Later we are interested in producing smooth morphisms from prorepresentable functors to predeformation categories $\mathcal{F}$. By the discussion in Remark 7.12 these morphisms correspond to certain formal objects of $\mathcal{F}$. More precisely, these are the so-called versal formal objects of $\mathcal{F}$.

06HR **Definition 8.9.** Let $\mathcal{F}$ be a category cofibered in groupoids. Let ξ be a formal object of $\mathcal{F}$ lying over $R \in \mathrm{Ob}(\widehat{\mathcal{C}}_\Lambda)$. We say ξ is *versal* if the corresponding morphism $\underline{\xi} : \underline{R}|_{\mathcal{C}_\Lambda} \to \mathcal{F}$ of Remark 7.12 is smooth.

06HS **Remark 8.10.** Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$, and let ξ be a formal object of $\mathcal{F}$. It follows from the definition of smoothness that versality of ξ is equivalent to the following condition: If

$$\begin{tikzcd} & y \arrow[d] \\ \xi \arrow[r] & x \end{tikzcd}$$

is a diagram in $\widehat{\mathcal{F}}$ such that $y \to x$ lies over a surjective map $B \to A$ of Artinian rings (we may assume it is a small extension), then there exists a morphism $\xi \to y$ such that

$$\begin{tikzcd} & y \arrow[d] \\ \xi \arrow[r] \arrow[ur] & x \end{tikzcd}$$

commutes. In particular, the condition that ξ be versal does not depend on the choices of pushforwards made in the construction of $\underline{\xi} : \underline{R}|_{\mathcal{C}_\Lambda} \to \mathcal{F}$ in Remark 7.12.

06HT **Lemma 8.11.** *Let $\mathcal{F}$ be a predeformation category. Let ξ be a versal formal object of $\mathcal{F}$. For any formal object η of $\widehat{\mathcal{F}}$, there exists a morphism $\xi \to \eta$.*

Proof. By assumption the morphism $\underline{\xi} : \underline{R}|_{\mathcal{C}_\Lambda} \to \mathcal{F}$ is smooth. Then $\iota(\xi) : \underline{R} \to \widehat{\mathcal{F}}$ is the completion of $\underline{\xi}$, see Remark 7.12. By Lemma 8.8 there exists an object f of $\underline{R}$ such that $\iota(\xi)(f) = \eta$. Then f is a ring map $f : R \to S$ in $\widehat{\mathcal{C}}_\Lambda$. And $\iota(\xi)(f) = \eta$ means that $f_*\xi \cong \eta$ which means exactly that there is a morphism $\xi \to \eta$ lying over f. $\square$

9. Smooth or unobstructed categories

0DYK Let $p : \mathcal{F} \to \mathcal{C}_\Lambda$ be a category cofibered in groupoids. We can consider $\mathcal{C}_\Lambda$ as a category cofibered in groupoids over $\mathcal{C}_\Lambda$ using the identity functor. In this way $p : \mathcal{F} \longrightarrow \mathcal{C}_\Lambda$ becomes a morphism of categories cofibered in groupoids over $\mathcal{C}_\Lambda$.

06HP **Definition 9.1.** Let $p : \mathcal{F} \to \mathcal{C}_\Lambda$ be a category cofibered in groupoids. We say $\mathcal{F}$ is *smooth* or *unobstructed* if its structure morphism p is smooth in the sense of Definition 8.1.

This is the "absolute" notion of smoothness for a category cofibered in groupoids over $\mathcal{C}_\Lambda$, although it would be more correct to say that $\mathcal{F}$ is smooth over Λ. One has to be careful with the phrase "$\mathcal{F}$ *is unobstructed*": it may happen that $\mathcal{F}$ has an obstruction theory with nonvanishing obstruction spaces even though $\mathcal{F}$ is smooth.

06HQ **Remark 9.2.** Suppose $\mathcal{F}$ is a predeformation category admitting a smooth morphism $\varphi : \mathcal{U} \to \mathcal{F}$ from a predeformation category $\mathcal{U}$. Then by Lemma 8.8 φ is essentially surjective, so by Lemma 8.7 $p : \mathcal{F} \to \mathcal{C}_\Lambda$ is smooth if and only if the composition $\mathcal{U} \xrightarrow{\varphi} \mathcal{F} \xrightarrow{p} \mathcal{C}_\Lambda$ is smooth, i.e. $\mathcal{F}$ is smooth if and only if $\mathcal{U}$ is smooth.

0DYL **Lemma 9.3.** *Let $R \in \mathrm{Ob}(\widehat{\mathcal{C}}_\Lambda)$. The following are equivalent*

1. *$\underline{R}|_{\mathcal{C}_\Lambda}$ is smooth,*
2. *$\Lambda \to R$ is formally smooth in the $\mathfrak{m}_R$-adic topology,*
3. *$\Lambda \to R$ is flat and $R \otimes_\Lambda k'$ is geometrically regular over k', and*
4. *$\Lambda \to R$ is flat and $k' \to R \otimes_\Lambda k'$ is formally smooth in the $\mathfrak{m}_R$-adic topology.*

In the classical case, these are also equivalent to

5. *R is isomorphic to $\Lambda[[x_1, \ldots, x_n]]$ for some n.*

Proof. Smoothness of $p : \underline{R}|_{\mathcal{C}_\Lambda} \to \mathcal{C}_\Lambda$ means that given $B \to A$ surjective in $\mathcal{C}_\Lambda$ and given $R \to A$ we can find the dotted arrow in the diagram

$$\begin{tikzcd} R \arrow[r] \arrow[rd, dashed] & A \\ \Lambda \arrow[r] \arrow[u] & B \arrow[u] \end{tikzcd}$$

This is certainly true if $\Lambda \to R$ is formally smooth in the $\mathfrak{m}_R$-adic topology, see More on Algebra, Definitions 37.3 and 37.1. Conversely, if this holds, then we see that $\Lambda \to R$ is formally smooth in the $\mathfrak{m}_R$-adic topology by More on Algebra, Lemma 38.1. Thus (1) and (2) are equivalent.

The equivalence of (2), (3), and (4) is More on Algebra, Proposition 40.5. The equivalence with (5) follows for example from Lemma 8.6 and the fact that $\mathcal{C}_\Lambda$ is the same as $\underline{\Lambda}|_{\mathcal{C}_\Lambda}$ in the classical case. $\square$

0DZK **Lemma 9.4.** *Let $\mathcal{F}$ be a predeformation category. Let ξ be a versal formal object of $\mathcal{F}$ lying over $R \in \mathrm{Ob}(\widehat{\mathcal{C}}_\Lambda)$. The following are equivalent*

(1) *$\mathcal{F}$ is unobstructed, and*
(2) *$\Lambda \to R$ is formally smooth in the $\mathfrak{m}_R$-adic topology.*

In the classical case these are also equivalent to

(3) *$R \cong \Lambda[[x_1, \ldots, x_n]]$ for some n.*

Proof. If (1) holds, i.e., if $\mathcal{F}$ is unobstructed, then the composition

$$\underline{R}|_{\mathcal{C}_\Lambda} \xrightarrow{\xi} \mathcal{F} \to \mathcal{C}_\Lambda$$

is smooth, see Lemma 8.7. Hence we see that (2) holds by Lemma 9.3. Conversely, if (2) holds, then the composition is smooth and moreover the first arrow is essentially surjective by Lemma 8.11. Hence we find that the second arrow is smooth by Lemma 8.7 which means that $\mathcal{F}$ is unobstructed by definition. The equivalence with (3) in the classical case follows from Lemma 9.3. □

06SM **Lemma 9.5.** *There exists an $R \in \mathrm{Ob}(\widehat{\mathcal{C}}_\Lambda)$ such that the equivalent conditions of Lemma 9.3 hold and moreover $H_1(L_{k/\Lambda}) = \mathfrak{m}_R/\mathfrak{m}_R^2$ and $\Omega_{R/\Lambda} \otimes_R k = \Omega_{k/\Lambda}$.*

Proof. In the classical case we choose $R = \Lambda$. More generally, if the residue field extension k/k' is separable, then there exists a unique finite étale extension $\Lambda^\wedge \to R$ (Algebra, Lemmas 153.9 and 153.7) of the completion $\Lambda^\wedge$ of Λ inducing the extension k/k' on residue fields.

In the general case we proceed as follows. Choose a smooth Λ-algebra P and a Λ-algebra surjection $P \to k$. (For example, let P be a polynomial algebra.) Denote $\mathfrak{m}_P$ the kernel of $P \to k$. The Jacobi-Zariski sequence, see (3.10.2) and Algebra, Lemma 134.4, is an exact sequence

$$0 \to H_1(NL_{k/\Lambda}) \to \mathfrak{m}_P/\mathfrak{m}_P^2 \to \Omega_{P/\Lambda} \otimes_P k \to \Omega_{k/\Lambda} \to 0$$

We have the 0 on the left because P/k is smooth, hence $NL_{P/\Lambda}$ is quasi-isomorphic to a finite projective module placed in degree 0, hence $H_1(NL_{P/\Lambda} \otimes_P k) = 0$. Suppose $f \in \mathfrak{m}_P$ maps to a nonzero element of $\Omega_{P/\Lambda} \otimes_P k$. Setting $P' = P/(f)$ we have a Λ-algebra surjection $P' \to k$. Observe that P' is smooth at $\mathfrak{m}_{P'}$: this follows from More on Morphisms, Lemma 38.1. Thus after replacing P by a principal localization of P', we see that $\dim(\mathfrak{m}_P/\mathfrak{m}_P^2)$ decreases. Repeating finitely many times, we may assume the map $\mathfrak{m}_P/\mathfrak{m}_P^2 \to \Omega_{P/\Lambda} \otimes_P k$ is zero so that the exact sequence breaks into isomorphisms $H_1(L_{k/\Lambda}) = \mathfrak{m}_P/\mathfrak{m}_P^2$ and $\Omega_{P/\Lambda} \otimes_P k = \Omega_{k/\Lambda}$.

Let R be the $\mathfrak{m}_P$-adic completion of P. Then R is an object of $\widehat{\mathcal{C}}_\Lambda$. Namely, it is a complete local Noetherian ring (see Algebra, Lemma 97.6) and its residue field is identified with k. We claim that R works.

First observe that the map $P \to R$ induces isomorphisms $\mathfrak{m}_P/\mathfrak{m}_P^2 = \mathfrak{m}_R/\mathfrak{m}_R^2$ and $\Omega_{P/\Lambda} \otimes_P k = \Omega_{R/\Lambda} \otimes_R k$. This is true because both $\mathfrak{m}_P/\mathfrak{m}_P^2$ and $\Omega_{P/\Lambda} \otimes_P k$ only depend on the Λ-algebra $P/\mathfrak{m}_P^2$, see Algebra, Lemma 131.11, the same holds for R and we have $P/\mathfrak{m}_P^2 = R/\mathfrak{m}_R^2$. Using the functoriality of the Jacobi-Zariski sequence (3.10.3) we deduce that $H_1(L_{k/\Lambda}) = \mathfrak{m}_R/\mathfrak{m}_R^2$ and $\Omega_{R/\Lambda} \otimes_R k = \Omega_{k/\Lambda}$ as the same is true for P.

Finally, since $\Lambda \to P$ is smooth we see that $\Lambda \to P$ is formally smooth by Algebra, Proposition 138.13. Then $\Lambda \to P$ is formally smooth for the $\mathfrak{m}_P$-adic topology by More on Algebra, Lemma 37.2. This property is inherited by the completion R by More on Algebra, Lemma 37.4 and the proof is complete. In fact, it turns out that whenever $\underline{R}|_{\mathcal{C}_\Lambda}$ is smooth, then R is isomorphic to a completion of a smooth algebra over Λ, but we won't use this. $\square$

06SN **Example 9.6.** Here is a more explicit example of an R as in Lemma 9.5. Let p be a prime number and let $n \in \mathbf{N}$. Let $\Lambda = \mathbf{F}_p(t_1, t_2, \ldots, t_n)$ and let $k = \mathbf{F}_p(x_1, \ldots, x_n)$ with map $\Lambda \to k$ given by $t_i \mapsto x_i^p$. Then we can take

$$R = \Lambda[x_1, \ldots, x_n]^\wedge_{(x_1^p - t_1, \ldots, x_n^p - t_n)}$$

We cannot do "better" in this example, i.e., we cannot approximate $\mathcal{C}_\Lambda$ by a smaller smooth object of $\widehat{\mathcal{C}}_\Lambda$ (one can argue that the dimension of R has to be at least n since the map $\Omega_{R/\Lambda} \otimes_R k \to \Omega_{k/\Lambda}$ is surjective). We will discuss this phenomenon later in more detail.

10. Schlessinger's conditions

06HV In the following we often consider fibre products $A_1 \times_A A_2$ of rings in the category $\mathcal{C}_\Lambda$. We have seen in Example 3.7 that such a fibre product may not always be an object of $\mathcal{C}_\Lambda$. However, in virtually all cases below one of the two maps $A_i \to A$ is surjective and $A_1 \times_A A_2$ will be an object of $\mathcal{C}_\Lambda$ by Lemma 3.8. We will use this result without further mention.

We denote by $k[\epsilon]$ the ring of dual numbers over k. More generally, for a k-vector space V, we denote by $k[V]$ the k-algebra whose underlying vector space is $k \oplus V$ and whose multiplication is given by $(a, v) \cdot (a', v') = (aa', av' + a'v)$. When $V = k$, $k[V]$ is the ring of dual numbers over k. For any finite dimensional k-vector space V the ring $k[V]$ is in $\mathcal{C}_\Lambda$.

06HW **Definition 10.1.** Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. We define *conditions (S1) and (S2)* on $\mathcal{F}$ as follows:

(S1) Every diagram in $\mathcal{F}$

$$\begin{tikzcd} & x_2 \arrow[d] \\ x_1 \arrow[r] & x \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} & A_2 \arrow[d] \\ A_1 \arrow[r] & A \end{tikzcd}$$

in $\mathcal{C}_\Lambda$ with $A_2 \to A$ surjective can be completed to a commutative diagram

$$\begin{tikzcd} y \arrow[r] \arrow[d] & x_2 \arrow[d] \\ x_1 \arrow[r] & x \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} A_1 \times_A A_2 \arrow[r] \arrow[d] & A_2 \arrow[d] \\ A_1 \arrow[r] & A. \end{tikzcd}$$

(S2) The condition of (S1) holds for diagrams in $\mathcal{F}$ lying over a diagram in $\mathcal{C}_\Lambda$ of the form

$$\begin{tikzcd} & k[\epsilon] \arrow[d] \\ A \arrow[r] & k. \end{tikzcd}$$

Moreover, if we have two commutative diagrams in $\mathcal{F}$

$$\begin{tikzcd} y \arrow[r, "c"] \arrow[d, "a"'] & x_\epsilon \arrow[d, "e"] \\ x \arrow[r, "d"] & x_0 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} y' \arrow[r, "c'"] \arrow[d, "a'"'] & x_\epsilon \arrow[d, "e"] \\ x \arrow[r, "d"] & x_0 \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} A \times_k k[\epsilon] \arrow[r] \arrow[d] & k[\epsilon] \arrow[d] \\ A \arrow[r] & k \end{tikzcd}$$

then there exists a morphism $b : y \to y'$ in $\mathcal{F}(A \times_k k[\epsilon])$ such that $a = a' \circ b$.

We can partly explain the meaning of conditions (S1) and (S2) in terms of fibre categories. Suppose that $f_1 : A_1 \to A$ and $f_2 : A_2 \to A$ are ring maps in $\mathcal{C}_\Lambda$ with f_2 surjective. Denote $p_i : A_1 \times_A A_2 \to A_i$ the projection maps. Assume a choice of pushforwards for $\mathcal{F}$ has been made. Then the commutative diagram of rings translates into a 2-commutative diagram

$$\begin{tikzcd} \mathcal{F}(A_1 \times_A A_2) \arrow[r, "p_{2,*}"] \arrow[d, "p_{1,*}"'] & \mathcal{F}(A_2) \arrow[d, "f_{2,*}"] \\ \mathcal{F}(A_1) \arrow[r, "f_{1,*}"] & \mathcal{F}(A) \end{tikzcd}$$

of fibre categories whence a functor

06SP (10.1.1) $$\mathcal{F}(A_1 \times_A A_2) \to \mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)$$

into the 2-fibre product of categories. Condition (S1) requires that this functor be essentially surjective. The first part of condition (S2) requires that this functor be a essentially surjective if f_2 equals the map $k[\epsilon] \to k$. Moreover in this case, the second part of (S2) implies that two objects which become isomorphic in the target are isomorphic in the source (but it is *not* equivalent to this statement). The advantage of stating the conditions as in the definition is that no choices have to be made.

06HX **Lemma 10.2.** *Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. Then $\mathcal{F}$ satisfies (S1) if the condition of (S1) is assumed to hold only when $A_2 \to A$ is a small extension.*

Proof. Proof omitted. Hints: apply Lemma 3.3 and use induction similar to the proof of Lemma 8.2. ☐

06HY **Remark 10.3.** When $\mathcal{F}$ is cofibered in sets, conditions (S1) and (S2) are exactly conditions (H1) and (H2) from Schlessinger's paper [Sch68]. Namely, for a functor $F : \mathcal{C}_\Lambda \to Sets$, conditions (S1) and (S2) state:

(S1) If $A_1 \to A$ and $A_2 \to A$ are maps in $\mathcal{C}_\Lambda$ with $A_2 \to A$ surjective, then the induced map $F(A_1 \times_A A_2) \to F(A_1) \times_{F(A)} F(A_2)$ is surjective.

(S2) If $A \to k$ is a map in $\mathcal{C}_\Lambda$, then the induced map $F(A \times_k k[\epsilon]) \to F(A) \times_{F(k)} F(k[\epsilon])$ is bijective.

The injectivity of the map $F(A \times_k k[\epsilon]) \to F(A) \times_{F(k)} F(k[\epsilon])$ comes from the second part of condition (S2) and the fact that morphisms are identities.

06HZ **Lemma 10.4.** *Let $\mathcal{F}$ be a category cofibred in groupoids over $\mathcal{C}_\Lambda$. If $\mathcal{F}$ satisfies (S2), then the condition of (S2) also holds when $k[\epsilon]$ is replaced by $k[V]$ for any finite dimensional k-vector space V.*

Proof. In the case that $\mathcal{F}$ is cofibred in sets, i.e., corresponds to a functor $F : \mathcal{C}_\Lambda \to \textit{Sets}$ this follows from the description of (S2) for F in Remark 10.3 and the fact that $k[V] \cong k[\epsilon] \times_k \ldots \times_k k[\epsilon]$ with $\dim_k V$ factors. The case of functors is what we will use in the rest of this chapter.

We prove the general case by induction on $\dim(V)$. If $\dim(V) = 1$, then $k[V] \cong k[\epsilon]$ and the result holds by assumption. If $\dim(V) > 1$ we write $V = V' \oplus k\epsilon$. Pick a diagram

$$\begin{tikzcd} & x_V \arrow[d] \\ x \arrow[r] & x_0 \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} & k[V] \arrow[d] \\ A \arrow[r] & k \end{tikzcd}$$

Choose a morphism $x_V \to x_{V'}$ lying over $k[V] \to k[V']$ and a morphism $x_V \to x_\epsilon$ lying over $k[V] \to k[\epsilon]$. Note that the morphism $x_V \to x_0$ factors as $x_V \to x_{V'} \to x_0$ and as $x_V \to x_\epsilon \to x_0$. By induction hypothesis we can find a diagram

$$\begin{tikzcd} y' \arrow[r] \arrow[d] & x_{V'} \arrow[d] \\ x \arrow[r] & x_0 \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} A \times_k k[V'] \arrow[r] \arrow[d] & k[V'] \arrow[d] \\ A \arrow[r] & k \end{tikzcd}$$

This gives us a commutative diagram

$$\begin{tikzcd} & x_\epsilon \arrow[d] \\ y' \arrow[r] & x_0 \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} & k[\epsilon] \arrow[d] \\ A \times_k k[V'] \arrow[r] & k \end{tikzcd}$$

Hence by (S2) we get a commutative diagram

$$\begin{tikzcd} y \arrow[r] \arrow[d] & x_\epsilon \arrow[d] \\ y' \arrow[r] & x_0 \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} (A \times_k k[V']) \times_k k[\epsilon] \arrow[r] \arrow[d] & k[\epsilon] \arrow[d] \\ A \times_k k[V'] \arrow[r] & k \end{tikzcd}$$

Note that $(A \times_k k[V']) \times_k k[\epsilon] = A \times_k k[V' \oplus k\epsilon] = A \times_k k[V]$. We claim that y fits into the correct commutative diagram. To see this we let $y \to y_V$ be a morphism lying over $A \times_k k[V] \to k[V]$. We can factor the morphisms $y \to y' \to x_{V'}$ and $y \to x_\epsilon$ through the morphism $y \to y_V$ (by the axioms of categories cofibred in groupoids). Hence we see that both y_V and x_V fit into commutative diagrams

$$\begin{tikzcd} y_V \arrow[r] \arrow[d] & x_\epsilon \arrow[d] \\ x_{V'} \arrow[r] & x_0 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} x_V \arrow[r] \arrow[d] & x_\epsilon \arrow[d] \\ x_{V'} \arrow[r] & x_0 \end{tikzcd}$$

and hence by the second part of (S2) there exists an isomorphism $y_V \to x_V$ compatible with $y_V \to x_{V'}$ and $x_V \to x_{V'}$ and in particular compatible with the maps to x_0. The composition $y \to y_V \to x_V$ then fits into the required commutative diagram

$$\begin{tikzcd} y \arrow[r] \arrow[d] & x_V \arrow[d] \\ x \arrow[r] & x_0 \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} A \times_k k[V] \arrow[r] \arrow[d] & k[V] \arrow[d] \\ A \arrow[r] & k \end{tikzcd}$$

In this way we see that the first part of $(S2)$ holds with $k[\epsilon]$ replaced by $k[V]$.

To prove the second part suppose given two commutative diagrams

$$\begin{tikzcd} y \arrow[r] \arrow[d] & x_V \arrow[d] \\ x \arrow[r] & x_0 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} y' \arrow[r] \arrow[d] & x_V \arrow[d] \\ x \arrow[r] & x_0 \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} A \times_k k[V] \arrow[r] \arrow[d] & k[V] \arrow[d] \\ A \arrow[r] & k \end{tikzcd}$$

We will use the morphisms $x_V \to x_{V'} \to x_0$ and $x_V \to x_\epsilon \to x_0$ introduced in the first paragraph of the proof. Choose morphisms $y \to y_{V'}$ and $y' \to y'_{V'}$ lying over $A \times_k k[V] \to A \times_k k[V']$. The axioms of a cofibred category imply we can find commutative diagrams

$$\begin{tikzcd} y_{V'} \arrow[r] \arrow[d] & x_{V'} \arrow[d] \\ x \arrow[r] & x_0 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} y'_{V'} \arrow[r] \arrow[d] & x_{V'} \arrow[d] \\ x \arrow[r] & x_0 \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} A \times_k k[V'] \arrow[r] \arrow[d] & k[V'] \arrow[d] \\ A \arrow[r] & k \end{tikzcd}$$

By induction hypothesis we obtain an isomorphism $b : y_{V'} \to y'_{V'}$ compatible with the morphisms $y_{V'} \to x$ and $y'_{V'} \to x$, in particular compatible with the morphisms to x_0. Then we have commutative diagrams

$$\begin{tikzcd} y \arrow[r] \arrow[d] & x_\epsilon \arrow[d] \\ y'_{V'} \arrow[r] & x_0 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} y' \arrow[r] \arrow[d] & x_\epsilon \arrow[d] \\ y'_{V'} \arrow[r] & x_0 \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} A \times_k k[\epsilon] \arrow[r] \arrow[d] & k[\epsilon] \arrow[d] \\ A \arrow[r] & k \end{tikzcd}$$

where the morphism $y \to y'_{V'}$ is the composition $y \to y_{V'} \xrightarrow{b} y'_{V'}$ and where the morphisms $y \to x_\epsilon$ and $y' \to x_\epsilon$ are the compositions of the maps $y \to x_V$ and $y' \to x_V$ with the morphism $x_V \to x_\epsilon$. Then the second part of (S2) guarantees the existence of an isomorphism $y \to y'$ compatible with the maps to $y'_{V'}$, in particular compatible with the maps to x (because b was compatible with the maps to x). $\square$

06I0 **Lemma 10.5.** *Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$.*

(1) *If $\mathcal{F}$ satisfies (S1), then so does $\overline{\mathcal{F}}$.*
(2) *If $\mathcal{F}$ satisfies (S2), then so does $\overline{\mathcal{F}}$ provided at least one of the following conditions is satisfied*
   (a) *$\mathcal{F}$ is a predeformation category,*
   (b) *the category $\mathcal{F}(k)$ is a set or a setoid, or*
   (c) *for any morphism $x_\epsilon \to x_0$ of $\mathcal{F}$ lying over $k[\epsilon] \to k$ the pushforward map $Aut_{k[\epsilon]}(x_\epsilon) \to Aut_k(x_0)$ is surjective.*

Proof. Assume $\mathcal{F}$ has (S1). Suppose we have ring maps $f_i : A_i \to A$ in $\mathcal{C}_\Lambda$ with f_2 surjective. Let $x_i \in \mathcal{F}(A_i)$ such that the pushforwards $f_{1,*}(x_1)$ and $f_{2,*}(x_2)$ are isomorphic. Then we can denote x an object of $\mathcal{F}$ over A isomorphic to both of these and we obtain a diagram as in (S1). Hence we find an object y of $\mathcal{F}$ over

$A_1 \times_A A_2$ whose pushforward to A_1, resp. A_2 is isomorphic to x_1, resp. x_2. In this way we see that (S1) holds for $\overline{\mathcal{F}}$.

Assume $\mathcal{F}$ has (S2). The first part of (S2) for $\overline{\mathcal{F}}$ follows as in the argument above. The second part of (S2) for $\overline{\mathcal{F}}$ signifies that the map

$$\overline{\mathcal{F}}(A \times_k k[\epsilon]) \to \overline{\mathcal{F}}(A) \times_{\overline{\mathcal{F}}(k)} \overline{\mathcal{F}}(k[\epsilon])$$

is injective for any ring A in $\mathcal{C}_\Lambda$. Suppose that $y, y' \in \mathcal{F}(A \times_k k[\epsilon])$. Using the axioms of cofibred categories we can choose commutative diagrams

$$\begin{tikzcd} y \arrow[r, "c"'] \arrow[d, "a"'] & x_\epsilon \arrow[d, "e"] \\ x \arrow[r, "d"] & x_0 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} y' \arrow[r, "c'"'] \arrow[d, "a'"'] & x'_\epsilon \arrow[d, "e'"] \\ x' \arrow[r, "d'"] & x'_0 \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} A \times_k k[\epsilon] \arrow[r] \arrow[d] & k[\epsilon] \arrow[d] \\ A \arrow[r] & k \end{tikzcd}$$

Assume that there exist isomorphisms $\alpha : x \to x'$ in $\mathcal{F}(A)$ and $\beta : x_\epsilon \to x'_\epsilon$ in $\mathcal{F}(k[\epsilon])$. This also means there exists an isomorphism $\gamma : x_0 \to x'_0$ compatible with α. To prove (S2) for $\overline{\mathcal{F}}$ we have to show that there exists an isomorphism $y \to y'$ in $\mathcal{F}(A \times_k k[\epsilon])$. By (S2) for $\mathcal{F}$ such a morphism will exist if we can choose the isomorphisms α and β and γ such that

$$\begin{tikzcd} x \arrow[r] \arrow[d, "\alpha"] & x_0 \arrow[d, "\gamma"] & x_\epsilon \arrow[l, "e"'] \arrow[d, "\beta"] \\ x' \arrow[r] & x'_0 & x'_\epsilon \arrow[l, "e'"] \end{tikzcd}$$

is commutative (because then we can replace x by x' and x_ϵ by x'_ϵ in the previous displayed diagram). The left hand square commutes by our choice of γ. We can factor $e' \circ \beta$ as $\gamma' \circ e$ for some second map $\gamma' : x_0 \to x'_0$. Now the question is whether we can arrange it so that $\gamma = \gamma'$? This is clear if $\mathcal{F}(k)$ is a set, or a setoid. Moreover, if $\mathrm{Aut}_{k[\epsilon]}(x_\epsilon) \to \mathrm{Aut}_k(x_0)$ is surjective, then we can adjust the choice of β by precomposing with an automorphism of x_ϵ whose image is $\gamma^{-1} \circ \gamma'$ to make things work. □

06SQ **Lemma 10.6.** *Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. Let $x_0 \in \mathrm{Ob}(\mathcal{F}(k))$. Let $\mathcal{F}_{x_0}$ be the category cofibred in groupoids over $\mathcal{C}_\Lambda$ constructed in Remark 6.4.*

1. *If $\mathcal{F}$ satisfies (S1), then so does $\mathcal{F}_{x_0}$.*
2. *If $\mathcal{F}$ satisfies (S2), then so does $\mathcal{F}_{x_0}$.*

Proof. Any diagram as in Definition 10.1 in $\mathcal{F}_{x_0}$ gives rise to a diagram in $\mathcal{F}$ and the output of condition (S1) or (S2) for this diagram in $\mathcal{F}$ can be viewed as an output for $\mathcal{F}_{x_0}$ as well. □

06IS **Lemma 10.7.** *Let $p : \mathcal{F} \to \mathcal{C}_\Lambda$ be a category cofibered in groupoids. Consider a diagram of $\mathcal{F}$*

$$\begin{tikzcd} y \arrow[r] \arrow[d, "a"'] & x_\epsilon \arrow[d, "e"] \\ x \arrow[r, "d"] & x_0 \end{tikzcd} \quad \textit{lying over} \quad \begin{tikzcd} A \times_k k[\epsilon] \arrow[r] \arrow[d] & k[\epsilon] \arrow[d] \\ A \arrow[r] & k. \end{tikzcd}$$

in $\mathcal{C}_\Lambda$. Assume $\mathcal{F}$ satisfies (S2). Then there exists a morphism $s : x \to y$ with $a \circ s = \mathrm{id}_x$ if and only if there exists a morphism $s_\epsilon : x \to x_\epsilon$ with $e \circ s_\epsilon = d$.

Proof. The "only if" direction is clear. Conversely, assume there exists a morphism $s_\epsilon : x \to x_\epsilon$ with $e \circ s_\epsilon = d$. Note that $p(s_\epsilon) : A \to k[\epsilon]$ is a ring map compatible with the map $A \to k$. Hence we obtain

$$\sigma = (\mathrm{id}_A, p(s_\epsilon)) : A \to A \times_k k[\epsilon].$$

Choose a pushforward $x \to \sigma_* x$. By construction we can factor s_ϵ as $x \to \sigma_* x \to x_\epsilon$. Moreover, as σ is a section of $A \times_k k[\epsilon] \to A$, we get a morphism $\sigma_* x \to x$ such that $x \to \sigma_* x \to x$ is id_x. Because $e \circ s_\epsilon = d$ we find that the diagram

$$\begin{tikzcd} \sigma_* x \arrow[r] \arrow[d] & x_\epsilon \arrow[d, "e"] \\ x \arrow[r, "d"] & x_0 \end{tikzcd}$$

is commutative. Hence by (S2) we obtain a morphism $\sigma_* x \to y$ such that $\sigma_* x \to y \to x$ is the given map $\sigma_* x \to x$. The solution to the problem is now to take $a : x \to y$ equal to the composition $x \to \sigma_* x \to y$. □

06IT **Lemma 10.8.** *Consider a commutative diagram in a predeformation category $\mathcal{F}$*

$$\begin{tikzcd} y \arrow[r] \arrow[d] & x_2 \arrow[d, "a_2"] \\ x_1 \arrow[r, "a_1"] & x \end{tikzcd} \quad \textit{lying over} \quad \begin{tikzcd} A_1 \times_A A_2 \arrow[r] \arrow[d] & A_2 \arrow[d, "f_2"] \\ A_1 \arrow[r, "f_1"] & A \end{tikzcd}$$

in $\mathcal{C}_\Lambda$ where $f_2 : A_2 \to A$ is a small extension. Assume there is a map $h : A_1 \to A_2$ such that $f_2 = f_1 \circ h$. Let $I = \mathrm{Ker}(f_2)$. Consider the ring map

$$g : A_1 \times_A A_2 \longrightarrow k[I] = k \oplus I, \quad (u, v) \longmapsto \overline{u} \oplus (v - h(u))$$

Choose a pushforward $y \to g_* y$. Assume $\mathcal{F}$ satisfies (S2). If there exists a morphism $x_1 \to g_* y$, then there exists a morphism $b : x_1 \to x_2$ such that $a_1 = a_2 \circ b$.

Proof. Note that $\mathrm{id}_{A_1} \times g : A_1 \times_A A_2 \to A_1 \times_k k[I]$ is an isomorphism and that $k[I] \cong k[\epsilon]$. Hence we have a diagram

$$\begin{tikzcd} y \arrow[r] \arrow[d] & g_* y \arrow[d] \\ x_1 \arrow[r] & x_0 \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} A_1 \times_k k[\epsilon] \arrow[r] \arrow[d] & k[\epsilon] \arrow[d] \\ A_1 \arrow[r] & k. \end{tikzcd}$$

where x_0 is an object of $\mathcal{F}$ lying over k (every object of $\mathcal{F}$ has a unique morphism to x_0, see discussion following Definition 6.2). If we have a morphism $x_1 \to g_* y$ then Lemma 10.7 provides us with a section $s : x_1 \to y$ of the map $y \to x_1$. Composing this with the map $y \to x_2$ we obtain $b : x_1 \to x_2$ which has the property that $a_1 = a_2 \circ b$ because the diagram of the lemma commutes and because s is a section. □

11. Tangent spaces of functors

06I2 Let R be a ring. We write Mod_R for the category of R-modules and Mod_R^{fg} for the category of finitely generated R-modules.

06I3 **Definition 11.1.** Let $L : \text{Mod}_R^{fg} \to \text{Mod}_R$, resp. $L : \text{Mod}_R \to \text{Mod}_R$ be a functor. We say that L is *R-linear* if for every pair of objects M, N of Mod_R^{fg}, resp. Mod_R the map

$$L : \text{Hom}_R(M, N) \longrightarrow \text{Hom}_R(L(M), L(N))$$

is a map of R-modules.

06I4 **Remark 11.2.** One can define the notion of an R-linearity for any functor between categories enriched over Mod_R. We made the definition specifically for functors $L : \text{Mod}_R^{fg} \to \text{Mod}_R$ and $L : \text{Mod}_R \to \text{Mod}_R$ because these are the cases that we have needed so far.

06I5 **Remark 11.3.** If $L : \text{Mod}_R^{fg} \to \text{Mod}_R$ is an R-linear functor, then L preserves finite products and sends the zero module to the zero module, see Homology, Lemma 3.7. On the other hand, if a functor $\text{Mod}_R^{fg} \to \textit{Sets}$ preserves finite products and sends the zero module to a one element set, then it has a unique lift to a R-linear functor, see Lemma 11.4.

06I6 **Lemma 11.4.** *Let $L : Mod_R^{fg} \to Sets$, resp. $L : Mod_R \to Sets$ be a functor. Suppose $L(0)$ is a one element set and L preserves finite products. Then there exists a unique R-linear functor $\widetilde{L} : Mod_R^{fg} \to Mod_R$, resp. $\widetilde{L} : Mod_R^{fg} \to Mod_R$, such that*

$$\begin{tikzcd} & Mod_R \arrow[rd, "forget"] & \\ Mod_R^{fg} \arrow[ru, "\widetilde{L}"] \arrow[rr, "L"] & & Sets \end{tikzcd} \quad \textit{resp.} \quad \begin{tikzcd} & Mod_R \arrow[rd, "forget"] & \\ Mod_R \arrow[ru, "\widetilde{L}"] \arrow[rr, "L"] & & Sets \end{tikzcd}$$

commutes.

Proof. We only prove this in case $L : \text{Mod}_R^{fg} \to \textit{Sets}$. Let M be a finitely generated R-module. We define $\widetilde{L}(M)$ to be the set $L(M)$ with the following R-module structure.

Multiplication: If $r \in R$, multiplication by r on $L(M)$ is defined to be the map $L(M) \to L(M)$ induced by the multiplication map $r\cdot : M \to M$.

Addition: The sum map $M \times M \to M : (m_1, m_2) \mapsto m_1 + m_2$ induces a map $L(M \times M) \to L(M)$. By assumption $L(M \times M)$ is canonically isomorphic to $L(M) \times L(M)$. Addition on $L(M)$ is defined by the map $L(M) \times L(M) \cong L(M \times M) \to L(M)$.

Zero: There is a unique map $0 \to M$. The zero element of $L(M)$ is the image of $L(0) \to L(M)$.

We omit the verification that this defines an R-module $\widetilde{L}(M)$, the unique such that is R-linearly functorial in M. $\square$

06I7 **Lemma 11.5.** *Let $L_1, L_2 : Mod_R^{fg} \to Sets$ be functors that take 0 to a one element set and preserve finite products. Let $t : L_1 \to L_2$ be a morphism of functors. Then t induces a morphism $\widetilde{t} : \widetilde{L}_1 \to \widetilde{L}_2$ between the functors guaranteed by Lemma 11.4,* *which is given simply by* $\widetilde{t}_M = t_M : \widetilde{L}_1(M) \to \widetilde{L}_2(M)$ *for each* $M \in \mathrm{Ob}(\mathit{Mod}_R^{fg})$*. In other words,* $t_M : \widetilde{L}_1(M) \to \widetilde{L}_2(M)$ *is a map of* R*-modules.*

Proof. Omitted. $\square$

In the case $R = K$ is a field, a K-linear functor $L : \mathrm{Mod}_K^{fg} \to \mathrm{Mod}_K$ is determined by its value $L(K)$.

06I8 **Lemma 11.6.** *Let* K *be a field. Let* $L : \mathit{Mod}_K^{fg} \to \mathit{Mod}_K$ *be a* K*-linear functor. Then* L *is isomorphic to the functor* $L(K) \otimes_K - : \mathit{Mod}_K^{fg} \to \mathit{Mod}_K$*.*

Proof. For $V \in \mathrm{Ob}(\mathrm{Mod}_K^{fg})$, the isomorphism $L(K) \otimes_K V \to L(V)$ is given on pure tensors by $x \otimes v \mapsto L(f_v)(x)$, where $f_v : K \to V$ is the K-linear map sending $1 \mapsto v$. When $V = K$, this is the isomorphism $L(K) \otimes_K K \to L(K)$ given by multiplication by K. For general V, it is an isomorphism by the case $V = K$ and the fact that L commutes with finite products (Remark 11.3). $\square$

For a ring R and an R-module M, let $R[M]$ be the R-algebra whose underlying R-module is $R \oplus M$ and whose multiplication is given by $(r, m) \cdot (r', m') = (rr', rm' + r'm)$. When $M = R$ this is the ring of dual numbers over R, which we denote by $R[\epsilon]$.

Now let S be a ring and assume R is an S-algebra. Then the assignment $M \mapsto R[M]$ determines a functor $\mathrm{Mod}_R \to S\text{-Alg}/R$, where $S\text{-Alg}/R$ denotes the category of S-algebras over R. Note that $S\text{-Alg}/R$ admits finite products: if $A_1 \to R$ and $A_2 \to R$ are two objects, then $A_1 \times_R A_2$ is a product.

06I9 **Lemma 11.7.** *Let* R *be an* S*-algebra. Then the functor* $\mathit{Mod}_R \to S\text{-}\mathit{Alg}/R$ *described above preserves finite products.*

Proof. This is merely the statement that if M and N are R-modules, then the map $R[M \times N] \to R[M] \times_R R[N]$ is an isomorphism in $S\text{-Alg}/R$. $\square$

06IA **Lemma 11.8.** *Let* R *be an* S*-algebra, and let* $\mathcal{C}$ *be a strictly full subcategory of* $S\text{-}\mathit{Alg}/R$ *containing* $R[M]$ *for all* $M \in \mathrm{Ob}(\mathit{Mod}_R^{fg})$*. Let* $F : \mathcal{C} \to \mathit{Sets}$ *be a functor. Suppose that* $F(R)$ *is a one element set and that for any* $M, N \in \mathrm{Ob}(\mathit{Mod}_R^{fg})$*, the induced map*

$$F(R[M] \times_R R[N]) \to F(R[M]) \times F(R[N])$$

is a bijection. Then $F(R[M])$ *has a natural* R*-module structure for any* $M \in \mathrm{Ob}(\mathit{Mod}_R^{fg})$*.*

Proof. Note that $R \cong R[0]$ and $R[M] \times_R R[N] \cong R[M \times N]$ hence R and $R[M] \times_R R[N]$ are objects of $\mathcal{C}$ by our assumptions on $\mathcal{C}$. Thus the conditions on F make sense. The functor $\mathrm{Mod}_R \to S\text{-Alg}/R$ of Lemma 11.7 restricts to a functor $\mathrm{Mod}_R^{fg} \to \mathcal{C}$ by the assumption on $\mathcal{C}$. Let L be the composition $\mathrm{Mod}_R^{fg} \to \mathcal{C} \to \mathit{Sets}$, i.e., $L(M) = F(R[M])$. Then L preserves finite products by Lemma 11.7 and the assumption on F. Hence Lemma 11.4 shows that $L(M) = F(R[M])$ has a natural R-module structure for any $M \in \mathrm{Ob}(\mathrm{Mod}_R^{fg})$. $\square$

06IB **Definition 11.9.** Let $\mathcal{C}$ be a category as in Lemma 11.8. Let $F : \mathcal{C} \to \mathit{Sets}$ be a functor such that $F(R)$ is a one element set. The *tangent space* TF *of* F is $F(R[\epsilon])$.

When $F : \mathcal{C} \to \mathit{Sets}$ satisfies the hypotheses of Lemma 11.8, the tangent space TF has a natural R-module structure.

06SR **Example 11.10.** Since $\mathcal{C}_\Lambda$ contains all $k[V]$ for finite dimensional vector spaces V we see that Definition 11.9 applies with $S = \Lambda$, $R = k$, $\mathcal{C} = \mathcal{C}_\Lambda$, and $F : \mathcal{C}_\Lambda \to \textit{Sets}$ a predeformation functor. The tangent space is $TF = F(k[\epsilon])$.

06IC **Example 11.11.** Let us work out the tangent space of Example 11.10 when $F : \mathcal{C}_\Lambda \to \textit{Sets}$ is a prorepresentable functor, say $F = \underline{S}|_{\mathcal{C}_\Lambda}$ for $S \in \mathrm{Ob}(\widehat{\mathcal{C}}_\Lambda)$. Then F commutes with arbitrary limits and thus satisfies the hypotheses of Lemma 11.8. We compute

$$TF = F(k[\epsilon]) = \mathrm{Mor}_{\mathcal{C}_\Lambda}(S, k[\epsilon]) = \mathrm{Der}_\Lambda(S, k)$$

and more generally for a finite dimensional k-vector space V we have

$$F(k[V]) = \mathrm{Mor}_{\mathcal{C}_\Lambda}(S, k[V]) = \mathrm{Der}_\Lambda(S, V).$$

Explicitly, a Λ-algebra map $f : S \to k[V]$ compatible with the augmentations $q : S \to k$ and $k[V] \to k$ corresponds to the derivation D defined by $s \mapsto f(s) - q(s)$. Conversely, a Λ-derivation $D : S \to V$ corresponds to $f : S \to k[V]$ in $\mathcal{C}_\Lambda$ defined by the rule $f(s) = q(s) + D(s)$. Since these identifications are functorial we see that the k-vector spaces structures on TF and $\mathrm{Der}_\Lambda(S, k)$ correspond (see Lemma 11.5). It follows that $\dim_k TF$ is finite by Lemma 4.5.

06SS **Example 11.12.** The computation of Example 11.11 simplifies in the classical case. Namely, in this case the tangent space of the functor $F = \underline{S}|_{\mathcal{C}_\Lambda}$ is simply the relative cotangent space of S over Λ, in a formula $TF = T_{S/\Lambda}$. In fact, this works more generally when the field extension k/k' is separable. See Exercises, Exercise 35.2.

06ID **Lemma 11.13.** *Let $F, G : \mathcal{C} \to \textit{Sets}$ be functors satisfying the hypotheses of Lemma 11.8. Let $t : F \to G$ be a morphism of functors. For any $M \in \mathrm{Ob}(\textit{Mod}_R^{fg})$, the map $t_{R[M]} : F(R[M]) \to G(R[M])$ is a map of R-modules, where $F(R[M])$ and $G(R[M])$ are given the R-module structure from Lemma 11.8. In particular, $t_{R[\epsilon]} : TF \to TG$ is a map of R-modules.*

Proof. Follows from Lemma 11.5. $\square$

06ST **Example 11.14.** Suppose that $f : R \to S$ is a ring map in $\widehat{\mathcal{C}}_\Lambda$. Set $F = \underline{R}|_{\mathcal{C}_\Lambda}$ and $G = \underline{S}|_{\mathcal{C}_\Lambda}$. The ring map f induces a transformation of functors $G \to F$. By Lemma 11.13 we get a k-linear map $TG \to TF$. This is the map

$$TG = \mathrm{Der}_\Lambda(S, k) \longrightarrow \mathrm{Der}_\Lambda(R, k) = TF$$

as follows from the canonical identifications $F(k[V]) = \mathrm{Der}_\Lambda(R, V)$ and $G(k[V]) = \mathrm{Der}_\Lambda(S, V)$ of Example 11.11 and the rule for computing the map on tangent spaces.

06IE **Lemma 11.15.** *Let $F : \mathcal{C} \to \textit{Sets}$ be a functor satisfying the hypotheses of Lemma 11.8. Assume $R = K$ is a field. Then $F(K[V]) \cong TF \otimes_K V$ for any finite dimensional K-vector space V.*

Proof. Follows from Lemma 11.6. $\square$

12. Tangent spaces of predeformation categories

06I1 We will define tangent spaces of predeformation functors using the general Definition 11.9. We have spelled this out in Example 11.10. It applies to predeformation categories by looking at the associated functor of isomorphism classes.

06IG **Definition 12.1.** Let $\mathcal{F}$ be a predeformation category. The *tangent space $T\mathcal{F}$ of $\mathcal{F}$* is the set $\overline{\mathcal{F}}(k[\epsilon])$ of isomorphism classes of objects in the fiber category $\mathcal{F}(k[\epsilon])$.

Thus $T\mathcal{F}$ is nothing but the tangent space of the associated functor $\overline{\mathcal{F}} : \mathcal{C}_\Lambda \to \textit{Sets}$. It has a natural vector space structure when $\mathcal{F}$ satisfies (S2), or, in fact, as long as $\overline{\mathcal{F}}$ does.

06IH **Lemma 12.2.** *Let $\mathcal{F}$ be a predeformation category such that $\overline{\mathcal{F}}$ satisfies (S2)[2]. Then $T\mathcal{F}$ has a natural k-vector space structure. For any finite dimensional vector space V we have $\overline{\mathcal{F}}(k[V]) = T\mathcal{F} \otimes_k V$ functorially in V.*

Proof. Let us write $F = \overline{\mathcal{F}} : \mathcal{C}_\Lambda \to \textit{Sets}$. This is a predeformation functor and F satisfies (S2). By Lemma 10.4 (and the translation of Remark 10.3) we see that

$$F(A \times_k k[V]) \longrightarrow F(A) \times F(k[V])$$

is a bijection for every finite dimensional vector space V and every $A \in \mathrm{Ob}(\mathcal{C}_\Lambda)$. In particular, if $A = k[W]$ then we see that $F(k[W] \times_k k[V]) = F(k[W]) \times F(k[V])$. In other words, the hypotheses of Lemma 11.8 hold and we see that $TF = T\mathcal{F}$ has a natural k-vector space structure. The final assertion follows from Lemma 11.15. $\square$

A morphism of predeformation categories induces a map on tangent spaces.

06II **Definition 12.3.** Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism predeformation categories. The *differential $d\varphi : T\mathcal{F} \to T\mathcal{G}$ of φ* is the map obtained by evaluating the morphism of functors $\overline{\varphi} : \overline{\mathcal{F}} \to \overline{\mathcal{G}}$ at $A = k[\epsilon]$.

06IJ **Lemma 12.4.** *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of predeformation categories. Assume $\overline{\mathcal{F}}$ and $\overline{\mathcal{G}}$ both satisfy (S2). Then $d\varphi : T\mathcal{F} \to T\mathcal{G}$ is k-linear.*

Proof. In the proof of Lemma 12.2 we have seen that $\overline{\mathcal{F}}$ and $\overline{\mathcal{G}}$ satisfy the hypotheses of Lemma 11.8. Hence the lemma follows from Lemma 11.13. $\square$

06IK **Remark 12.5.** We can globalize the notions of tangent space and differential to arbitrary categories cofibered in groupoids as follows. Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$, and let $x \in \mathrm{Ob}(\mathcal{F}(k))$. As in Remark 6.4, we get a predeformation category $\mathcal{F}_x$. We define

$$T_x\mathcal{F} = T\mathcal{F}_x$$

to be the *tangent space of $\mathcal{F}$ at x*. If $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of categories cofibered in groupoids over $\mathcal{C}_\Lambda$ and $x \in \mathrm{Ob}(\mathcal{F}(k))$, then there is an induced morphism $\varphi_x : \mathcal{F}_x \to \mathcal{G}_{\varphi(x)}$. We define the *differential $d_x\varphi : T_x\mathcal{F} \to T_{\varphi(x)}\mathcal{G}$ of φ at x* to be the map $d\varphi_x : T\mathcal{F}_x \to T\mathcal{G}_{\varphi(x)}$. If both $\mathcal{F}$ and $\mathcal{G}$ satisfy (S2) then all of these tangent spaces have a natural k-vector space structure and all the differentials $d_x\varphi : T_x\mathcal{F} \to T_{\varphi(x)}\mathcal{G}$ are k-linear (use Lemmas 10.6 and 12.4).

The following observations are uninteresting in the classical case or when k/k' is a separable field extension, because then $\mathrm{Der}_\Lambda(k, k)$ and $\mathrm{Der}_\Lambda(k, V)$ are zero. There is a canonical identification

$$\mathrm{Mor}_{\mathcal{C}_\Lambda}(k, k[\epsilon]) = \mathrm{Der}_\Lambda(k, k).$$

[2]For example if $\mathcal{F}$ satisfies (S2), see Lemma 10.5.

Namely, for $D \in \text{Der}_\Lambda(k, k)$ let $f_D : k \to k[\epsilon]$ be the map $a \mapsto a + D(a)\epsilon$. More generally, given a finite dimensional vector space V over k we have

$$\text{Mor}_{\mathcal{C}_\Lambda}(k, k[V]) = \text{Der}_\Lambda(k, V)$$

and we will use the same notation f_D for the map associated to the derivation D. We also have

$$\text{Mor}_{\mathcal{C}_\Lambda}(k[W], k[V]) = \text{Hom}_k(V, W) \oplus \text{Der}_\Lambda(k, V)$$

where (φ, D) corresponds to the map $f_{\varphi,D} : a + w \mapsto a + \varphi(w) + D(a)$. We will sometimes write $f_{1,D} : a+v \to a+v+D(a)$ for the automorphism of $k[V]$ determined by the derivation $D : k \to V$. Note that $f_{1,D} \circ f_{1,D'} = f_{1,D+D'}$.

Let $\mathcal{F}$ be a predeformation category over $\mathcal{C}_\Lambda$. Let $x_0 \in \text{Ob}(\mathcal{F}(k))$. By the above there is a canonical map

$$\gamma_V : \text{Der}_\Lambda(k, V) \longrightarrow \overline{\mathcal{F}}(k[V])$$

defined by $D \mapsto f_{D,*}(x_0)$. Moreover, there is an action

$$a_V : \text{Der}_\Lambda(k, V) \times \overline{\mathcal{F}}(k[V]) \longrightarrow \overline{\mathcal{F}}(k[V])$$

defined by $(D, x) \mapsto f_{1,D,*}(x)$. These two maps are compatible, i.e., $f_{1,D,*}f_{D',*}x_0 = f_{D+D',*}x_0$ as follows from a computation of the compositions of these maps. Note that the maps γ_V and a_V are independent of the choice of x_0 as there is a unique x_0 up to isomorphism.

06SU **Lemma 12.6.** *Let $\mathcal{F}$ be a predeformation category over $\mathcal{C}_\Lambda$. If $\overline{\mathcal{F}}$ has (S2) then the maps γ_V are k-linear and we have $a_V(D, x) = x + \gamma_V(D)$.*

Proof. In the proof of Lemma 12.2 we have seen that the functor $V \mapsto \overline{\mathcal{F}}(k[V])$ transforms 0 to a singleton and products to products. The same is true of the functor $V \mapsto \text{Der}_\Lambda(k, V)$. Hence γ_V is linear by Lemma 11.5. Let $D : k \to V$ be a Λ-derivation. Set $D_1 : k \to V^{\oplus 2}$ equal to $a \mapsto (D(a), 0)$. Then

$$\begin{tikzcd} k[V \times V] \arrow[r, "+"'] \arrow[d, "f_{1,D_1}"] & k[V] \arrow[d, "f_{1,D}"] \\ k[V \times V] \arrow[r, "+"] & k[V] \end{tikzcd}$$

commutes. Unwinding the definitions and using that $\overline{F}(V \times V) = \overline{F}(V) \times \overline{F}(V)$ this means that $a_D(x_1) + x_2 = a_D(x_1 + x_2)$ for all $x_1, x_2 \in \overline{F}(V)$. Thus it suffices to show that $a_V(D, 0) = 0 + \gamma_V(D)$ where $0 \in \overline{F}(V)$ is the zero vector. By definition this is the element $f_{0,*}(x_0)$. Since $f_D = f_{1,D} \circ f_0$ the desired result follows. $\square$

A special case of the constructions above are the map

06SV (12.6.1) $$\gamma : \text{Der}_\Lambda(k, k) \longrightarrow T\mathcal{F}$$

and the action

06SW (12.6.2) $$a : \text{Der}_\Lambda(k, k) \times T\mathcal{F} \longrightarrow T\mathcal{F}$$

defined for any predeformation category $\mathcal{F}$. Note that if $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of predeformation categories, then we get commutative diagrams

$$\begin{tikzcd} \mathrm{Der}_\Lambda(k,k) \arrow[r, "\gamma"'] \arrow[dr, "\gamma"'] & T\mathcal{F} \arrow[d, "d\varphi"'] \\ & T\mathcal{G} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathrm{Der}_\Lambda(k,k) \times T\mathcal{F} \arrow[r, "a"'] \arrow[d, "1\times d\varphi"'] & T\mathcal{F} \arrow[d, "d\varphi"] \\ \mathrm{Der}_\Lambda(k,k) \times T\mathcal{G} \arrow[r, "a"] & T\mathcal{G} \end{tikzcd}$$

13. Versal formal objects

06SX The existence of a versal formal object forces $\mathcal{F}$ to have property (S1).

06SY **Lemma 13.1.** *Let $\mathcal{F}$ be a predeformation category. Assume $\mathcal{F}$ has a versal formal object. Then $\mathcal{F}$ satisfies (S1).*

Proof. Let ξ be a versal formal object of $\mathcal{F}$. Let

$$\begin{tikzcd} & x_2 \arrow[d] \\ x_1 \arrow[r] & x \end{tikzcd}$$

be a diagram in $\mathcal{F}$ such that $x_2 \to x$ lies over a surjective ring map. Since the natural morphism $\widehat{\mathcal{F}}|_{\mathcal{C}_\Lambda} \xrightarrow{\sim} \mathcal{F}$ is an equivalence (see Remark 7.7), we can consider this diagram also as a diagram in $\widehat{\mathcal{F}}$. By Lemma 8.11 there exists a morphism $\xi \to x_1$, so by Remark 8.10 we also get a morphism $\xi \to x_2$ making the diagram

$$\begin{tikzcd} \xi \arrow[r] \arrow[d] & x_2 \arrow[d] \\ x_1 \arrow[r] & x \end{tikzcd}$$

commute. If $x_1 \to x$ and $x_2 \to x$ lie above ring maps $A_1 \to A$ and $A_2 \to A$ then taking the pushforward of ξ to $A_1 \times_A A_2$ gives an object y as required by (S1). $\square$

In the case that our cofibred category satisfies (S1) and (S2) we can characterize the versal formal objects as follows.

06IU **Lemma 13.2.** *Let $\mathcal{F}$ be a predeformation category satisfying (S1) and (S2). Let ξ be a formal object of $\mathcal{F}$ corresponding to $\underline{\xi} : \underline{R}|_{\mathcal{C}_\Lambda} \to \mathcal{F}$, see Remark 7.12. Then ξ is versal if and only if the following two conditions hold:*

1. *the map $d\underline{\xi} : T\underline{R}|_{\mathcal{C}_\Lambda} \to T\mathcal{F}$ on tangent spaces is surjective, and*
2. *given a diagram in $\widehat{\mathcal{F}}$*

$$\begin{tikzcd} & y \arrow[d] \\ \xi \arrow[r] & x \end{tikzcd} \quad \textit{lying over} \quad \begin{tikzcd} & B \arrow[d, "f"] \\ R \arrow[r] & A \end{tikzcd}$$

in $\widehat{\mathcal{C}}_\Lambda$ with $B \to A$ a small extension of Artinian rings, then there exists a ring map $R \to B$ such that

$$\begin{tikzcd} & B \arrow[d, "f"] \\ R \arrow[ur] \arrow[r] & A \end{tikzcd}$$

commutes.

Proof. If ξ is versal then (1) holds by Lemma 8.8 and (2) holds by Remark 8.10. Assume (1) and (2) hold. By Remark 8.10 we must show that given a diagram in $\widehat{\mathcal{F}}$ as in (2), there exists $\xi \to y$ such that

$$\begin{tikzcd} & y \arrow[d] \\ \xi \arrow[ur] \arrow[r] & x \end{tikzcd}$$

commutes. Let $b : R \to B$ be the map guaranteed by (2). Denote $y' = b_*\xi$ and choose a factorization $\xi \to y' \to x$ lying over $R \to B \to A$ of the given morphism $\xi \to x$. By (S1) we obtain a commutative diagram

$$\begin{tikzcd} z \arrow[r] \arrow[d] & y \arrow[d] \\ y' \arrow[r] & x \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} B \times_A B \arrow[r] \arrow[d] & B \arrow[d, "f"] \\ B \arrow[r, "f"] & A. \end{tikzcd}$$

Set $I = \mathrm{Ker}(f)$. Let $\overline{g} : B \times_A B \to k[I]$ be the ring map $(u, v) \mapsto \overline{u} \oplus (v - u)$, cf. Lemma 10.8. By (1) there exists a morphism $\xi \to \overline{g}_*z$ which lies over a ring map $i : R \to k[\epsilon]$. Choose an Artinian quotient $b_1 : R \to B_1$ such that both $b : R \to B$ and $i : R \to k[\epsilon]$ factor through $R \to B_1$, i.e., giving $h : B_1 \to B$ and $i' : B_1 \to k[\epsilon]$. Choose a pushforward $y_1 = b_{1,*}\xi$, a factorization $\xi \to y_1 \to y'$ lying over $R \to B_1 \to B$ of $\xi \to y'$, and a factorization $\xi \to y_1 \to \overline{g}_*z$ lying over $R \to B_1 \to k[\epsilon]$ of $\xi \to \overline{g}_*z$. Applying (S1) once more we obtain

$$\begin{tikzcd} z_1 \arrow[r] \arrow[d] & z \arrow[r] \arrow[d] & y \arrow[d] \\ y_1 \arrow[r] & y' \arrow[r] & x \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} B_1 \times_A B \arrow[r] \arrow[d] & B \times_A B \arrow[r] \arrow[d] & B \arrow[d, "f"] \\ B_1 \arrow[r] & B \arrow[r] & A. \end{tikzcd}$$

Note that the map $g : B_1 \times_A B \to k[I]$ of Lemma 10.8 (defined using h) is the composition of $B_1 \times_A B \to B \times_A B$ and the map $\overline{g}$ above. By construction there exists a morphism $y_1 \to g_*z_1 \cong \overline{g}_*z$! Hence Lemma 10.8 applies (to the outer rectangles in the diagrams above) to give a morphism $y_1 \to y$ and precomposing with $\xi \to y_1$ gives the desired morphism $\xi \to y$. $\square$

If $\mathcal{F}$ has property (S1) then the "largest quotient where a lift exists" exists. Here is a precise statement.

06SZ **Lemma 13.3.** *Let $\mathcal{F}$ be a category cofibred in groupoids over $\mathcal{C}_\Lambda$ which has (S1). Let $B \to A$ be a surjection in $\mathcal{C}_\Lambda$ with kernel I annihilated by $\mathfrak{m}_B$. Let $x \in \mathcal{F}(A)$. The set of ideals*

$$\mathcal{J} = \{J \subset I \mid \textit{there exists an } y \to x \textit{ lying over } B/J \to A\}$$

has a smallest element.

Proof. Note that $\mathcal{J}$ is nonempty as $I \in \mathcal{J}$. Also, if $J \in \mathcal{J}$ and $J \subset J' \subset I$ then $J' \in \mathcal{J}$ because we can pushforward the object y to an object y' over B/J'. Let J and K be elements of the displayed set. We claim that $J \cap K \in \mathcal{J}$ which will prove the lemma. Since I is a k-vector space we can find an ideal $J \subset J' \subset I$ such that $J \cap K = J' \cap K$ and such that $J' + K = I$. By the above we may replace J by J' and assume that $J + K = I$. In this case

$$A/(J \cap K) = A/J \times_{A/I} A/K.$$

Hence the existence of an element $z \in \mathcal{F}(A/(J \cap K))$ mapping to x follows, via (S1), from the existence of the elements we have assumed exist over A/J and A/K. $\square$

We will improve on the following result later.

06IW **Lemma 13.4.** *Let $\mathcal{F}$ be a category cofibred in groupoids over $\mathcal{C}_\Lambda$. Assume the following conditions hold:*

1. *$\mathcal{F}$ is a predeformation category.*
2. *$\mathcal{F}$ satisfies (S1).*
3. *$\mathcal{F}$ satisfies (S2).*
4. *$\dim_k T\mathcal{F}$ is finite.*

Then $\mathcal{F}$ has a versal formal object.

Proof. Assume (1), (2), (3), and (4) hold. Choose an object $R \in \mathrm{Ob}(\widehat{\mathcal{C}}_\Lambda)$ such that $\underline{R}|_{\mathcal{C}_\Lambda}$ is smooth. See Lemma 9.5. Let $r = \dim_k T\mathcal{F}$ and put $S = R[[X_1, \ldots, X_r]]$.

We are going to inductively construct for $n \geq 2$ pairs $(J_n, f_{n-1} : \xi_n \to \xi_{n-1})$ where $J_n \subset S$ is an decreasing sequence of ideals and $f_{n-1} : \xi_n \to \xi_{n-1}$ is a morphism of $\mathcal{F}$ lying over the projection $S/J_n \to S/J_{n-1}$.

Step 1. Let $J_1 = \mathfrak{m}_S$. Let ξ_1 be the unique (up to unique isomorphism) object of $\mathcal{F}$ over $k = S/J_1 = S/\mathfrak{m}_S$

Step 2. Let $J_2 = \mathfrak{m}_S^2 + \mathfrak{m}_R S$. Then $S/J_2 = k[V]$ with $V = kX_1 \oplus \ldots \oplus kX_r$ By (S2) for $\overline{\mathcal{F}}$ we get a bijection

$$\overline{\mathcal{F}}(S/J_2) \longrightarrow T\mathcal{F} \otimes_k V,$$

see Lemmas 10.5 and 12.2. Choose a basis $\theta_1, \ldots, \theta_r$ for $T\mathcal{F}$ and set $\xi_2 = \sum \theta_i \otimes X_i \in \mathrm{Ob}(\mathcal{F}(S/J_2))$. The point of this choice is that

$$d\xi_2 : \mathrm{Mor}_{\mathcal{C}_\Lambda}(S/J_2, k[\epsilon]) \longrightarrow T\mathcal{F}$$

is surjective. Let $f_1 : \xi_2 \to \xi_1$ be the unique morphism.

Induction step. Assume $(J_n, f_{n-1} : \xi_n \to \xi_{n-1})$ has been constructed for some $n \geq 2$. There is a minimal element J_{n+1} of the set of ideals $J \subset S$ satisfying: (a) $\mathfrak{m}_S J_n \subset J \subset J_n$ and (b) there exists a morphism $\xi_{n+1} \to \xi_n$ lying over $S/J \to S/J_n$, see Lemma 13.3. Let $f_n : \xi_{n+1} \to \xi_n$ be any morphism of $\mathcal{F}$ lying over $S/J_{n+1} \to S/J_n$.

Set $J = \bigcap J_n$. Set $\overline{S} = S/J$. Set $\overline{J}_n = J_n/J$. By Lemma 4.7 the sequence of ideals $(\overline{J}_n)$ induces the $\mathfrak{m}_{\overline{S}}$-adic topology on $\overline{S}$. Since (ξ_n, f_n) is an object of $\widehat{\mathcal{F}}_\mathcal{I}(\overline{S})$, where $\mathcal{I}$ is the filtration $(\overline{J}_n)$ of $\overline{S}$, we see that (ξ_n, f_n) induces an object ξ of $\widehat{\mathcal{F}}(\overline{S})$. see Lemma 7.4.

We prove ξ is versal. For versality it suffices to check conditions (1) and (2) of Lemma 13.2. Condition (1) follows from our choice of ξ_2 in Step 2 above. Suppose given a diagram in $\widehat{\mathcal{F}}$

$$\begin{tikzcd} & y \arrow[d] & & & B \arrow[d, "f"] \\ \xi \arrow[r] & x & \text{lying over} & \overline{S} \arrow[r] & A \end{tikzcd}$$

in $\widehat{\mathcal{C}}_\Lambda$ with $f : B \to A$ a small extension of Artinian rings. We have to show there is a map $\overline{S} \to B$ fitting into the diagram on the right. Choose n such that $\overline{S} \to A$ factors through $\overline{S} \to S/J_n$. This is possible as the sequence $(\overline{J}_n)$ induces the $\mathfrak{m}_{\overline{S}}$-adic topology as we saw above. The pushforward of ξ along $\overline{S} \to S/J_n$ is ξ_n. We may factor $\xi \to x$ as $\xi \to \xi_n \to x$ hence we get a diagram in $\mathcal{F}$

$$\begin{tikzcd} & y \arrow[d] & & & B \arrow[d, "f"] \\ \xi_n \arrow[r] & x & \text{lying over} & S/J_n \arrow[r] & A. \end{tikzcd}$$

To check condition (2) of Lemma 13.2 it suffices to complete the diagram

$$\begin{tikzcd} S/J_{n+1} \arrow[r, dashed] \arrow[d] & B \arrow[d, "f"] \\ S/J_n \arrow[r] & A \end{tikzcd}$$

or equivalently, to complete the diagram

$$\begin{tikzcd} & S/J_n \times_A B \arrow[d, "p_1"] \\ S/J_{n+1} \arrow[r] \arrow[ur, dashed] & S/J_n. \end{tikzcd}$$

If p_1 has a section we are done. If not, by Lemma 3.8 (2) p_1 is a small extension, so by Lemma 3.12 (4) p_1 is an essential surjection. Recall that $S = R[[X_1, \ldots, X_r]]$ and that we chose R such that $\underline{R}|_{\mathcal{C}_\Lambda}$ is smooth. Hence there exists a map $h : R \to B$ lifting the map $R \to S \to S/J_n \to A$. By the universal property of a power series ring there is an R-algebra map $h : S = R[[X_1, \ldots, X_r]] \to B$ lifting the given map $S \to S/J_n \to A$. This induces a map $g : S \to S/J_n \times_A B$ making the solid square in the diagram

$$\begin{tikzcd} S \arrow[r, "g"'] \arrow[d] & S/J_n \times_A B \arrow[d, "p_1"] \\ S/J_{n+1} \arrow[r] \arrow[ur, dashed] & S/J_n \end{tikzcd}$$

commute. Then g is a surjection since p_1 is an essential surjection. We claim the ideal $K = \text{Ker}(g)$ of S satisfies conditions (a) and (b) of the construction of J_{n+1} in the induction step above. Namely, $K \subset J_n$ is clear and $\mathfrak{m}_S J_n \subset K$ as p_1 is a small extension; this proves (a). By (S1) applied to

$$\begin{tikzcd} & y \arrow[d] \\ \xi_n \arrow[r] & x, \end{tikzcd}$$

there exists a lifting of ξ_n to $S/K \cong S/J_n \times_A B$, so (b) holds. Since J_{n+1} was the minimal ideal with properties (a) and (b) this implies $J_{n+1} \subset K$. Thus the desired map $S/J_{n+1} \to S/K \cong S/J_n \times_A B$ exists. □

0D3G **Remark 13.5.** Let $F : \mathcal{C}_\Lambda \to \textit{Sets}$ be a predeformation functor satisfying (S1) and (S2). The condition $\dim_k TF < \infty$ is precisely condition (H3) from Schlessinger's paper. Recall that (S1) and (S2) correspond to conditions (H1) and (H2), see Remark 10.3. Thus Lemma 13.4 tells us

$$(H1) + (H2) + (H3) \Rightarrow \text{ there exists a versal formal object}$$

for predeformation functors. We will make the link with hulls in Remark 15.6.

14. Minimal versal formal objects

06T0 We do a little bit of work to try and understand (non)uniqueness of versal formal objects. It turns out that if a predeformation category has a versal formal object, then it has a minimal versal formal object and any two such are isomorphic. Moreover, all versal formal objects are "more or less" the same up to replacing the base ring by a power series extension.

Let $\mathcal{F}$ be a category cofibred in groupoids over $\mathcal{C}_\Lambda$. For every object x of $\mathcal{F}$ lying over $A \in \mathrm{Ob}(\mathcal{C}_\Lambda)$ consider the category $\mathcal{S}_x$ with objects

$$\mathrm{Ob}(\mathcal{S}_x) = \{x' \to x \mid x' \to x \text{ lies over } A' \subset A\}$$

and morphisms are morphisms over x. For every $y \to x$ in $\mathcal{F}$ lying over $f : B \to A$ in $\mathcal{C}_\Lambda$ there is a functor $f_* : \mathcal{S}_y \to \mathcal{S}_x$ defined as follows: Given $y' \to y$ lying over $B' \subset B$ set $A' = f(B')$ and let $y' \to x'$ be over $B' \to f(B')$ be the pushforward of y'. By the axioms of a category cofibred in groupoids we obtain a unique morphism $x' \to x$ lying over $f(B') \to A$ such that

$$\begin{tikzcd} y' \arrow[r] \arrow[d] & x' \arrow[d] \\ y \arrow[r] & x \end{tikzcd}$$

commutes. Then $x' \to x$ is an object of $\mathcal{S}_x$. We say an object $x' \to x$ of $\mathcal{S}_x$ is *minimal* if any morphism $(x'_1 \to x) \to (x' \to x)$ in $\mathcal{S}_x$ is an isomorphism, i.e., x' and x'_1 are defined over the same subring of A. Since A has finite length as a Λ-module we see that minimal objects always exist.

06T1 **Lemma 14.1.** *Let $\mathcal{F}$ be a category cofibred in groupoids over $\mathcal{C}_\Lambda$ which has (S1).*

1. *For $y \to x$ in $\mathcal{F}$ a minimal object in $\mathcal{S}_y$ maps to a minimal object of $\mathcal{S}_x$.*
2. *For $y \to x$ in $\mathcal{F}$ lying over a surjection $f : B \to A$ in $\mathcal{C}_\Lambda$ every minimal object of $\mathcal{S}_x$ is the image of a minimal object of $\mathcal{S}_y$.*

Proof. Proof of (1). Say $y \to x$ lies over $f : B \to A$. Let $y' \to y$ lying over $B' \subset B$ be a minimal object of $\mathcal{S}_y$. Let

$$\begin{array}{ccc} y' & \longrightarrow & x' \\ \downarrow & & \downarrow \\ y & \longrightarrow & x \end{array} \quad \text{lying over} \quad \begin{array}{ccc} B' & \longrightarrow & f(B') \\ \downarrow & & \downarrow \\ B & \longrightarrow & A \end{array}$$

be as in the construction of f_* above. Suppose that $(x'' \to x) \to (x' \to x)$ is a morphism of $\mathcal{S}_x$ with $x'' \to x'$ lying over $A'' \subset f(B')$. By (S1) there exists $y'' \to y'$ lying over $B' \times_{f(B')} A'' \to B'$. Since $y' \to y$ is minimal we conclude that $B' \times_{f(B')} A'' \to B'$ is an isomorphism, which implies that $A'' = f(B')$, i.e., $x' \to x$ is minimal.

Proof of (2). Suppose $f : B \to A$ is surjective and $y \to x$ lies over f. Let $x' \to x$ be a minimal object of $\mathcal{S}_x$ lying over $A' \subset A$. By (S1) there exists $y' \to y$ lying over $B' = f^{-1}(A') = B \times_A A' \to B$ whose image in $\mathcal{S}_x$ is $x' \to x$. So $f_*(y' \to y) = x' \to x$. Choose a morphism $(y'' \to y) \to (y' \to y)$ in $\mathcal{S}_y$ with $y'' \to y$ a minimal object (this is possible by the remark on lengths above the lemma). Then $f_*(y'' \to y)$ is an object of $\mathcal{S}_x$ which maps to $x' \to x$ (by functoriality of f_*) hence is isomorphic to $x' \to x$ by minimality of $x' \to x$. □

06T2 **Lemma 14.2.** *Let $\mathcal{F}$ be a category cofibred in groupoids over $\mathcal{C}_\Lambda$ which has (S1). Let ξ be a versal formal object of $\mathcal{F}$ lying over R. There exists a morphism $\xi' \to \xi$ lying over $R' \subset R$ with the following minimality properties*

(1) *for every $f : R \to A$ with $A \in \mathrm{Ob}(\mathcal{C}_\Lambda)$ the pushforwards*

$$\begin{array}{ccc} \xi' & \longrightarrow & x' \\ \downarrow & & \downarrow \\ \xi & \longrightarrow & x \end{array} \quad \textit{lying over} \quad \begin{array}{ccc} R' & \longrightarrow & f(R') \\ \downarrow & & \downarrow \\ R & \longrightarrow & A \end{array}$$

produce a minimal object $x' \to x$ of $\mathcal{S}_x$, and

(2) *for any morphism of formal objects $\xi'' \to \xi'$ the corresponding morphism $R'' \to R'$ is surjective.*

Proof. Write $\xi = (R, \xi_n, f_n)$. Set $R'_1 = k$ and $\xi'_1 = \xi_1$. Suppose that we have constructed minimal objects $\xi'_m \to \xi_m$ of $\mathcal{S}_{\xi_m}$ lying over $R'_m \subset R/\mathfrak{m}_R^m$ for $m \leq n$ and morphisms $f'_m : \xi'_{m+1} \to \xi'_m$ compatible with f_m for $m \leq n-1$. By Lemma 14.1 (2) there exists a minimal object $\xi'_{n+1} \to \xi_{n+1}$ lying over $R'_{n+1} \subset R/\mathfrak{m}_R^{n+1}$ whose image is $\xi'_n \to \xi_n$ over $R'_n \subset R/\mathfrak{m}_R^n$. This produces the commutative diagram

$$\begin{array}{ccc} \xi'_{n+1} & \xrightarrow[f'_n]{} & \xi'_n \\ \downarrow & & \downarrow \\ \xi_{n+1} & \xrightarrow{f_n} & \xi_n \end{array}$$

by construction. Moreover the ring map $R'_{n+1} \to R'_n$ is surjective. Set $R' = \lim_n R'_n$. Then $R' \to R$ is injective.

However, it isn't a priori clear that R' is Noetherian. To prove this we use that ξ is versal. Namely, versality implies that there exists a morphism $\xi \to \xi'_n$ in $\widehat{\mathcal{F}}$, see Lemma 8.11. The corresponding map $R \to R'_n$ has to be surjective (as $\xi'_n \to \xi_n$

is minimal in $\mathcal{S}_{\xi_n}$). Thus the dimensions of the cotangent spaces are bounded and Lemma 4.8 implies R' is Noetherian, i.e., an object of $\widehat{\mathcal{C}}_\Lambda$. By Lemma 7.4 (plus the result on filtrations of Lemma 4.8) the sequence of elements ξ'_n defines a formal object ξ' over R' and we have a map $\xi' \to \xi$.

By construction (1) holds for $R \to R/\mathfrak{m}_R^n$ for each n. Since each $R \to A$ as in (1) factors through $R \to R/\mathfrak{m}_R^n \to A$ we see that (1) for $x' \to x$ over $f(R) \subset A$ follows from the minimality of $\xi'_n \to \xi_n$ over $R'_n \to R/\mathfrak{m}_R^n$ by Lemma 14.1 (1).

If $R'' \to R'$ as in (2) is not surjective, then $R'' \to R' \to R'_n$ would not be surjective for some n and $\xi'_n \to \xi_n$ wouldn't be minimal, a contradiction. This contradiction proves (2). □

06T3 **Lemma 14.3.** *Let $\mathcal{F}$ be a category cofibred in groupoids over $\mathcal{C}_\Lambda$ which has (S1). Let ξ be a versal formal object of $\mathcal{F}$ lying over R. Let $\xi' \to \xi$ be a morphism of formal objects lying over $R' \subset R$ as constructed in Lemma 14.2. Then*

$$R \cong R'[[x_1, \ldots, x_r]]$$

is a power series ring over R'. Moreover, ξ' is a versal formal object too.

Proof. By Lemma 8.11 there exists a morphism $\xi \to \xi'$. By Lemma 14.2 the corresponding map $f : R \to R'$ induces a surjection $f|_{R'} : R' \to R'$. This is an isomorphism by Algebra, Lemma 31.10. Hence $I = \mathrm{Ker}(f)$ is an ideal of R such that $R = R' \oplus I$. Let $x_1, \ldots, x_n \in I$ be elements which form a basis for $I/\mathfrak{m}_R I$. Consider the map $S = R'[[X_1, \ldots, X_r]] \to R$ mapping X_i to x_i. For every $n \geq 1$ we get a surjection of Artinian R'-algebras $B = S/\mathfrak{m}_S^n \to R/\mathfrak{m}_R^n = A$. Denote $y \in \mathrm{Ob}(\mathcal{F}(B)$, resp. $x \in \mathrm{Ob}(\mathcal{F}(A))$ the pushforward of ξ' along $R' \to S \to B$, resp. $R' \to S \to A$. Note that x is also the pushforward of ξ along $R \to A$ as ξ is the pushforward of ξ' along $R' \to R$. Thus we have a solid diagram

$$\begin{tikzcd} & y \arrow[d] \\ \xi \arrow[ur, dotted] \arrow[r] & x \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} & S/\mathfrak{m}_S^n \arrow[d] \\ R \arrow[ur, dotted] \arrow[r] & R/\mathfrak{m}_R^n \end{tikzcd}$$

Because ξ is versal, using Remark 8.10 we obtain the dotted arrows fitting into these diagrams. In particular, the maps $S/\mathfrak{m}_S^n \to R/\mathfrak{m}_R^n$ have sections $h_n : R/\mathfrak{m}_R^n \to S/\mathfrak{m}_S^n$. It follows from Lemma 4.9 that $S \to R$ is an isomorphism.

As ξ is a pushforward of ξ' along $R' \to R$ we obtain from Remark 7.13 a commutative diagram

$$\begin{tikzcd} \underline{R}|_{\mathcal{C}_\Lambda} \arrow[rr] \arrow[dr, "\underline{\xi}"'] & & \underline{R'}|_{\mathcal{C}_\Lambda} \arrow[dl, "\underline{\xi'}"] \\ & \mathcal{F} & \end{tikzcd}$$

Since $R' \to R$ has a left inverse (namely $R \to R/I = R'$) we see that $\underline{R}|_{\mathcal{C}_\Lambda} \to \underline{R'}|_{\mathcal{C}_\Lambda}$ is essentially surjective. Hence by Lemma 8.7 we see that $\underline{\xi'}$ is smooth, i.e., ξ' is a versal formal object. □

Motivated by the preceding lemmas we make the following definition.

06T4 **Definition 14.4.** Let $\mathcal{F}$ be a predeformation category. We say a versal formal object ξ of $\mathcal{F}$ is *minimal*[3] if for any morphism of formal objects $\xi' \to \xi$ the underlying map on rings is surjective. Sometimes a minimal versal formal object is called *miniversal*.

The work in this section shows this definition is reasonable. First of all, the existence of a versal formal object implies that $\mathcal{F}$ has (S1). Then the preceding lemmas show there exists a minimal versal formal object. Finally, any two minimal versal formal objects are isomorphic. Here is a summary of our results (with detailed proofs).

06T5 **Lemma 14.5.** *Let $\mathcal{F}$ be a predeformation category which has a versal formal object. Then*

(1) *$\mathcal{F}$ has a minimal versal formal object,*
(2) *minimal versal objects are unique up to isomorphism, and*
(3) *any versal object is the pushforward of a minimal versal object along a power series ring extension.*

Proof. Suppose $\mathcal{F}$ has a versal formal object ξ over R. Then it satisfies (S1), see Lemma 13.1. Let $\xi' \to \xi$ over $R' \subset R$ be any of the morphisms constructed in Lemma 14.2. By Lemma 14.3 we see that ξ' is versal, hence it is a minimal versal formal object (by construction). This proves (1). Also, $R \cong R'[[x_1, \ldots, x_n]]$ which proves (3).

Suppose that ξ_i/R_i are two minimal versal formal objects. By Lemma 8.11 there exist morphisms $\xi_1 \to \xi_2$ and $\xi_2 \to \xi_1$. The corresponding ring maps $f : R_1 \to R_2$ and $g : R_2 \to R_1$ are surjective by minimality. Hence the compositions $g \circ f : R_1 \to R_1$ and $f \circ g : R_2 \to R_2$ are isomorphisms by Algebra, Lemma 31.10. Thus f and g are isomorphisms whence the maps $\xi_1 \to \xi_2$ and $\xi_2 \to \xi_1$ are isomorphisms (because $\widehat{\mathcal{F}}$ is cofibred in groupoids by Lemma 7.2). This proves (2) and finishes the proof of the lemma. $\square$

15. Miniversal formal objects and tangent spaces

06IL The general notion of minimality introduced in Definition 14.4 can sometimes be deduced from the behaviour on tangent spaces. Let ξ be a formal object of the predeformation category $\mathcal{F}$ and let $\underline{\xi} : \underline{R}|_{\mathcal{C}_\Lambda} \to \mathcal{F}$ be the corresponding morphism. Then we can consider the following the condition

06IM (15.0.1) $$d\underline{\xi} : \mathrm{Der}_\Lambda(R, k) \to T\mathcal{F} \text{ is bijective}$$

and the condition

06T6 (15.0.2) $$d\underline{\xi} : \mathrm{Der}_\Lambda(R, k) \to T\mathcal{F} \text{ is bijective on } \mathrm{Der}_\Lambda(k, k)\text{-orbits.}$$

Here we are using the identification $T\underline{R}|_{\mathcal{C}_\Lambda} = \mathrm{Der}_\Lambda(R, k)$ of Example 11.11 and the action (12.6.2) of derivations on the tangent spaces. If $k' \subset k$ is separable, then $\mathrm{Der}_\Lambda(k, k) = 0$ and the two conditions are equivalent. It turns out that, in the presence of condition (S2) a versal formal object is minimal if and only if $\underline{\xi}$ satisfies (15.0.2). Moreover, if $\underline{\xi}$ satisfies (15.0.1), then $\mathcal{F}$ satisfies (S2).

[3] This may be nonstandard terminology. Many authors tie this notion in with properties of tangent spaces. We will make the link in Section 15.

06IR **Lemma 15.1.** *Let $\mathcal{F}$ be a predeformation category. Let ξ be a versal formal object of $\mathcal{F}$ such that (15.0.2) holds. Then ξ is a minimal versal formal object. In particular, such ξ are unique up to isomorphism.*

Proof. If ξ is not minimal, then there exists a morphism $\xi' \to \xi$ lying over $R' \to R$ such that $R = R'[[x_1, \ldots, x_n]]$ with $n > 0$, see Lemma 14.5. Thus $d\underline{\xi}$ factors as

$$\mathrm{Der}_\Lambda(R, k) \to \mathrm{Der}_\Lambda(R', k) \to T\mathcal{F}$$

and we see that (15.0.2) cannot hold because $D : f \mapsto \partial/\partial x_1(f) \bmod \mathfrak{m}_R$ is an element of the kernel of the first arrow which is not in the image of $\mathrm{Der}_\Lambda(k, k) \to \mathrm{Der}_\Lambda(R, k)$. $\square$

06IV **Lemma 15.2.** *Let $\mathcal{F}$ be a predeformation category. Let ξ be a versal formal object of $\mathcal{F}$ such that (15.0.1) holds. Then*

(1) *$\mathcal{F}$ satisfies (S1).*
(2) *$\mathcal{F}$ satisfies (S2).*
(3) *$\dim_k T\mathcal{F}$ is finite.*

Proof. Condition (S1) holds by Lemma 13.1. The first part of (S2) holds since (S1) holds. Let

$$\begin{tikzcd} y \arrow[r, "c"'] \arrow[d, "a"'] & x_\epsilon \arrow[d, "e"] \\ x \arrow[r, "d"] & x_0 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} y' \arrow[r, "c'"'] \arrow[d, "a'"'] & x_\epsilon \arrow[d, "e"] \\ x \arrow[r, "d"] & x_0 \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} A \times_k k[\epsilon] \arrow[r] \arrow[d] & k[\epsilon] \arrow[d] \\ A \arrow[r] & k \end{tikzcd}$$

be diagrams as in the second part of (S2). As above we can find morphisms $b : \xi \to y$ and $b' : \xi \to y'$ such that

$$\begin{tikzcd} \xi \arrow[r, "b'"] \arrow[d, "b"'] & y' \arrow[d, "a'"] \\ y \arrow[r, "a"] & x \end{tikzcd}$$

commutes. Let $p : \mathcal{F} \to \mathcal{C}_\Lambda$ denote the structure morphism. Say $\widehat{p}(\xi) = R$, i.e., ξ lies over $R \in \mathrm{Ob}(\widehat{\mathcal{C}}_\Lambda)$. We see that the pushforward of ξ via $p(c) \circ p(b)$ is x_ϵ and that the pushforward of ξ via $p(c') \circ p(b')$ is x_ϵ. Since ξ satisfies (15.0.1), we see that $p(c) \circ p(b) = p(c') \circ p(b')$ as maps $R \to k[\epsilon]$. Hence $p(b) = p(b')$ as maps from $R \to A \times_k k[\epsilon]$. Thus we see that y and y' are isomorphic to the pushforward of ξ along this map and we get a unique morphism $y \to y'$ over $A \times_k k[\epsilon]$ compatible with b and b' as desired.

Finally, by Example 11.11 we see $\dim_k T\mathcal{F} = \dim_k T\underline{R}|_{\mathcal{C}_\Lambda}$ is finite. $\square$

06T7 **Example 15.3.** There exist predeformation categories which have a versal formal object satisfying (15.0.2) but which do not satisfy (S2). A quick example is to take $F = \underline{k[\epsilon]}/G$ where $G \subset \mathrm{Aut}_{\mathcal{C}_\Lambda}(k[\epsilon])$ is a finite nontrivial subgroup. Namely, the map $\underline{k[\epsilon]} \to F$ is smooth, but the tangent space of F does not have a natural k-vector space structure (as it is a quotient of a k-vector space by a finite group).

06T8 **Lemma 15.4.** *Let $\mathcal{F}$ be a predeformation category satisfying (S2) which has a versal formal object. Then its minimal versal formal object satisfies (15.0.2).*

Proof. Let ξ be a minimal versal formal object for $\mathcal{F}$, see Lemma 14.5. Say ξ lies over $R \in \mathrm{Ob}(\widehat{\mathcal{C}}_\Lambda)$. In order to parse (15.0.2) we point out that $T\mathcal{F}$ has a natural k-vector space structure (see Lemma 12.2), that $d\underline{\xi} : \mathrm{Der}_\Lambda(R, k) \to T\mathcal{F}$ is linear (see Lemma 12.4), and that the action of $\mathrm{Der}_\Lambda(k, k)$ is given by addition (see Lemma 12.6). Consider the diagram

$$\begin{tikzcd} & \mathrm{Hom}_k(\mathfrak{m}_R/\mathfrak{m}_R^2, k) & \\ K \arrow[r] & \mathrm{Der}_\Lambda(R, k) \arrow[u] \arrow[r, "d\underline{\xi}"] & T\mathcal{F} \\ & \mathrm{Der}_\Lambda(k, k) \arrow[u] \arrow[ur] & \end{tikzcd}$$

The vector space K is the kernel of $d\underline{\xi}$. Note that the middle column is exact in the middle as it is dual to the sequence (3.10.1). If (15.0.2) fails, then we can find a nonzero element $D \in K$ which does not map to zero in $\mathrm{Hom}_k(\mathfrak{m}_R/\mathfrak{m}_R^2, k)$. This means there exists an $t \in \mathfrak{m}_R$ such that $D(t) = 1$. Set $R' = \{a \in R \mid D(a) = 0\}$. As D is a derivation this is a subring of R. Since $D(t) = 1$ we see that $R' \to k$ is surjective (compare with the proof of Lemma 3.12). Note that $\mathfrak{m}_{R'} = \mathrm{Ker}(D : \mathfrak{m}_R \to k)$ is an ideal of R and $\mathfrak{m}_R^2 \subset \mathfrak{m}_{R'}$. Hence

$$\mathfrak{m}_R/\mathfrak{m}_R^2 = \mathfrak{m}_{R'}/\mathfrak{m}_R^2 + k\bar{t}$$

which implies that the map

$$R'/\mathfrak{m}_R^2 \times_k k[\epsilon] \to R/\mathfrak{m}_R^2$$

sending ϵ to $\bar{t}$ is an isomorphism. In particular there is a map $R/\mathfrak{m}_R^2 \to R'/\mathfrak{m}_R^2$.

Let $\xi \to y$ be a morphism lying over $R \to R/\mathfrak{m}_R^2$. Let $y \to x$ be a morphism lying over $R/\mathfrak{m}_R^2 \to R'/\mathfrak{m}_R^2$. Let $y \to x_\epsilon$ be a morphism lying over $R/\mathfrak{m}_R^2 \to k[\epsilon]$. Let x_0 be the unique (up to unique isomorphism) object of $\mathcal{F}$ over k. By the axioms of a category cofibred in groupoids we obtain a commutative diagram

$$\begin{tikzcd} y \arrow[r] \arrow[d] & x_\epsilon \arrow[d] \\ x \arrow[r] & x_0 \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} R'/\mathfrak{m}_R^2 \times_k k[\epsilon] \arrow[r] \arrow[d] & k[\epsilon] \arrow[d] \\ R'/\mathfrak{m}_R^2 \arrow[r] & k. \end{tikzcd}$$

Because $D \in K$ we see that x_ϵ is isomorphic to $0 \in \mathcal{F}(k[\epsilon])$, i.e., x_ϵ is the pushforward of x_0 via $k \to k[\epsilon], a \mapsto a$. Hence by Lemma 10.7 we see that there exists a morphism $x \to y$. Since $\mathrm{length}_\Lambda(R'/\mathfrak{m}_R^2) < \mathrm{length}_\Lambda(R/\mathfrak{m}_R^2)$ the corresponding ring map $R'/\mathfrak{m}_R^2 \to R/\mathfrak{m}_R^2$ is not surjective. This contradicts the minimality of ξ/R, see part (1) of Lemma 14.2. This contradiction shows that such a D cannot exist, hence we win. $\square$

06IX **Theorem 15.5.** *Let $\mathcal{F}$ be a predeformation category. Consider the following conditions*

(1) *$\mathcal{F}$ has a minimal versal formal object satisfying (15.0.1),*
(2) *$\mathcal{F}$ has a minimal versal formal object satisfying (15.0.2),*
(3) *the following conditions hold:*
   (a) *$\mathcal{F}$ satisfies (S1).*

(b) $\mathcal{F}$ *satisfies (S2).*
(c) $\dim_k T\mathcal{F}$ *is finite.*

We always have

$$(1) \Rightarrow (3) \Rightarrow (2).$$

If $k' \subset k$ *is separable, then all three are equivalent.*

Proof. Lemma 15.2 shows that (1) $\Rightarrow$ (3). Lemmas 13.4 and 15.4 show that (3) $\Rightarrow$ (2). If $k' \subset k$ is separable then $\mathrm{Der}_\Lambda(k, k) = 0$ and we see that (15.0.1) = (15.0.2), i.e., (1) is the same as (2).

An alternative proof of (3) $\Rightarrow$ (1) in the classical case is to add a few words to the proof of Lemma 13.4 to see that one can right away construct a versal object which satisfies (15.0.1) in this case. This avoids the use of Lemma 13.4 in the classical case. Details omitted. □

06IY **Remark 15.6.** Let $F : \mathcal{C}_\Lambda \to \textit{Sets}$ be a predeformation functor satisfying (S1) and (S2) and $\dim_k TF < \infty$. Recall that these conditions correspond to the conditions (H1), (H2), and (H3) from Schlessinger's paper, see Remark 13.5. Now, in the classical case (or if $k' \subset k$ is separable) following Schlessinger we introduce the notion of a hull: a *hull* is a versal formal object $\xi \in \widehat{F}(R)$ such that $d\underline{\xi} : T\underline{R}|_{\mathcal{C}_\Lambda} \to TF$ is an isomorphism, i.e., (15.0.1) holds. Thus Theorem 15.5 tells us

$$(H1) + (H2) + (H3) \Rightarrow \text{ there exists a hull}$$

in the classical case. In other words, our theorem recovers Schlessinger's theorem on the existence of hulls.

06IZ **Remark 15.7.** Let $\mathcal{F}$ be a predeformation category. Recall that $\mathcal{F} \to \overline{\mathcal{F}}$ is smooth, see Remark 8.5. Hence if $\xi \in \widehat{\mathcal{F}}(R)$ is a versal formal object, then the composition

$$\underline{R}|_{\mathcal{C}_\Lambda} \longrightarrow \mathcal{F} \longrightarrow \overline{\mathcal{F}}$$

is smooth (Lemma 8.7) and we conclude that the image $\overline{\xi}$ of ξ in $\overline{\mathcal{F}}$ is a versal formal object. If (15.0.1) holds, then $\overline{\xi}$ induces an isomorphism $T\underline{R}|_{\mathcal{C}_\Lambda} \to T\overline{\mathcal{F}}$ because $\mathcal{F} \to \overline{\mathcal{F}}$ identifies tangent spaces. Hence in this case $\overline{\xi}$ is a hull for $\overline{\mathcal{F}}$, see Remark 15.6. By Theorem 15.5 we can always find such a ξ if $k' \subset k$ is separable and $\mathcal{F}$ is a predeformation category satisfying (S1), (S2), and $\dim_k T\mathcal{F} < \infty$.

06T9 **Example 15.8.** In Lemma 9.5 we constructed objects $R \in \widehat{\mathcal{C}}_\Lambda$ such that $\underline{R}|_{\mathcal{C}_\Lambda}$ is smooth and such that

$$H_1(L_{k/\Lambda}) = \mathfrak{m}_R/\mathfrak{m}_R^2 \quad \text{and} \quad \Omega_{R/\Lambda} \otimes_R k = \Omega_{k/\Lambda}$$

Let us reinterpret this using the theorem above. Namely, consider $\mathcal{F} = \mathcal{C}_\Lambda$ as a category cofibred in groupoids over itself (using the identity functor). Then $\mathcal{F}$ is a predeformation category, satisfies (S1) and (S2), and we have $T\mathcal{F} = 0$. Thus $\mathcal{F}$ satisfies condition (3) of Theorem 15.5. The theorem implies that (2) holds, i.e., we can find a minimal versal formal object $\xi \in \widehat{\mathcal{F}}(S)$ over some $S \in \widehat{\mathcal{C}}_\Lambda$ satisfying (15.0.2). Lemma 9.3 shows that $\Lambda \to S$ is formally smooth in the $\mathfrak{m}_S$-adic topology (because $\underline{\xi} : \underline{R}|_{\mathcal{C}_\Lambda} \to \mathcal{F} = \mathcal{C}_\Lambda$ is smooth). Now condition (15.0.2) tells us that $\mathrm{Der}_\Lambda(S, k) \to 0$ is bijective on $\mathrm{Der}_\Lambda(k, k)$-orbits. This means the injection $\mathrm{Der}_\Lambda(k, k) \to \mathrm{Der}_\Lambda(S, k)$ is also surjective. In other words, we have $\Omega_{S/\Lambda} \otimes_S k = \Omega_{k/\Lambda}$. Since $\Lambda \to S$ is formally smooth in the $\mathfrak{m}_S$-adic topology, we can apply More on Algebra, Lemma 40.4 to conclude the exact sequence (3.10.2) turns into a pair of identifications

$$H_1(L_{k/\Lambda}) = \mathfrak{m}_S/\mathfrak{m}_S^2 \quad \text{and} \quad \Omega_{S/\Lambda} \otimes_S k = \Omega_{k/\Lambda}$$

Reading the argument backwards, we find that the R constructed in Lemma 9.5 carries a minimal versal object. By the uniqueness of minimal versal objects (Lemma 14.5) we also conclude $R \cong S$, i.e., the two constructions give the same answer.

16. Rim-Schlessinger conditions and deformation categories

06J1 There is a very natural property of categories fibred in groupoids over $\mathcal{C}_\Lambda$ which is easy to check in practice and which implies Schlessinger's properties (S1) and (S2) we have introduced earlier.

06J2 **Definition 16.1.** Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. We say that $\mathcal{F}$ satisfies *condition (RS)* if for every diagram in $\mathcal{F}$

$$\begin{tikzcd} & x_2 \arrow[d] \\ x_1 \arrow[r] & x \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} & A_2 \arrow[d] \\ A_1 \arrow[r] & A \end{tikzcd}$$

in $\mathcal{C}_\Lambda$ with $A_2 \to A$ surjective, there exists a fiber product $x_1 \times_x x_2$ in $\mathcal{F}$ such that the diagram

$$\begin{tikzcd} x_1 \times_x x_2 \arrow[r] \arrow[d] & x_2 \arrow[d] \\ x_1 \arrow[r] & x \end{tikzcd} \quad \text{lies over} \quad \begin{tikzcd} A_1 \times_A A_2 \arrow[r] \arrow[d] & A_2 \arrow[d] \\ A_1 \arrow[r] & A. \end{tikzcd}$$

06J3 **Lemma 16.2.** *Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$ satisfying (RS). Given a commutative diagram in $\mathcal{F}$*

$$\begin{tikzcd} y \arrow[r] \arrow[d] & x_2 \arrow[d] \\ x_1 \arrow[r] & x \end{tikzcd} \quad \textit{lying over} \quad \begin{tikzcd} A_1 \times_A A_2 \arrow[r] \arrow[d] & A_2 \arrow[d] \\ A_1 \arrow[r] & A. \end{tikzcd}$$

with $A_2 \to A$ surjective, then it is a fiber square.

Proof. Since $\mathcal{F}$ satisfies (RS), there exists a fiber product diagram

$$\begin{tikzcd} x_1 \times_x x_2 \arrow[r] \arrow[d] & x_2 \arrow[d] \\ x_1 \arrow[r] & x \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} A_1 \times_A A_2 \arrow[r] \arrow[d] & A_2 \arrow[d] \\ A_1 \arrow[r] & A. \end{tikzcd}$$

The induced map $y \to x_1 \times_x x_2$ lies over $\mathrm{id} : A_1 \times_A A_1 \to A_1 \times_A A_1$, hence it is an isomorphism. □

06J4 **Lemma 16.3.** *Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. Then $\mathcal{F}$ satisfies (RS) if the condition in Definition 16.1 is assumed to hold only when $A_2 \to A$ is a small extension.*

Proof. Apply Lemma 3.3. The proof is similar to that of Lemma 8.2. □

06J5 **Lemma 16.4.** *Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. The following are equivalent*

(1) *$\mathcal{F}$ satisfies (RS),*
(2) *the functor $\mathcal{F}(A_1 \times_A A_2) \to \mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)$ see (10.1.1) is an equivalence of categories whenever $A_2 \to A$ is surjective, and*
(3) *same as in (2) whenever $A_2 \to A$ is a small extension.*

Proof. Assume (1). By Lemma 16.2 we see that every object of $\mathcal{F}(A_1 \times_A A_2)$ is of the form $x_1 \times_x x_2$. Moreover

$$\mathrm{Mor}_{A_1 \times_A A_2}(x_1 \times_x x_2, y_1 \times_y y_2) = \mathrm{Mor}_{A_1}(x_1, y_1) \times_{\mathrm{Mor}_A(x,y)} \mathrm{Mor}_{A_2}(x_2, y_2).$$

Hence we see that $\mathcal{F}(A_1 \times_A A_2)$ is a 2-fibre product of $\mathcal{F}(A_1)$ with $\mathcal{F}(A_2)$ over $\mathcal{F}(A)$ by Categories, Remark 31.5. In other words, we see that (2) holds.

The implication (2) $\Rightarrow$ (3) is immediate.

Assume (3). Let $q_1 : A_1 \to A$ and $q_2 : A_2 \to A$ be given with q_2 a small extension. We will use the description of the 2-fibre product $\mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)$ from Categories, Remark 31.5. Hence let $y \in \mathcal{F}(A_1 \times_A A_2)$ correspond to $(x_1, x_2, x, a_1 : x_1 \to x, a_2 : x_2 \to x)$. Let z be an object of $\mathcal{F}$ lying over C. Then

$$\begin{aligned}
\mathrm{Mor}_{\mathcal{F}}(z, y) &= \{(f, \alpha) \mid f : C \to A_1 \times_A A_2, \alpha : f_* z \to y\} \\
&= \{(f_1, f_2, \alpha_1, \alpha_2) \mid f_i : C \to A_i,\ \alpha_i : f_{i,*} z \to x_i, \\
&\qquad q_1 \circ f_1 = q_2 \circ f_2,\ q_{1,*}\alpha_1 = q_{2,*}\alpha_2\} \\
&= \mathrm{Mor}_{\mathcal{F}}(z, x_1) \times_{\mathrm{Mor}_{\mathcal{F}}(z,x)} \mathrm{Mor}_{\mathcal{F}}(z, x_2)
\end{aligned}$$

whence y is a fibre product of x_1 and x_2 over x. Thus we see that $\mathcal{F}$ satisfies (RS) in case $A_2 \to A$ is a small extension. Hence (RS) holds by Lemma 16.3. $\square$

06J6 **Remark 16.5.** When $\mathcal{F}$ is cofibered in sets, condition (RS) is exactly condition (H4) from Schlessinger's paper [Sch68, Theorem 2.11]. Namely, for a functor $F : \mathcal{C}_\Lambda \to \textit{Sets}$, condition (RS) states: If $A_1 \to A$ and $A_2 \to A$ are maps in $\mathcal{C}_\Lambda$ with $A_2 \to A$ surjective, then the induced map $F(A_1 \times_A A_2) \to F(A_1) \times_{F(A)} F(A_2)$ is bijective.

06J7 **Lemma 16.6.** *Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. The condition (RS) for $\mathcal{F}$ implies both (S1) and (S2) for $\mathcal{F}$.*

Proof. Using the reformulation of Lemma 16.4 and the explanation of (S1) following Definition 10.1 it is immediate that (RS) implies (S1). This proves the first part of (S2). The second part of (S2) follows because Lemma 16.2 tells us that $y = x_1 \times_{d, x_0, e} x_2 = y'$ if y, y' are as in the second part of the definition of (S2) in Definition 10.1. (In fact the morphism $y \to y'$ is compatible with both a, a' and c, c'!) $\square$

The following lemma is the analogue of Lemma 10.5. Recall that if $\mathcal{F}$ is a category cofibred in groupoids over $\mathcal{C}_\Lambda$ and x is an object of $\mathcal{F}$ lying over A, then we denote $\mathrm{Aut}_A(x) = \mathrm{Mor}_A(x, x) = \mathrm{Mor}_{\mathcal{F}(A)}(x, x)$. If $x' \to x$ is a morphism of $\mathcal{F}$ lying over $A' \to A$ then there is a well defined map of groups $\mathrm{Aut}_{A'}(x') \to \mathrm{Aut}_A(x)$.

06J8 **Lemma 16.7.** *Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$ satisfying (RS). The following conditions are equivalent:*

(1) *$\overline{\mathcal{F}}$ satisfies (RS).*

(2) *Let $f_1 : A_1 \to A$ and $f_2 : A_2 \to A$ be ring maps in $\mathcal{C}_\Lambda$ with f_2 surjective. The induced map of sets of isomorphism classes*

$$\overline{\mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)} \to \overline{\mathcal{F}}(A_1) \times_{\overline{\mathcal{F}}(A)} \overline{\mathcal{F}}(A_2)$$

is injective.

(3) *For every morphism $x' \to x$ in $\mathcal{F}$ lying over a surjective ring map $A' \to A$, the map $Aut_{A'}(x') \to Aut_A(x)$ is surjective.*

(4) *For every morphism $x' \to x$ in $\mathcal{F}$ lying over a small extension $A' \to A$, the map $Aut_{A'}(x') \to Aut_A(x)$ is surjective.*

Proof. We prove that (1) is equivalent to (2) and (2) is equivalent to (3). The equivalence of (3) and (4) follows from Lemma 3.3.

Let $f_1 : A_1 \to A$ and $f_2 : A_2 \to A$ be ring maps in $\mathcal{C}_\Lambda$ with f_2 surjective. By Remark 16.5 we see $\overline{\mathcal{F}}$ satisfies (RS) if and only if the map

$$\overline{\mathcal{F}}(A_1 \times_A A_2) \to \overline{\mathcal{F}}(A_1) \times_{\overline{\mathcal{F}}(A)} \overline{\mathcal{F}}(A_2)$$

is bijective for any such f_1, f_2. This map is at least surjective since that is the condition of (S1) and $\overline{\mathcal{F}}$ satisfies (S1) by Lemmas 16.6 and 10.5. Moreover, this map factors as

$$\overline{\mathcal{F}}(A_1 \times_A A_2) \longrightarrow \overline{\mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)} \longrightarrow \overline{\mathcal{F}}(A_1) \times_{\overline{\mathcal{F}}(A)} \overline{\mathcal{F}}(A_2),$$

where the first map is a bijection since

$$\mathcal{F}(A_1 \times_A A_2) \longrightarrow \mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)$$

is an equivalence by (RS) for $\mathcal{F}$. Hence (1) is equivalent to (2).

Assume (2) holds. Let $x' \to x$ be a morphism in $\mathcal{F}$ lying over a surjective ring map $f : A' \to A$. Let $a \in \mathrm{Aut}_A(x)$. The objects

$$(x', x', a : x \to x),\ (x', x', \mathrm{id} : x \to x)$$

of $\mathcal{F}(A') \times_{\mathcal{F}(A)} \mathcal{F}(A')$ have the same image in $\overline{\mathcal{F}}(A') \times_{\overline{\mathcal{F}}(A)} \overline{\mathcal{F}}(A')$. By (2) there exists maps $b_1, b_2 : x' \to x'$ such that

$$\begin{tikzcd} x \arrow[r, "a"'] \arrow[d, "f_*b_1"'] & x \arrow[d, "f_*b_2"] \\ x \arrow[r, "\mathrm{id}"] & x \end{tikzcd}$$

commutes. Hence $b_2^{-1} \circ b_1 \in \mathrm{Aut}_{A'}(x')$ has image $a \in \mathrm{Aut}_A(x)$. Hence (3) holds.

Assume (3) holds. Suppose

$$(x_1, x_2, a : (f_1)_*x_1 \to (f_2)_*x_2),\ (x_1', x_2', a' : (f_1)_*x_1' \to (f_2)_*x_2')$$

are objects of $\mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)$ with the same image in $\overline{\mathcal{F}}(A_1) \times_{\overline{\mathcal{F}}(A)} \overline{\mathcal{F}}(A_2)$. Then there are morphisms $b_1 : x_1 \to x_1'$ in $\mathcal{F}(A_1)$ and $b_2 : x_2 \to x_2'$ in $\mathcal{F}(A_2)$. By (3) we can modify b_2 by an automorphism of x_2 over A_2 so that the diagram

$$\begin{tikzcd} (f_1)_*x_1 \arrow[r, "a"'] \arrow[d, "(f_1)_*b_1"'] & (f_2)_*x_2 \arrow[d, "(f_2)_*b_2"] \\ (f_1)_*x_1' \arrow[r, "a'"] & (f_2)_*x_2'. \end{tikzcd}$$

commutes. This proves $(x_1, x_2, a) \cong (x_1', x_2', a')$ in $\overline{\mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)}$. Hence (2) holds. ☐

Finally we define the notion of a deformation category.

06J9 **Definition 16.8.** A *deformation category* is a predeformation category $\mathcal{F}$ satisfying (RS). A morphism of deformation categories is a morphism of categories over $\mathcal{C}_\Lambda$.

06JA **Remark 16.9.** We say that a functor $F : \mathcal{C}_\Lambda \to \textit{Sets}$ is a *deformation functor* if the associated cofibered set is a deformation category, i.e. if $F(k)$ is a one element set and F satisfies (RS). If $\mathcal{F}$ is a deformation category, then $\overline{\mathcal{F}}$ is a predeformation functor but not necessarily a deformation functor, as Lemma 16.7 shows.

06JB **Example 16.10.** A prorepresentable functor F is a deformation functor. Namely, suppose $R \in \mathrm{Ob}(\widehat{\mathcal{C}}_\Lambda)$ and $F(A) = \mathrm{Mor}_{\widehat{\mathcal{C}}_\Lambda}(R, A)$. There is a unique morphism $R \to k$, so $F(k)$ is a one element set. Since

$$\mathrm{Hom}_\Lambda(R, A_1 \times_A A_2) = \mathrm{Hom}_\Lambda(R, A_1) \times_{\mathrm{Hom}_\Lambda(R,A)} \mathrm{Hom}_\Lambda(R, A_2)$$

the same is true for maps in $\widehat{\mathcal{C}}_\Lambda$ and we see that F has (RS).

The following is one of our typical remarks on passing from a category cofibered in groupoids to the predeformation category at a point over k: it says that this process preserves (RS).

06JC **Lemma 16.11.** *Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. Let $x_0 \in \mathrm{Ob}(\mathcal{F}(k))$. Let $\mathcal{F}_{x_0}$ be the category cofibred in groupoids over $\mathcal{C}_\Lambda$ constructed in Remark 6.4. If $\mathcal{F}$ satisfies (RS), then so does $\mathcal{F}_{x_0}$. In particular, $\mathcal{F}_{x_0}$ is a deformation category.*

Proof. Any diagram as in Definition 16.1 in $\mathcal{F}_{x_0}$ gives rise to a diagram in $\mathcal{F}$ and the output of (RS) for this diagram in $\mathcal{F}$ can be viewed as an output for $\mathcal{F}_{x_0}$ as well. ☐

The following lemma is the analogue of the fact that 2-fibre products of algebraic stacks are algebraic stacks.

06L4 **Lemma 16.12.** *Let*

$$\begin{tikzcd} \mathcal{H} \times_{\mathcal{F}} \mathcal{G} \arrow[r] \arrow[d] & \mathcal{G} \arrow[d, "g"] \\ \mathcal{H} \arrow[r, "f"] & \mathcal{F} \end{tikzcd}$$

be 2-fibre product of categories cofibered in groupoids over $\mathcal{C}_\Lambda$. If $\mathcal{F}, \mathcal{G}, \mathcal{H}$ all satisfy (RS), then $\mathcal{H} \times_{\mathcal{F}} \mathcal{G}$ satisfies (RS).

Proof. If A is an object of $\mathcal{C}_\Lambda$, then an object of the fiber category of $\mathcal{H} \times_{\mathcal{F}} \mathcal{G}$ over A is a triple (u, v, a) where $u \in \mathcal{H}(A)$, $v \in \mathcal{G}(A)$, and $a : f(u) \to g(v)$ is a morphism in $\mathcal{F}(A)$. Consider a diagram in $\mathcal{H} \times_{\mathcal{F}} \mathcal{G}$

$$\begin{tikzcd} & (u_2, v_2, a_2) \arrow[d] & & A_2 \arrow[d] \\ (u_1, v_1, a_1) \arrow[r] & (u, v, a) & \text{lying over} \quad A_1 \arrow[r] & A \end{tikzcd}$$

in $\mathcal{C}_\Lambda$ with $A_2 \to A$ surjective. Since $\mathcal{H}$ and $\mathcal{G}$ satisfy (RS), there are fiber products $u_1 \times_u u_2$ and $v_1 \times_v v_2$ lying over $A_1 \times_A A_2$. Since $\mathcal{F}$ satisfies (RS), Lemma 16.2 shows

$$\begin{tikzcd} f(u_1 \times_u u_2) \arrow[r] \arrow[d] & f(u_2) \arrow[d] \\ f(u_1) \arrow[r] & f(u) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} g(v_1 \times_v v_2) \arrow[r] \arrow[d] & g(v_2) \arrow[d] \\ g(v_1) \arrow[r] & g(v) \end{tikzcd}$$

are both fiber squares in $\mathcal{F}$. Thus we can view $a_1 \times_a a_2$ as a morphism from $f(u_1 \times_u u_2)$ to $g(v_1 \times_v v_2)$ over $A_1 \times_A A_2$. It follows that

$$\begin{tikzcd} (u_1 \times_u u_2, v_1 \times_v v_2, a_1 \times_a a_2) \arrow[r] \arrow[d] & (u_2, v_2, a_2) \arrow[d] \\ (u_1, v_1, a_1) \arrow[r] & (u, v, a) \end{tikzcd}$$

is a fiber square in $\mathcal{H} \times_{\mathcal{F}} \mathcal{G}$ as desired. ☐

17. Lifts of objects

06JD The content of this section is that the tangent space has a principal homogeneous action on the set of lifts along a small extension in the case of a deformation category.

06JE **Definition 17.1.** Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. Let $f : A' \to A$ be a map in $\mathcal{C}_\Lambda$. Let $x \in \mathcal{F}(A)$. The category *Lift*(x, f) of lifts of x along f is the category with the following objects and morphisms.

1. Objects: A *lift of x along f* is a morphism $x' \to x$ lying over f.
2. Morphisms: A *morphism of lifts* from $a_1 : x'_1 \to x$ to $a_2 : x'_2 \to x$ is a morphism $b : x'_1 \to x'_2$ in $\mathcal{F}(A')$ such that $a_2 = a_1 \circ b$.

The set $\text{Lift}(x, f)$ of lifts of x along f is the set of isomorphism classes of *Lift*(x, f).

06JF **Remark 17.2.** When the map $f : A' \to A$ is clear from the context, we may write *Lift*(x, A') and $\text{Lift}(x, A')$ in place of *Lift*(x, f) and $\text{Lift}(x, f)$.

06JG **Remark 17.3.** Let $\mathcal{F}$ be a category cofibred in groupoids over $\mathcal{C}_\Lambda$. Let $x_0 \in \text{Ob}(\mathcal{F}(k))$. Let V be a finite dimensional vector space. Then $\text{Lift}(x_0, k[V])$ is the set of isomorphism classes of $\mathcal{F}_{x_0}(k[V])$ where $\mathcal{F}_{x_0}$ is the predeformation category of objects in $\mathcal{F}$ lying over x_0, see Remark 6.4. Hence if $\mathcal{F}$ satisfies (S2), then so does $\mathcal{F}_{x_0}$ (see Lemma 10.6) and by Lemma 12.2 we see that

$$\text{Lift}(x_0, k[V]) = T\mathcal{F}_{x_0} \otimes_k V$$

as k-vector spaces.

06JH **Remark 17.4.** Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$ satisfying (RS). Let

$$\begin{tikzcd} A_1 \times_A A_2 \arrow[r] \arrow[d] & A_2 \arrow[d] \\ A_1 \arrow[r] & A \end{tikzcd}$$

be a fibre square in $\mathcal{C}_\Lambda$ such that either $A_1 \to A$ or $A_2 \to A$ is surjective. Let $x \in \text{Ob}(\mathcal{F}(A))$. Given lifts $x_1 \to x$ and $x_2 \to x$ of x to A_1 and A_2, we get by (RS) a lift $x_1 \times_x x_2 \to x$ of x to $A_1 \times_A A_2$. Conversely, by Lemma 16.2 any lift of x to $A_1 \times_A A_2$ is of this form. Hence a bijection

$$\mathrm{Lift}(x, A_1) \times \mathrm{Lift}(x, A_2) \longrightarrow \mathrm{Lift}(x, A_1 \times_A A_2).$$

Similarly, if $x_1 \to x$ is a fixed lifting of x to A_1, then there is a bijection

$$\mathrm{Lift}(x_1, A_1 \times_A A_2) \longrightarrow \mathrm{Lift}(x, A_2).$$

Now let

$$\begin{tikzcd} A_1' \times_A A_2 \arrow[r] \arrow[d] & A_1 \times_A A_2 \arrow[r] \arrow[d] & A_2 \arrow[d] \\ A_1' \arrow[r] & A_1 \arrow[r] & A \end{tikzcd}$$

be a composition of fibre squares in $\mathcal{C}_\Lambda$ with both $A_1' \to A_1$ and $A_1 \to A$ surjective. Let $x_1 \to x$ be a morphism lying over $A_1 \to A$. Then by the above we have bijections

$$\begin{aligned} \mathrm{Lift}(x_1, A_1' \times_A A_2) &= \mathrm{Lift}(x_1, A_1') \times \mathrm{Lift}(x_1, A_1 \times_A A_2) \\ &= \mathrm{Lift}(x_1, A_1') \times \mathrm{Lift}(x, A_2). \end{aligned}$$

06JI **Lemma 17.5.** *Let $\mathcal{F}$ be a deformation category. Let $A' \to A$ be a surjective ring map in $\mathcal{C}_\Lambda$ whose kernel I is annihilated by $\mathfrak{m}_{A'}$. Let $x \in \mathrm{Ob}(\mathcal{F}(A))$. If Lift$(x, A')$ is nonempty, then there is a free and transitive action of $T\mathcal{F} \otimes_k I$ on Lift(x, A').*

Proof. Consider the ring map $g : A' \times_A A' \to k[I]$ defined by the rule $g(a_1, a_2) = \overline{a_1} \oplus a_2 - a_1$ (compare with Lemma 10.8). There is an isomorphism

$$A' \times_A A' \xrightarrow{\sim} A' \times_k k[I]$$

given by $(a_1, a_2) \mapsto (a_1, g(a_1, a_2))$. This isomorphism commutes with the projections to A' on the first factor, and hence with the projections of $A' \times_A A'$ and $A' \times_k k[I]$ to A. Thus there is a bijection

06TA
$$\mathrm{Lift}(x, A' \times_A A') \longrightarrow \mathrm{Lift}(x, A' \times_k k[I]) \tag{17.5.1}$$

By Remark 17.4 there is a bijection

06TB
$$\mathrm{Lift}(x, A') \times \mathrm{Lift}(x, A') \longrightarrow \mathrm{Lift}(x, A' \times_A A') \tag{17.5.2}$$

There is a commutative diagram

$$\begin{tikzcd} A' \times_k k[I] \arrow[r] \arrow[d] & A \times_k k[I] \arrow[r] \arrow[d] & k[I] \arrow[d] \\ A' \arrow[r] & A \arrow[r] & k. \end{tikzcd}$$

Thus if we choose a pushforward $x \to x_0$ of x along $A \to k$, we obtain by the end of Remark 17.4 a bijection

06TC
$$\mathrm{Lift}(x, A' \times_k k[I]) \longrightarrow \mathrm{Lift}(x, A') \times \mathrm{Lift}(x_0, k[I]) \tag{17.5.3}$$

Composing (17.5.2), (17.5.1), and (17.5.3) we get a bijection

$$\Phi : \mathrm{Lift}(x, A') \times \mathrm{Lift}(x, A') \longrightarrow \mathrm{Lift}(x, A') \times \mathrm{Lift}(x_0, k[I]).$$

This bijection commutes with the projections on the first factors. By Remark 17.3 we see that $\mathrm{Lift}(x_0, k[I]) = T\mathcal{F} \otimes_k I$. If pr_2 is the second projection of $\mathrm{Lift}(x, A') \times \mathrm{Lift}(x, A')$, then we get a map

$$a = \mathrm{pr}_2 \circ \Phi^{-1} : \mathrm{Lift}(x, A') \times (T\mathcal{F} \otimes_k I) \longrightarrow \mathrm{Lift}(x, A').$$

Unwinding all the above we see that $a(x' \to x, \theta)$ is the unique lift $x'' \to x$ such that $g_*(x', x'') = \theta$ in $\mathrm{Lift}(x_0, k[I]) = T\mathcal{F} \otimes_k I$. To see this is an action of $T\mathcal{F} \otimes_k I$ on $\mathrm{Lift}(x, A')$ we have to show the following: if x', x'', x''' are lifts of x and $g_*(x', x'') = \theta$, $g_*(x'', x''') = \theta'$, then $g_*(x', x''') = \theta + \theta'$. This follows from the commutative diagram

$$\begin{tikzcd} A' \times_A A' \times_A A' \arrow[r, "(a_1,a_2,a_3) \mapsto (g(a_1,a_2), g(a_2,a_3))"'] \arrow[dr, "(a_1,a_2,a_3) \mapsto g(a_1,a_3)"'] & k[I] \times_k k[I] = k[I \times I] \arrow[d, "+"] \\ & k[I] \end{tikzcd}$$

The action is free and transitive because Φ is bijective. $\square$

06JJ **Remark 17.6.** The action of Lemma 17.5 is functorial. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of deformation categories. Let $A' \to A$ be a surjective ring map whose kernel I is annihilated by $\mathfrak{m}_{A'}$. Let $x \in \mathrm{Ob}(\mathcal{F}(A))$. In this situation φ induces the vertical arrows in the following commutative diagram

$$\begin{tikzcd} \mathrm{Lift}(x, A') \times (T\mathcal{F} \otimes_k I) \arrow[r] \arrow[d, "(\varphi, d\varphi \otimes \mathrm{id}_I)"'] & \mathrm{Lift}(x, A') \arrow[d, "\varphi"] \\ \mathrm{Lift}(\varphi(x), A') \times (T\mathcal{G} \otimes_k I) \arrow[r] & \mathrm{Lift}(\varphi(x), A') \end{tikzcd}$$

The commutativity follows as each of the maps (17.5.2), (17.5.1), and (17.5.3) of the proof of Lemma 17.5 gives rise to a similar commutative diagram.

18. Schlessinger's theorem on prorepresentable functors

06JK We deduce Schlessinger's theorem characterizing prorepresentable functors on $\mathcal{C}_\Lambda$.

06JL **Lemma 18.1.** *Let $F, G : \mathcal{C}_\Lambda \to Sets$ be deformation functors. Let $\varphi : F \to G$ be a smooth morphism which induces an isomorphism $d\varphi : TF \to TG$ of tangent spaces. Then φ is an isomorphism.*

Proof. We prove $F(A) \to G(A)$ is a bijection for all $A \in \mathrm{Ob}(\mathcal{C}_\Lambda)$ by induction on $\mathrm{length}_A(A)$. For $A = k$ the statement follows from the assumption that F and G are deformation functors. Suppose that the statement holds for rings of length less than n and let A' be a ring of length n. Choose a small extension $f : A' \to A$. We have a commutative diagram

$$\begin{tikzcd} F(A') \arrow[r] \arrow[d, "F(f)"'] & G(A') \arrow[d, "G(f)"] \\ F(A) \arrow[r, "\sim"] & G(A) \end{tikzcd}$$

where the map $F(A) \to G(A)$ is a bijection. By smoothness of $F \to G$, $F(A') \to G(A')$ is surjective (Lemma 8.8). Thus we can check bijectivity by checking it on fibers $F(f)^{-1}(x) \to G(f)^{-1}(\varphi(x))$ for $x \in F(A)$ such that $F(f)^{-1}(x)$ is nonempty.

These fibers are precisely $\mathrm{Lift}(x, A')$ and $\mathrm{Lift}(\varphi(x), A')$ and by assumption we have an isomorphism $d\varphi \otimes \mathrm{id} : TF \otimes_k \mathrm{Ker}(f) \to TG \otimes_k \mathrm{Ker}(f)$. Thus, by Lemma 17.5 and Remark 17.6, for $x \in F(A)$ such that $F(f)^{-1}(x)$ is nonempty the map $F(f)^{-1}(x) \to G(f)^{-1}(\varphi(x))$ is a map of sets commuting with free transitive actions by $TF \otimes_k \mathrm{Ker}(f)$. Hence it is bijective. $\square$

Note that in case $k' \subset k$ is separable condition (c) in the theorem below is empty.

06JM **Theorem 18.2.** *Let $F : \mathcal{C}_\Lambda \to Sets$ be a functor. Then F is prorepresentable if and only if (a) F is a deformation functor, (b) $\dim_k TF$ is finite, and (c) $\gamma : Der_\Lambda(k, k) \to TF$ is injective.*

Proof. Assume F is prorepresentable by $R \in \widehat{\mathcal{C}}_\Lambda$. We see F is a deformation functor by Example 16.10. We see $\dim_k TF$ is finite by Example 11.11. Finally, $\mathrm{Der}_\Lambda(k, k) \to TF$ is identified with $\mathrm{Der}_\Lambda(k, k) \to \mathrm{Der}_\Lambda(R, k)$ by Example 11.14 which is injective because $R \to k$ is surjective.

Conversely, assume (a), (b), and (c) hold. By Lemma 16.6 we see that (S1) and (S2) hold. Hence by Theorem 15.5 there exists a minimal versal formal object ξ of F such that (15.0.2) holds. Say ξ lies over R. The map

$$d\underline{\xi} : \mathrm{Der}_\Lambda(R, k) \to T\mathcal{F}$$

is bijective on $\mathrm{Der}_\Lambda(k, k)$-orbits. Since the action of $\mathrm{Der}_\Lambda(k, k)$ on the left hand side is free by (c) and Lemma 12.6 we see that the map is bijective. Thus we see that $\underline{\xi}$ is an isomorphism by Lemma 18.1. $\square$

19. Infinitesimal automorphisms

06JN Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. Given a morphism $x' \to x$ in $\mathcal{F}$ lying over $A' \to A$, there is an induced homomorphism

$$\mathrm{Aut}_{A'}(x') \to \mathrm{Aut}_A(x).$$

Lemma 16.7 says that the cokernel of this homomorphism determines whether condition (RS) on $\mathcal{F}$ passes to $\overline{\mathcal{F}}$. In this section we study the kernel of this homomorphism. We will see that it also gives a measure of how far $\mathcal{F}$ is from $\overline{\mathcal{F}}$.

06JP **Definition 19.1.** Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. Let $x' \to x$ be a morphism in $\mathcal{F}$ lying over $A' \to A$. The kernel

$$\mathrm{Inf}(x'/x) = \mathrm{Ker}(\mathrm{Aut}_{A'}(x') \to \mathrm{Aut}_A(x))$$

is the *group of infinitesimal automorphisms of x' over x.*

06JQ **Definition 19.2.** Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. Let $x_0 \in \mathrm{Ob}(\mathcal{F}(k))$. Assume a choice of pushforward $x_0 \to x'_0$ of x_0 along the map $k \to k[\epsilon], a \mapsto a$ has been made. Then there is a unique map $x'_0 \to x_0$ such that $x_0 \to x'_0 \to x_0$ is the identity on x_0. Then

$$\mathrm{Inf}_{x_0}(\mathcal{F}) = \mathrm{Inf}(x'_0/x_0)$$

is the *group of infinitesimal automorphisms of x_0*

06JR **Remark 19.3.** Up to canonical isomorphism $\mathrm{Inf}_{x_0}(\mathcal{F})$ does not depend on the choice of pushforward $x_0 \to x'_0$ because any two pushforwards are canonically isomorphic. Moreover, if $y_0 \in \mathcal{F}(k)$ and $x_0 \cong y_0$ in $\mathcal{F}(k)$, then $\mathrm{Inf}_{x_0}(\mathcal{F}) \cong \mathrm{Inf}_{y_0}(\mathcal{F})$ where the isomorphism depends (only) on the choice of an isomorphism $x_0 \to y_0$. In particular, $\mathrm{Aut}_k(x_0)$ acts on $\mathrm{Inf}_{x_0}(\mathcal{F})$.

06JS **Remark 19.4.** Assume $\mathcal{F}$ is a predeformation category. Then

(1) for $x_0 \in \mathrm{Ob}(\mathcal{F}(k))$ the automorphism group $\mathrm{Aut}_k(x_0)$ is trivial and hence $\mathrm{Inf}_{x_0}(\mathcal{F}) = \mathrm{Aut}_{k[\epsilon]}(x'_0)$, and
(2) for $x_0, y_0 \in \mathrm{Ob}(\mathcal{F}(k))$ there is a unique isomorphism $x_0 \to y_0$ and hence a canonical identification $\mathrm{Inf}_{x_0}(\mathcal{F}) = \mathrm{Inf}_{y_0}(\mathcal{F})$.

Since $\mathcal{F}(k)$ is nonempty, choosing $x_0 \in \mathrm{Ob}(\mathcal{F}(k))$ and setting

$$\mathrm{Inf}(\mathcal{F}) = \mathrm{Inf}_{x_0}(\mathcal{F})$$

we get a well defined *group of infinitesimal automorphisms of* $\mathcal{F}$. With this notation we have $\mathrm{Inf}(\mathcal{F}_{x_0}) = \mathrm{Inf}_{x_0}(\mathcal{F})$. Please compare with the equality $T\mathcal{F}_{x_0} = T_{x_0}\mathcal{F}$ in Remark 12.5.

We will see that $\mathrm{Inf}_{x_0}(\mathcal{F})$ has a natural k-vector space structure when $\mathcal{F}$ satisfies (RS). At the same time, we will see that if $\mathcal{F}$ satisfies (RS), then the infinitesimal automorphisms $\mathrm{Inf}(x'/x)$ of a morphism $x' \to x$ lying over a small extension are governed by $\mathrm{Inf}_{x_0}(\mathcal{F})$, where x_0 is a pushforward of x to $\mathcal{F}(k)$. In order to do this, we introduce the automorphism functor for any object $x \in \mathrm{Ob}(\mathcal{F})$ as follows.

06JT **Definition 19.5.** Let $p : \mathcal{F} \to \mathcal{C}$ be a category cofibered in groupoids over an arbitrary base category $\mathcal{C}$. Assume a choice of pushforwards has been made. Let $x \in \mathrm{Ob}(\mathcal{F})$ and let $U = p(x)$. Let $U/\mathcal{C}$ denote the category of objects under U. The *automorphism functor of* x is the functor $Aut(x) : U/\mathcal{C} \to Sets$ sending an object $f : U \to V$ to $\mathrm{Aut}_V(f_*x)$ and sending a morphism

$$\begin{tikzcd} V' \arrow[rr] & & V \\ & U \arrow[ul, "f'"] \arrow[ur, "f"'] & \end{tikzcd}$$

to the homomorphism $\mathrm{Aut}_{V'}(f'_*x) \to \mathrm{Aut}_V(f_*x)$ coming from the unique morphism $f'_*x \to f_*x$ lying over $V' \to V$ and compatible with $x \to f'_*x$ and $x \to f_*x$.

We will be concerned with the automorphism functors of objects in a category cofibered in groupoids $\mathcal{F}$ over $\mathcal{C}_\Lambda$. If $A \in \mathrm{Ob}(\mathcal{C}_\Lambda)$, then the category $A/\mathcal{C}_\Lambda$ is nothing but the category $\mathcal{C}_A$, i.e. the category defined in Section 3 where we take $\Lambda = A$ and $k = A/\mathfrak{m}_A$. Hence the automorphism functor of an object $x \in \mathrm{Ob}(\mathcal{F}(A))$ is a functor $Aut(x) : \mathcal{C}_A \to Sets$.

The following lemma could be deduced from Lemma 16.12 by thinking about the "inertia" of a category cofibred in groupoids, see for example Stacks, Section 7 and Categories, Section 34. However, it is easier to see it directly.

06JU **Lemma 19.6.** *Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$ satisfying (RS). Let $x \in \mathrm{Ob}(\mathcal{F}(A))$. Then $Aut(x) : \mathcal{C}_A \to Sets$ satisfies (RS).*

Proof. It follows that $Aut(x)$ satisfies (RS) from the fully faithfulness of the functor $\mathcal{F}(A_1 \times_A A_2) \to \mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)$ in Lemma 16.4. □

06JV **Lemma 19.7.** *Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$ satisfying (RS). Let $x \in \mathrm{Ob}(\mathcal{F}(A))$. Let x_0 be a pushforward of x to $\mathcal{F}(k)$.*

(1) *$T_{id_{x_0}} Aut(x)$ has a natural k-vector space structure such that addition agrees with composition in $T_{id_{x_0}} Aut(x)$. In particular, composition in $T_{id_{x_0}} Aut(x)$ is commutative.*

(2) *There is a canonical isomorphism* $T_{id_{x_0}} Aut(x) \to T_{id_{x_0}} Aut(x_0)$ *of* k*-vector spaces.*

Proof. We apply Remark 6.4 to the functor $Aut(x) : \mathcal{C}_A \to Sets$ and the element $\mathrm{id}_{x_0} \in Aut(x)(k)$ to get a predeformation functor $F = Aut(x)_{\mathrm{id}_{x_0}}$. By Lemmas 19.6 and 16.11 F is a deformation functor. By definition $T_{\mathrm{id}_{x_0}} Aut(x) = TF = F(k[\epsilon])$ which has a natural k-vector space structure specified by Lemma 11.8.

Addition is defined as the composition

$$F(k[\epsilon]) \times F(k[\epsilon]) \longrightarrow F(k[\epsilon] \times_k k[\epsilon]) \longrightarrow F(k[\epsilon])$$

where the first map is the inverse of the bijection guaranteed by (RS) and the second is induced by the k-algebra map $k[\epsilon] \times_k k[\epsilon] \to k[\epsilon]$ which maps $(\epsilon, 0)$ and $(0, \epsilon)$ to ϵ. If $A \to B$ is a ring map in $\mathcal{C}_\Lambda$, then $F(A) \to F(B)$ is a homomorphism where $F(A) = Aut(x)_{\mathrm{id}_{x_0}}(A)$ and $F(B) = Aut(x)_{\mathrm{id}_{x_0}}(B)$ are groups under composition. We conclude that $+ : F(k[\epsilon]) \times F(k[\epsilon]) \to F(k[\epsilon])$ is a homomorphism where $F(k[\epsilon])$ is regarded as a group under composition. With $\mathrm{id} \in F(k[\epsilon])$ the unit element we see that $+(v, \mathrm{id}) = +(\mathrm{id}, v) = v$ for any $v \in F(k[\epsilon])$ because (id, v) is the pushforward of v along the ring map $k[\epsilon] \to k[\epsilon] \times_k k[\epsilon]$ with $\epsilon \mapsto (\epsilon, 0)$. In general, given a group G with multiplication $\circ$ and $+ : G \times G \to G$ is a homomorphism such that $+(g, 1) = +(1, g) = g$, where 1 is the identity of G, then $+ = \circ$. This shows addition in the k-vector space structure on $F(k[\epsilon])$ agrees with composition.

Finally, (2) is a matter of unwinding the definitions. Namely $T_{\mathrm{id}_{x_0}} Aut(x)$ is the set of automorphisms α of the pushforward of x along $A \to k \to k[\epsilon]$ which are trivial modulo ϵ. On the other hand $T_{\mathrm{id}_{x_0}} Aut(x_0)$ is the set of automorphisms of the pushforward of x_0 along $k \to k[\epsilon]$ which are trivial modulo ϵ. Since x_0 is the pushforward of x along $A \to k$ the result is clear. □

06JW **Remark 19.8.** We point out some basic relationships between infinitesimal automorphism groups, liftings, and tangent spaces to automorphism functors. Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. Let $x' \to x$ be a morphism lying over a ring map $A' \to A$. Then from the definitions we have an equality

$$\mathrm{Inf}(x'/x) = \mathrm{Lift}(\mathrm{id}_x, A')$$

where the liftings are of id_x as an object of $Aut(x')$. If $x_0 \in \mathrm{Ob}(\mathcal{F}(k))$ and x'_0 is the pushforward to $\mathcal{F}(k[\epsilon])$, then applying this to $x'_0 \to x_0$ we get

$$\mathrm{Inf}_{x_0}(\mathcal{F}) = \mathrm{Lift}(\mathrm{id}_{x_0}, k[\epsilon]) = T_{\mathrm{id}_{x_0}} Aut(x_0),$$

the last equality following directly from the definitions.

06JX **Lemma 19.9.** *Let* $\mathcal{F}$ *be a category cofibered in groupoids over* $\mathcal{C}_\Lambda$ *satisfying (RS). Let* $x_0 \in \mathrm{Ob}(\mathcal{F}(k))$*. Then* $Inf_{x_0}(\mathcal{F})$ *is equal as a set to* $T_{id_{x_0}} Aut(x_0)$*, and so has a natural* k*-vector space structure such that addition agrees with composition of automorphisms.*

Proof. The equality of sets is as in the end of Remark 19.8 and the statement about the vector space structure follows from Lemma 19.7. □

07W6 **Lemma 19.10.** *Let* $\varphi : \mathcal{F} \to \mathcal{G}$ *be a morphism of categories cofibred in groupoids over* $\mathcal{C}_\Lambda$ *satisfying (RS). Let* $x_0 \in \mathrm{Ob}(\mathcal{F}(k))$*. Then* φ *induces a* k*-linear map* $Inf_{x_0}(\mathcal{F}) \to Inf_{\varphi(x_0)}(\mathcal{G})$*.*

Proof. It is clear that φ induces a morphism from $Aut(x_0) \to Aut(\varphi(x_0))$ which maps the identity to the identity. Hence this follows from the result for tangent spaces, see Lemma 12.4. $\square$

06JY **Lemma 19.11.** *Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$ satisfying (RS). Let $x' \to x$ be a morphism lying over a surjective ring map $A' \to A$ with kernel I annihilated by $\mathfrak{m}_{A'}$. Let x_0 be a pushforward of x to $\mathcal{F}(k)$. Then Inf(x'/x) has a free and transitive action by $T_{id_{x_0}} Aut(x') \otimes_k I = Inf_{x_0}(\mathcal{F}) \otimes_k I$.*

Proof. This is just the analogue of Lemma 17.5 in the setting of automorphism sheaves. To be precise, we apply Remark 6.4 to the functor $Aut(x') : \mathcal{C}_{A'} \to Sets$ and the element $\mathrm{id}_{x_0} \in Aut(x)(k)$ to get a predeformation functor $F = Aut(x')_{\mathrm{id}_{x_0}}$. By Lemmas 19.6 and 16.11 F is a deformation functor. Hence Lemma 17.5 gives a free and transitive action of $TF \otimes_k I$ on $\mathrm{Lift}(\mathrm{id}_x, A')$, because as $\mathrm{Lift}(\mathrm{id}_x, A')$ is a group it is always nonempty. Note that we have equalities of vector spaces

$$TF = T_{\mathrm{id}_{x_0}} Aut(x') \otimes_k I = \mathrm{Inf}_{x_0}(\mathcal{F}) \otimes_k I$$

by Lemma 19.7. The equality $\mathrm{Inf}(x'/x) = \mathrm{Lift}(\mathrm{id}_x, A')$ of Remark 19.8 finishes the proof. $\square$

06JZ **Lemma 19.12.** *Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$ satisfying (RS). Let $x' \to x$ be a morphism in $\mathcal{F}$ lying over a surjective ring map. Let x_0 be a pushforward of x to $\mathcal{F}(k)$. If $Inf_{x_0}(\mathcal{F}) = 0$ then Inf$(x'/x) = 0$.*

Proof. Follows from Lemmas 3.3 and 19.11. $\square$

06K0 **Lemma 19.13.** *Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$ satisfying (RS). Let $x_0 \in \mathrm{Ob}(\mathcal{F}(k))$. Then $Inf_{x_0}(\mathcal{F}) = 0$ if and only if the natural morphism $\mathcal{F}_{x_0} \to \overline{\mathcal{F}_{x_0}}$ of categories cofibered in groupoids is an equivalence.*

Proof. The morphism $\mathcal{F}_{x_0} \to \overline{\mathcal{F}_{x_0}}$ is an equivalence if and only if $\mathcal{F}_{x_0}$ is fibered in setoids, cf. Categories, Section 39 (a setoid is by definition a groupoid in which the only automorphism of any object is the identity). We prove that $\mathrm{Inf}_{x_0}(\mathcal{F}) = 0$ if and only if this condition holds for $\mathcal{F}_{x_0}$. Obviously if $\mathcal{F}_{x_0}$ is fibered in setoids then $\mathrm{Inf}_{x_0}(\mathcal{F}) = 0$. Conversely assume $\mathrm{Inf}_{x_0}(\mathcal{F}) = 0$. Let A be an object of $\mathcal{C}_\Lambda$. Then by Lemma 19.12, $\mathrm{Inf}(x/x_0) = 0$ for any object $x \to x_0$ of $\mathcal{F}_{x_0}(A)$. Since by definition $\mathrm{Inf}(x/x_0)$ equals the group of automorphisms of $x \to x_0$ in $\mathcal{F}_{x_0}(A)$, this proves $\mathcal{F}_{x_0}(A)$ is a setoid. $\square$

20. Applications

0DYM We collect some results on deformation categories we will use later.

06L5 **Lemma 20.1.** *Let $f : \mathcal{H} \to \mathcal{F}$ and $g : \mathcal{G} \to \mathcal{F}$ be 1-morphisms of deformation categories. Then*

1. *$\mathcal{W} = \mathcal{H} \times_\mathcal{F} \mathcal{G}$ is a deformation category, and*
2. *we have a 6-term exact sequence of vector spaces*

$$0 \to Inf(\mathcal{W}) \to Inf(\mathcal{H}) \oplus Inf(\mathcal{G}) \to Inf(\mathcal{F}) \to T\mathcal{W} \to T\mathcal{H} \oplus T\mathcal{G} \to T\mathcal{F}$$

Proof. Part (1) follows from Lemma 16.12 and the fact that $\mathcal{W}(k)$ is the fibre product of two setoids with a unique isomorphism class over a setoid with a unique isomorphism class.

Part (2). Let $w_0 \in \mathrm{Ob}(\mathcal{W}(k))$ and let x_0, y_0, z_0 be the image of w_0 in $\mathcal{F}, \mathcal{H}, \mathcal{G}$. Then $\mathrm{Inf}(\mathcal{W}) = \mathrm{Inf}_{w_0}(\mathcal{W})$ and simlarly for $\mathcal{H}$, $\mathcal{G}$, and $\mathcal{F}$, see Remark 19.4. We apply Lemmas 12.4 and 19.10 to get all the linear maps except for the "boundary map" $\delta : \mathrm{Inf}_{x_0}(\mathcal{F}) \to T\mathcal{W}$. We will insert suitable signs later.

Construction of δ. Choose a pushforward $w_0 \to w_0'$ along $k \to k[\epsilon]$. Denote x_0', y_0', z_0' the images of w_0' in $\mathcal{F}, \mathcal{H}, \mathcal{G}$. In particular we obtain isomorphisms $b' : f(y_0') \to x_0'$ and $c' : x_0' \to g(z_0')$. Denote $b : f(y_0) \to x_0$ and $c : x_0 \to g(z_0)$ the pushforwards along $k[\epsilon] \to k$. Observe that this means $w_0' = (k[\epsilon], y_0', z_0', c' \circ b')$ and $w_0 = (k, y_0, z_0, c \circ b)$ in terms of the explicit form of the fibre product of categories, see Remarks 5.2 (13). Given $\alpha : x_0' \to x_0'$ we set $\delta(\alpha) = (k[\epsilon], y_0', z_0', c' \circ \alpha \circ b')$ which is indeed an object of $\mathcal{W}$ over $k[\epsilon]$ and comes with a morphism $(k[\epsilon], y_0', z_0', c' \circ \alpha \circ b') \to w_0$ over $k[\epsilon] \to k$ as α pushes forward to the identity over k. More generally, for any k-vector space V we can define a map

$$\mathrm{Lift}(\mathrm{id}_{x_0}, k[V]) \longrightarrow \mathrm{Lift}(w_0, k[V])$$

using exactly the same formulae. This construction is functorial in the vector space V (details omitted). Hence δ is k-linear by an application of Lemma 11.5.

Having constructed these maps it is straightforward to show the sequence is exact. Injectivity of the first map comes from the fact that $f \times g : \mathcal{W} \to \mathcal{H} \times \mathcal{G}$ is faithful. If $(\beta, \gamma) \in \mathrm{Inf}_{y_0}(\mathcal{H}) \oplus \mathrm{Inf}_{z_0}(\mathcal{G})$ map to the same element of $\mathrm{Inf}_{x_0}(\mathcal{F})$ then (β, γ) defines an automorphism of $w_0' = (k[\epsilon], y_0', z_0', c' \circ b')$ whence exactness at the second spot. If α as above gives the trivial deformation $(k[\epsilon], y_0', z_0', c' \circ \alpha \circ b')$ of w_0, then the isomorphism $w_0' = (k[\epsilon], y_0', z_0', c' \circ b') \to (k[\epsilon], y_0', z_0', c' \circ \alpha \circ b')$ produces a pair (β, γ) which is a preimage of α. If $w = (k[\epsilon], y, z, \phi)$ is a deformation of w_0 such that $y_0' \cong y$ and $z \cong z_0'$ then the map

$$f(y_0') \to f(y) \xrightarrow{\phi} g(z) \to g(z_0')$$

is an α which maps to w under δ. Finally, if y and z are deformations of y_0 and z_0 and there exists an isomorphism $\phi : f(y) \to g(z)$ of deformations of $f(y_0) = x_0 = g(z_0)$ then we get a preimage $w = (k[\epsilon], y, z, \phi)$ of (x, y) in $T\mathcal{W}$. This finishes the proof. □

0DYN **Lemma 20.2.** *Let $\mathcal{H}_1 \to \mathcal{G}$, $\mathcal{H}_2 \to \mathcal{G}$, and $\mathcal{G} \to \mathcal{F}$ be maps of categories cofibred in groupoids over $\mathcal{C}_\Lambda$. Assume*

1. *$\mathcal{F}$ and $\mathcal{G}$ are deformation categories,*
2. *$T\mathcal{G} \to T\mathcal{F}$ is injective, and*
3. *$\mathrm{Inf}(\mathcal{G}) \to \mathrm{Inf}(\mathcal{F})$ is surjective.*

Then $\mathcal{H}_1 \times_\mathcal{G} \mathcal{H}_2 \to \mathcal{H}_1 \times_\mathcal{F} \mathcal{H}_2$ is smooth.

Proof. Denote $p_i : \mathcal{H}_i \to \mathcal{G}$ and $q : \mathcal{G} \to \mathcal{F}$ be the given maps. Let $A' \to A$ be a small extension in $\mathcal{C}_\Lambda$. An object of $\mathcal{H}_1 \times_\mathcal{F} \mathcal{H}_2$ over A' is a triple (x_1', x_2', a') where x_i' is an object of $\mathcal{H}_i$ over A' and $a' : q(p_1(x_1')) \to q(p_2(x_2'))$ is a morphism of the fibre category of $\mathcal{F}$ over A'. By pushforward along $A' \to A$ we get (x_1, x_2, a). Lifting this to an object of $\mathcal{H}_1 \times_\mathcal{G} \mathcal{H}_2$ over A means finding a morphism $b : p_1(x_1) \to p_2(x_2)$ over A with $q(b) = a$. Thus we have to show that we can lift b to a morphism $b' : p_1(x_1') \to p_2(x_2')$ whose image under q is a'.

Observe that we can think of

$$p_1(x_1') \to p_1(x_1) \xrightarrow{b} p_2(x_2) \quad \text{and} \quad p_2(x_2') \to p_2(x_2)$$

as two objects of $\mathit{Lift}(p_2(x_2), A' \to A)$. The functor q sends these objects to the two objects

$$q(p_1(x_1')) \to q(p_1(x_1)) \xrightarrow{b} q(p_2(x_2)) \quad \text{and} \quad q(p_2(x_2')) \to q(p_2(x_2))$$

of $\mathit{Lift}(q(p_2(x_2)), A' \to A)$ which are isomorphic using the map $a' : q(p_1(x_1')) \to q(p_2(x_2'))$. On the other hand, the functor

$$q : \mathit{Lift}(p_2(x_2), A' \to A) \to \mathit{Lift}(q(p_2(x_2)), A' \to A)$$

defines a injection on isomorphism classes by Lemma 17.5 and our assumption on tangent spaces. Thus we see that there is a morphism $b' : p_1(x_1') \to p_2(x_2')$ whose pushforward to A is b. However, we may need to adjust our choice of b' to achieve $q(b') = a'$. For this it suffices to see that $q : \text{Inf}(p_2(x_2')/p_2(x_2)) \to \text{Inf}(q(p_2(x_2'))/q(p_2(x_2)))$ is surjective. This follows from our assumption on infinitesimal automorphisms and Lemma 19.11. □

0DYP **Lemma 20.3.** *Let $f : \mathcal{F} \to \mathcal{G}$ be a map of deformation categories. Let $x_0 \in \text{Ob}(\mathcal{F}(k))$ with image $y_0 \in \text{Ob}(\mathcal{G}(k))$. If*

1. *the map $T\mathcal{F} \to T\mathcal{G}$ is surjective, and*
2. *for every small extension $A' \to A$ in $\mathcal{C}_\Lambda$ and $x \in \mathcal{F}(A)$ with image $y \in \mathcal{G}(A)$ if there is a lift of y to A', then there is a lift of x to A',*

then $\mathcal{F} \to \mathcal{G}$ is smooth (and vice versa).

Proof. Let $A' \to A$ be a small extension. Let $x \in \mathcal{F}(A)$. Let $y' \to f(x)$ be a morphism in $\mathcal{G}$ over $A' \to A$. Consider the functor $\text{Lift}(A', x) \to \text{Lift}(A', f(x))$ induced by f. We have to show that there exists an object $x' \to x$ of $\text{Lift}(A', x)$ mapping to $y' \to f(x)$, see Lemma 8.2. By condition (2) we know that $\text{Lift}(A', x)$ is not the empty category. By condition (2) and Lemma 17.5 we conlude that the map on isomorphism classes is surjective as desired. □

0E3R **Lemma 20.4.** *Let $\mathcal{F} \to \mathcal{G} \to \mathcal{H}$ be maps of categories cofibred in groupoids over $\mathcal{C}_\Lambda$. If*

1. *$\mathcal{F}$, $\mathcal{G}$ are deformation categories*
2. *the map $T\mathcal{F} \to T\mathcal{G}$ is surjective, and*
3. *$\mathcal{F} \to \mathcal{H}$ is smooth.*

Then $\mathcal{F} \to \mathcal{G}$ is smooth.

Proof. Let $A' \to A$ be a small extension in $\mathcal{C}_\Lambda$ and let $x \in \mathcal{F}(A)$ with image $y \in \mathcal{G}(A)$. Assume there is a lift $y' \in \mathcal{G}(A')$. According to Lemma 20.3 all we have to do is check that x has a lift too. Take the image $z' \in \mathcal{H}(A')$ of y'. Since $\mathcal{F} \to \mathcal{H}$ is smooth, there is an $x' \in \mathcal{F}(A')$ mapping to both $x \in \mathcal{F}(A)$ and $z' \in \mathcal{H}(A')$, see Definition 8.1. This finishes the proof. □

21. Groupoids in functors on an arbitrary category

06K2 We begin with generalities on groupoids in functors on an arbitrary category. In the next section we will pass to the category $\mathcal{C}_\Lambda$. For clarity we shall sometimes refer to an ordinary groupoid, i.e., a category whose morphisms are all isomorphisms, as a groupoid category.

06K3 **Definition 21.1.** Let $\mathcal{C}$ be a category. The *category of groupoids in functors on $\mathcal{C}$* is the category with the following objects and morphisms.

(1) Objects: A *groupoid in functors on* $\mathcal{C}$ is a quintuple (U, R, s, t, c) where $U, R : \mathcal{C} \to \textit{Sets}$ are functors and $s, t : R \to U$ and $c : R \times_{s,U,t} R \to R$ are morphisms with the following property: For any object T of $\mathcal{C}$, the quintuple
$$(U(T), R(T), s, t, c)$$
is a groupoid category.

(2) Morphisms: A *morphism* $(U, R, s, t, c) \to (U', R', s', t', c')$ *of groupoids in functors on* $\mathcal{C}$ consists of morphisms $U \to U'$ and $R \to R'$ with the following property: For any object T of $\mathcal{C}$, the induced maps $U(T) \to U'(T)$ and $R(T) \to R'(T)$ define a functor between groupoid categories
$$(U(T), R(T), s, t, c) \to (U'(T), R'(T), s', t', c').$$

06K4 **Remark 21.2.** A groupoid in functors on $\mathcal{C}$ amounts to the data of a functor $\mathcal{C} \to \textit{Groupoids}$, and a morphism of groupoids in functors on $\mathcal{C}$ amounts to a morphism of the corresponding functors $\mathcal{C} \to \textit{Groupoids}$ (where *Groupoids* is regarded as a 1-category). However, for our purposes it is more convenient to use the terminology of groupoids in functors. In fact, thinking of a groupoid in functors as the corresponding functor $\mathcal{C} \to \textit{Groupoids}$, or equivalently as the category cofibered in groupoids associated to that functor, can lead to confusion (Remark 23.2).

06K5 **Remark 21.3.** Let (U, R, s, t, c) be a groupoid in functors on a category $\mathcal{C}$. There are unique morphisms $e : U \to R$ and $i : R \to R$ such that for every object T of $\mathcal{C}$, $e : U(T) \to R(T)$ sends $x \in U(T)$ to the identity morphism on x and $i : R(T) \to R(T)$ sends $a \in U(T)$ to the inverse of a in the groupoid category $(U(T), R(T), s, t, c)$. We will sometimes refer to s, t, c, e, and i as "source", "target", "composition", "identity", and "inverse".

06K6 **Definition 21.4.** Let $\mathcal{C}$ be a category. A groupoid in functors on $\mathcal{C}$ is *representable* if it is isomorphic to one of the form $(\underline{U}, \underline{R}, s, t, c)$ where U and R are objects of $\mathcal{C}$ and the pushout $R \amalg_{s,U,t} R$ exists.

06K7 **Remark 21.5.** Hence a representable groupoid in functors on $\mathcal{C}$ is given by objects U and R of $\mathcal{C}$ and morphisms $s, t : U \to R$ and $c : R \to R \amalg_{s,U,t} R$ such that $(\underline{U}, \underline{R}, s, t, c)$ satisfies the condition of Definition 21.1. The reason for requiring the existence of the pushout $R \amalg_{s,U,t} R$ is so that the composition morphism c is defined at the level of morphisms in $\mathcal{C}$. This requirement will always be satisfied below when we consider representable groupoids in functors on $\widehat{\mathcal{C}}_\Lambda$, since by Lemma 4.3 the category $\widehat{\mathcal{C}}_\Lambda$ admits pushouts.

06K8 **Remark 21.6.** We will say "*let* $(\underline{U}, \underline{R}, s, t, c)$ *be a groupoid in functors on* $\mathcal{C}$" to mean that we have a representable groupoid in functors. Thus this means that U and R are objects of $\mathcal{C}$, there are morphisms $s, t : U \to R$, the pushout $R \amalg_{s,U,t} R$ exists, there is a morphism $c : R \to R \amalg_{s,U,t} R$, and $(\underline{U}, \underline{R}, s, t, c)$ is a groupoid in functors on $\mathcal{C}$.

We introduce notation for restriction of groupoids in functors. This will be relevant below in situations where we restrict from $\widehat{\mathcal{C}}_\Lambda$ to $\mathcal{C}_\Lambda$.

06K9 **Definition 21.7.** Let (U, R, s, t, c) be a groupoid in functors on a category $\mathcal{C}$. Let $\mathcal{C}'$ be a subcategory of $\mathcal{C}$. The *restriction* $(U, R, s, t, c)|_{\mathcal{C}'}$ *of* (U, R, s, t, c) *to* $\mathcal{C}'$ is the groupoid in functors on $\mathcal{C}'$ given by $(U|_{\mathcal{C}'}, R|_{\mathcal{C}'}, s|_{\mathcal{C}'}, t|_{\mathcal{C}'}, c|_{\mathcal{C}'})$.

06KA **Remark 21.8.** In the situation of Definition 21.7, we often denote $s|_{\mathcal{C}'}, t|_{\mathcal{C}'}, c|_{\mathcal{C}'}$ simply by s, t, c.

06KB **Definition 21.9.** Let (U, R, s, t, c) be a groupoid in functors on a category $\mathcal{C}$.

(1) The assignment $T \mapsto (U(T), R(T), s, t, c)$ determines a functor $\mathcal{C} \to$ *Groupoids*. The *quotient category cofibered in groupoids* $[U/R] \to \mathcal{C}$ is the category cofibered in groupoids over $\mathcal{C}$ associated to this functor (as in Remarks 5.2 (9)).
(2) The *quotient morphism* $U \to [U/R]$ is the morphism of categories cofibered in groupoids over $\mathcal{C}$ defined by the rules
   (a) $x \in U(T)$ maps to the object $(T, x) \in \mathrm{Ob}([U/R](T))$, and
   (b) $x \in U(T)$ and $f : T \to T'$ give rise to the morphism $(f, \mathrm{id}_{U(f)(x)}) : (T, x) \to (T, U(f)(x))$ lying over $f : T \to T'$.

22. Groupoids in functors on the base category

06KC In this section we discuss groupoids in functors on $\mathcal{C}_\Lambda$. Our eventual goal is to show that prorepresentable groupoids in functors on $\mathcal{C}_\Lambda$ serve as "presentations" for well-behaved deformation categories in the same way that smooth groupoids in algebraic spaces serve as presentations for algebraic stacks, cf. Algebraic Stacks, Section 16.

06KD **Definition 22.1.** A groupoid in functors on $\mathcal{C}_\Lambda$ is *prorepresentable* if it is isomorphic to $(\underline{R_0}, \underline{R_1}, s, t, c)|_{\mathcal{C}_\Lambda}$ for some representable groupoid in functors $(\underline{R_0}, \underline{R_1}, s, t, c)$ on the category $\widehat{\mathcal{C}}_\Lambda$.

Let (U, R, s, t, c) be a groupoid in functors on $\mathcal{C}_\Lambda$. Taking completions, we get a quintuple $(\widehat{U}, \widehat{R}, \widehat{s}, \widehat{t}, \widehat{c})$. By Remark 7.10 completion as a functor on $\mathrm{CofSet}(\mathcal{C}_\Lambda)$ is a right adjoint, so it commutes with limits. In particular, there is a canonical isomorphism

$$R \widehat{\times_{s,U,t}} R \longrightarrow \widehat{R} \times_{\widehat{s},\widehat{U},\widehat{t}} \widehat{R},$$

so $\widehat{c}$ can be regarded as a functor $\widehat{R} \times_{\widehat{s},\widehat{U},\widehat{t}} \widehat{R} \to \widehat{R}$. Then $(\widehat{U}, \widehat{R}, \widehat{s}, \widehat{t}, \widehat{c})$ is a groupoid in functors on $\widehat{\mathcal{C}}_\Lambda$, with identity and inverse morphisms being the completions of those of (U, R, s, t, c).

06KE **Definition 22.2.** Let (U, R, s, t, c) be a groupoid in functors on $\mathcal{C}_\Lambda$. The *completion* $(U, R, s, t, c)^\wedge$ *of* (U, R, s, t, c) is the groupoid in functors $(\widehat{U}, \widehat{R}, \widehat{s}, \widehat{t}, \widehat{c})$ on $\widehat{\mathcal{C}}_\Lambda$ described above.

06KF **Remark 22.3.** Let (U, R, s, t, c) be a groupoid in functors on $\mathcal{C}_\Lambda$. Then there is a canonical isomorphism $(U, R, s, t, c)^\wedge|_{\mathcal{C}_\Lambda} \cong (U, R, s, t, c)$, see Remark 7.7. On the other hand, let (U, R, s, t, c) be a groupoid in functors on $\widehat{\mathcal{C}}_\Lambda$ such that $U, R : \widehat{\mathcal{C}}_\Lambda \to$ *Sets* both commute with limits, e.g. if U, R are representable. Then there is a canonical isomorphism $((U, R, s, t, c)|_{\mathcal{C}_\Lambda})^\wedge \cong (U, R, s, t, c)$. This follows from Remark 7.11.

06KG **Lemma 22.4.** *Let (U, R, s, t, c) be a groupoid in functors on $\mathcal{C}_\Lambda$.*

(1) *(U, R, s, t, c) is prorepresentable if and only if its completion is representable as a groupoid in functors on $\widehat{\mathcal{C}}_\Lambda$.*
(2) *(U, R, s, t, c) is prorepresentable if and only if U and R are prorepresentable.*

Proof. Part (1) follows from Remark 22.3. For (2), the "only if" direction is clear from the definition of a prorepresentable groupoid in functors. Conversely, assume U and R are prorepresentable, say $U \cong \underline{R_0}|_{\mathcal{C}_\Lambda}$ and $R \cong \underline{R_1}|_{\mathcal{C}_\Lambda}$ for objects R_0 and R_1 of $\widehat{\mathcal{C}}_\Lambda$. Since $\underline{R_0} \cong \widehat{\underline{R_0}|_{\mathcal{C}_\Lambda}}$ and $\underline{R_1} \cong \widehat{\underline{R_1}|_{\mathcal{C}_\Lambda}}$ by Remark 7.11 we see that the completion $(U, R, s, t, c)^\wedge$ is a groupoid in functors of the form $(\underline{R_0}, \underline{R_1}, \widehat{s}, \widehat{t}, \widehat{c})$. By Lemma 4.3 the pushout $\underline{R_1} \times_{\widehat{s}, \underline{R_1}, \widehat{t}} \underline{R_1}$ exists. Hence $(\underline{R_0}, \underline{R_1}, \widehat{s}, \widehat{t}, \widehat{c})$ is a representable groupoid in functors on $\widehat{\mathcal{C}}_\Lambda$. Finally, the restriction $(\underline{R_0}, \underline{R_1}, s, t, c)|_{\mathcal{C}_\Lambda}$ gives back (U, R, s, t, c) by Remark 22.3 hence (U, R, s, t, c) is prorepresentable by definition. □

23. Smooth groupoids in functors on the base category

06KH The notion of smoothness for groupoids in functors on $\mathcal{C}_\Lambda$ is defined as follows.

06KI **Definition 23.1.** Let (U, R, s, t, c) be a groupoid in functors on $\mathcal{C}_\Lambda$. We say (U, R, s, t, c) is *smooth* if $s, t : R \to U$ are smooth.

06KJ **Remark 23.2.** We note that this terminology is potentially confusing: if (U, R, s, t, c) is a smooth groupoid in functors, then the quotient $[U/R]$ need not be a smooth category cofibred in groupoids as defined in Definition 9.1. However smoothness of (U, R, s, t, c) does imply (in fact is equivalent to) smoothness of the quotient morphism $U \to [U/R]$ as we shall see in Lemma 23.4.

06KK **Remark 23.3.** Let $(\underline{R_0}, \underline{R_1}, s, t, c)|_{\mathcal{C}_\Lambda}$ be a prorepresentable groupoid in functors on $\mathcal{C}_\Lambda$. Then $(\underline{R_0}, \underline{R_1}, s, t, c)|_{\mathcal{C}_\Lambda}$ is smooth if and only if R_1 is a power series over R_0 via both s and t. This follows from Lemma 8.6.

06KL **Lemma 23.4.** *Let (U, R, s, t, c) be a groupoid in functors on $\mathcal{C}_\Lambda$. The following are equivalent:*

(1) *The groupoid in functors (U, R, s, t, c) is smooth.*
(2) *The morphism $s : R \to U$ is smooth.*
(3) *The morphism $t : R \to U$ is smooth.*
(4) *The quotient morphism $U \to [U/R]$ is smooth.*

Proof. Statement (2) is equivalent to (3) since the inverse $i : R \to R$ of (U, R, s, t, c) is an isomorphism and $t = s \circ i$. By definition (1) is equivalent to (2) and (3) together, hence it is equivalent to either of them individually.

Finally we prove (2) is equivalent to (4). Unwinding the definitions:

(2) Smoothness of $s : R \to U$ amounts to the following condition: If $f : B \to A$ is a surjective ring map in $\mathcal{C}_\Lambda$, $a \in R(A)$, and $y \in U(B)$ such that $s(a) = U(f)(y)$, then there exists $a' \in R(B)$ such that $R(f)(a') = a$ and $s(a') = y$.
(4) Smoothness of $U \to [U/R]$ amounts to the following condition: If $f : B \to A$ is a surjective ring map in $\mathcal{C}_\Lambda$ and $(f, a) : (B, y) \to (A, x)$ is a morphism of $[U/R]$, then there exists $x' \in U(B)$ and $b \in R(B)$ with $s(b) = x'$, $t(b) = y$ such that $c(a, R(f)(b)) = e(x)$. Here $e : U \to R$ denotes the identity and the notation (f, a) is as in Remarks 5.2 (9); in particular $a \in R(A)$ with $s(a) = U(f)(y)$ and $t(a) = x$.

If (4) holds and f, a, y as in (2) are given, let $x = t(a)$ so that we have a morphism $(f, a) : (B, y) \to (A, x)$. Then (4) produces x' and b, and $a' = i(b)$ satisfies the requirements of (2). Conversely, assume (2) holds and let $(f, a) : (B, y) \to (A, x)$ as in (4) be given. Then (2) produces $a' \in R(B)$, and $x' = t(a')$ and $b = i(a')$ satisfy the requirements of (4). □

24. Deformation categories as quotients of groupoids in functors

06KS We discuss conditions on a groupoid in functors on $\mathcal{C}_\Lambda$ which guarantee that the quotient is a deformation category, and we calculate the tangent and infinitesimal automorphism spaces of such a quotient.

06KT **Lemma 24.1.** *Let (U, R, s, t, c) be a smooth groupoid in functors on $\mathcal{C}_\Lambda$. Assume U and R satisfy (RS). Then $[U/R]$ satisfies (RS).*

Proof. Let

$$\begin{tikzcd} & (A_2, x_2) \arrow[d, "(f_2, a_2)"] \\ (A_1, x_1) \arrow[r, "(f_1, a_1)"] & (A, x) \end{tikzcd}$$

be a diagram in $[U/R]$ such that $f_2 : A_2 \to A$ is surjective. The notation is as in Remarks 5.2 (9). Hence $f_1 : A_1 \to A, f_2 : A_2 \to A$ are maps in $\mathcal{C}_\Lambda$, $x \in U(A)$, $x_1 \in U(A_1)$, $x_2 \in U(A_2)$, and $a_1, a_2 \in R(A)$ with $s(a_1) = U(f_1)(x_1)$, $t(a_1) = x$ and $s(a_2) = U(f_2)(x_2)$, $t(a_2) = x$. We construct a fiber product lying over $A_1 \times_A A_2$ for this diagram in $[U/R]$ as follows.

Let $a = c(i(a_1), a_2)$, where $i : R \to R$ is the inverse morphism. Then $a \in R(A)$, $x_2 \in U(A_2)$ and $s(a) = U(f_2)(x_2)$. Hence an element $(a, x_2) \in R(A) \times_{s, U(A), U(f_2)} U(A_2)$. By smoothness of $s : R \to U$ there is an element $\widetilde{a} \in R(A_2)$ with $R(f_2)(\widetilde{a}) = a$ and $s(\widetilde{a}) = x_2$. In particular $U(f_2)(t(\widetilde{a})) = t(a) = U(f_1)(x_1)$. Thus x_1 and $t(\widetilde{a})$ define an element

$$(x_1, t(\widetilde{a})) \in U(A_1) \times_{U(A)} U(A_2).$$

By the assumption that U satisfies (RS), we have an identification $U(A_1) \times_{U(A)} U(A_2) = U(A_1 \times_A A_2)$. Let us denote $x_1 \times t(\widetilde{a}) \in U(A_1 \times_A A_2)$ the element corresponding to $(x_1, t(\widetilde{a})) \in U(A_1) \times_{U(A)} U(A_2)$. Let p_1, p_2 be the projections of $A_1 \times_A A_2$. We claim

$$\begin{tikzcd} (A_1 \times_A A_2, x_1 \times t(\widetilde{a})) \arrow[r, "(p_2, i(\widetilde{a}))"'] \arrow[d, "(p_1, e(x_1))"'] & (A_2, x_2) \arrow[d, "(f_2, a_2)"] \\ (A_1, x_1) \arrow[r, "(f_1, a_1)"] & (A, x) \end{tikzcd}$$

is a fiber square in $[U/R]$. (Note $e : U \to R$ denotes the identity.)

The diagram is commutative because $c(a_2, R(f_2)(i(\widetilde{a}))) = c(a_2, i(a)) = a_1$. To check it is a fiber square, let

$$\begin{tikzcd} (B, z) \arrow[r, "(g_2, b_2)"'] \arrow[d, "(g_1, b_1)"'] & (A_2, x_2) \arrow[d, "(f_2, a_2)"] \\ (A_1, x_1) \arrow[r, "(f_1, a_1)"] & (A, x) \end{tikzcd}$$

be a commutative diagram in $[U/R]$. We will show there is a unique morphism $(g, b) : (B, z) \to (A_1 \times_A A_2, x_1 \times t(\widetilde{a}))$ compatible with the morphisms to (A_1, x_1) and (A_2, x_2). We must take $g = (g_1, g_2) : B \to A_1 \times_A A_2$. Since by assumption R satisfies (RS), we have an identification $R(A_1 \times_A A_2) = R(A_1) \times_{R(A)} R(A_2)$. Hence we can write $b = (b'_1, b'_2)$ for some $b'_1 \in R(A_1)$, $b'_2 \in R(A_2)$ which agree in $R(A)$. Then $((g_1, g_2), (b'_1, b'_2)) : (B, z) \to (A_1 \times_A A_2, x_1 \times t(\widetilde{a}))$ will commute with the projections if and only if $b'_1 = b_1$ and $b'_2 = c(\widetilde{a}, b_2)$ proving unicity and existence. $\square$

06KU **Lemma 24.2.** *Let (U, R, s, t, c) be a smooth groupoid in functors on $\mathcal{C}_\Lambda$. Assume U and R are deformation functors. Then:*

1. *The quotient $[U/R]$ is a deformation category.*
2. *The tangent space of $[U/R]$ is*
$$T[U/R] = \text{Coker}(ds - dt : TR \to TU).$$
3. *The space of infinitesimal automorphisms of $[U/R]$ is*
$$\textit{Inf}([U/R]) = \text{Ker}(ds \oplus dt : TR \to TU \oplus TU).$$

Proof. Since U and R are deformation functors $[U/R]$ is a predeformation category. Since (RS) holds for deformation functors by definition we see that (RS) holds for [U/R] by Lemma 24.1. Hence $[U/R]$ is a deformation category. Statements (2) and (3) follow directly from the definitions. $\square$

25. Presentations of categories cofibered in groupoids

06KW A presentation is defined as follows.

06KX **Definition 25.1.** Let $\mathcal{F}$ be a category cofibered in groupoids over a category $\mathcal{C}$. Let (U, R, s, t, c) be a groupoid in functors on $\mathcal{C}$. A *presentation of $\mathcal{F}$ by (U, R, s, t, c)* is an equivalence $\varphi : [U/R] \to \mathcal{F}$ of categories cofibered in groupoids over $\mathcal{C}$.

The following two general lemmas will be used to get presentations.

06KY **Lemma 25.2.** *Let $\mathcal{F}$ be category cofibered in groupoids over a category $\mathcal{C}$. Let $U : \mathcal{C} \to Sets$ be a functor. Let $f : U \to \mathcal{F}$ be a morphism of categories cofibered in groupoids over $\mathcal{C}$. Define R, s, t, c as follows:*

1. *$R : \mathcal{C} \to Sets$ is the functor $U \times_{f,\mathcal{F},f} U$.*
2. *$t, s : R \to U$ are the first and second projections, respectively.*
3. *$c : R \times_{s,U,t} R \to R$ is the morphism given by projection onto the first and last factors of $U \times_{f,\mathcal{F},f} U \times_{f,\mathcal{F},f} U$ under the canonical isomorphism $R \times_{s,U,t} R \to U \times_{f,\mathcal{F},f} U \times_{f,\mathcal{F},f} U$.*

Then (U, R, s, t, c) is a groupoid in functors on $\mathcal{C}$.

Proof. Omitted. $\square$

06KZ **Lemma 25.3.** *Let $\mathcal{F}$ be category cofibered in groupoids over a category $\mathcal{C}$. Let $U : \mathcal{C} \to Sets$ be a functor. Let $f : U \to \mathcal{F}$ be a morphism of categories cofibered in groupoids over $\mathcal{C}$. Let (U, R, s, t, c) be the groupoid in functors on $\mathcal{C}$ constructed from $f : U \to \mathcal{F}$ in Lemma 25.2. Then there is a natural morphism $[f] : [U/R] \to \mathcal{F}$ such that:*

1. *$[f] : [U/R] \to \mathcal{F}$ is fully faithful.*
2. *$[f] : [U/R] \to \mathcal{F}$ is an equivalence if and only if $f : U \to \mathcal{F}$ is essentially surjective.*

Proof. Omitted. $\square$

26. Presentations of deformation categories

06L0 According to the next lemma, a smooth morphism from a predeformation functor to a predeformation category $\mathcal{F}$ gives rise to a presentation of $\mathcal{F}$ by a smooth groupoid in functors.

06L1 **Lemma 26.1.** *Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. Let $U : \mathcal{C}_\Lambda \to Sets$ be a functor. Let $f : U \to \mathcal{F}$ be a smooth morphism of categories cofibered in groupoids. Then:*

1. *If (U, R, s, t, c) is the groupoid in functors on $\mathcal{C}_\Lambda$ constructed from $f : U \to \mathcal{F}$ in Lemma 25.2, then (U, R, s, t, c) is smooth.*
2. *If $f : U(k) \to \mathcal{F}(k)$ is essentially surjective, then the morphism $[f] : [U/R] \to \mathcal{F}$ of Lemma 25.3 is an equivalence.*

Proof. From the construction of Lemma 25.2 we have a commutative diagram

$$\begin{tikzcd} R = U \times_{f,\mathcal{F},f} U \arrow[r, "s"'] \arrow[d, "t"'] & U \arrow[d, "f"] \\ U \arrow[r, "f"] & \mathcal{F} \end{tikzcd}$$

where t, s are the first and second projections. So t, s are smooth by Lemma 8.7. Hence (1) holds.

If the assumption of (2) holds, then by Lemma 8.8 the morphism $f : U \to \mathcal{F}$ is essentially surjective. Hence by Lemma 25.3 the morphism $[f] : [U/R] \to \mathcal{F}$ is an equivalence. $\square$

06L6 **Lemma 26.2.** *Let $\mathcal{F}$ be a deformation category. Let $U : \mathcal{C}_\Lambda \to Sets$ be a deformation functor. Let $f : U \to \mathcal{F}$ be a morphism of categories cofibered in groupoids. Then $U \times_{f,\mathcal{F},f} U$ is a deformation functor with tangent space fitting into an exact sequence of k-vector spaces*

$$0 \to Inf(\mathcal{F}) \to T(U \times_{f,\mathcal{F},f} U) \to TU \oplus TU$$

Proof. Follows from Lemma 20.1 and the fact that $\mathrm{Inf}(U) = (0)$. $\square$

06L7 **Lemma 26.3.** *Let $\mathcal{F}$ be a deformation category. Let $U : \mathcal{C}_\Lambda \to Sets$ be a prorepresentable functor. Let $f : U \to \mathcal{F}$ be a morphism of categories cofibered in groupoids. Let (U, R, s, t, c) be the groupoid in functors on $\mathcal{C}_\Lambda$ constructed from $f : U \to \mathcal{F}$ in Lemma 25.2. If $\dim_k Inf(\mathcal{F}) < \infty$, then (U, R, s, t, c) is prorepresentable.*

Proof. Note that U is a deformation functor by Example 16.10. By Lemma 26.2 we see that $R = U \times_{f,\mathcal{F},f} U$ is a deformation functor whose tangent space $TR = T(U \times_{f,\mathcal{F},f} U)$ sits in an exact sequence $0 \to \mathrm{Inf}(\mathcal{F}) \to TR \to TU \oplus TU$. Since we have assumed the first space has finite dimension and since TU has finite dimension by Example 11.11 we see that $\dim TR < \infty$. The map $\gamma : \mathrm{Der}_\Lambda(k, k) \to TR$ see (12.6.1) is injective because its composition with $TR \to TU$ is injective by Theorem 18.2 for the prorepresentable functor U. Thus R is prorepresentable by Theorem 18.2. It follows from Lemma 22.4 that (U, R, s, t, c) is prorepresentable. $\square$

06L8 **Theorem 26.4.** *Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. Then $\mathcal{F}$ admits a presentation by a smooth prorepresentable groupoid in functors on $\mathcal{C}_\Lambda$ if and only if the following conditions hold:*

1. *$\mathcal{F}$ is a deformation category.*

(2) $\dim_k T\mathcal{F}$ *is finite.*
(3) $\dim_k Inf(\mathcal{F})$ *is finite.*

Proof. Recall that a prorepresentable functor is a deformation functor, see Example 16.10. Thus if $\mathcal{F}$ is equivalent to a smooth prorepresentable groupoid in functors, then conditions (1), (2), and (3) follow from Lemma 24.2 (1), (2), and (3).

Conversely, assume conditions (1), (2), and (3) hold. Condition (1) implies that (S1) and (S2) are satisfied, see Lemma 16.6. By Lemma 13.4 there exists a versal formal object ξ. Setting $U = \underline{R}|_{\mathcal{C}_\Lambda}$ the associated map $\underline{\xi} : U \to \mathcal{F}$ is smooth (this is the definition of a versal formal object). Let (U, R, s, t, c) be the groupoid in functors constructed in Lemma 25.2 from the map $\underline{\xi}$. By Lemma 26.1 we see that (U, R, s, t, c) is a smooth groupoid in functors and that $[U/R] \to \mathcal{F}$ is an equivalence. By Lemma 26.3 we see that (U, R, s, t, c) is prorepresentable. Hence $[U/R] \to \mathcal{F}$ is the desired presentation of $\mathcal{F}$. $\square$

27. Remarks regarding minimality

06TD The main theorem of this chapter is Theorem 26.4 above. It describes completely those categories cofibred in groupoids over $\mathcal{C}_\Lambda$ which have a presentation by a smooth prorepresentable groupoid in functors. In this section we briefly discuss how the minimality discussed in Sections 14 and 15 can be used to obtain a “minimal” smooth prorepresentable presentation.

06KM **Definition 27.1.** Let (U, R, s, t, c) be a smooth prorepresentable groupoid in functors on $\mathcal{C}_\Lambda$.

(1) We say (U, R, s, t, c) is *normalized* if the groupoid $(U(k[\epsilon]), R(k[\epsilon]), s, t, c)$ is totally disconnected, i.e., there are no morphisms between distinct objects.
(2) We say (U, R, s, t, c) is *minimal* if the $U \to [U/R]$ is given by a minimal versal formal object of $[U/R]$.

The difference between the two notions is related to the difference between conditions (15.0.1) and (15.0.2) and disappears when $k' \subset k$ is separable. Also a normalized smooth prorepresentable groupoid in functors is minimal as the following lemma shows. Here is a precise statement.

06KN **Lemma 27.2.** *Let (U, R, s, t, c) be a smooth prorepresentable groupoid in functors on $\mathcal{C}_\Lambda$.*

(1) *(U, R, s, t, c) is normalized if and only if the morphism $U \to [U/R]$ induces an isomorphism on tangent spaces, and*
(2) *(U, R, s, t, c) is minimal if and only if the kernel of $TU \to T[U/R]$ is contained in the image of $Der_\Lambda(k, k) \to TU$.*

Proof. Part (1) follows immediately from the definitions. To see part (2) set $\mathcal{F} = [U/R]$. Since $\mathcal{F}$ has a presentation it is a deformation category, see Theorem 26.4. In particular it satisfies (RS), (S1), and (S2), see Lemma 16.6. Recall that minimal versal formal objects are unique up to isomorphism, see Lemma 14.5. By Theorem 15.5 a minimal versal object induces a map $\underline{\xi} : \underline{R}|_{\mathcal{C}_\Lambda} \to \mathcal{F}$ satisfying (15.0.2). Since $U \cong \underline{R}|_{\mathcal{C}_\Lambda}$ over $\mathcal{F}$ we see that $TU \to T\mathcal{F} = T[U/R]$ satisfies the property as stated in the lemma. $\square$

The quotient of a minimal prorepresentable groupoid in functors on $\mathcal{C}_\Lambda$ does not admit autoequivalences which are not automorphisms. To prove this, we first note the following lemma.

06KP **Lemma 27.3.** *Let $U : \mathcal{C}_\Lambda \to Sets$ be a prorepresentable functor. Let $\varphi : U \to U$ be a morphism such that $d\varphi : TU \to TU$ is an isomorphism. Then φ is an isomorphism.*

Proof. If $U \cong \underline{R}|_{\mathcal{C}_\Lambda}$ for some $R \in \mathrm{Ob}(\widehat{\mathcal{C}}_\Lambda)$, then completing φ gives a morphism $\underline{R} \to \underline{R}$. If $f : R \to R$ is the corresponding morphism in $\widehat{\mathcal{C}}_\Lambda$, then f induces an isomorphism $\mathrm{Der}_\Lambda(R, k) \to \mathrm{Der}_\Lambda(R, k)$, see Example 11.14. In particular f is a surjection by Lemma 4.6. As a surjective endomorphism of a Noetherian ring is an isomorphism (see Algebra, Lemma 31.10) we conclude f, hence $\underline{R} \to \underline{R}$, hence $\varphi : U \to U$ is an isomorphism. $\square$

06KQ **Lemma 27.4.** *Let (U, R, s, t, c) be a minimal smooth prorepresentable groupoid in functors on $\mathcal{C}_\Lambda$. If $\varphi : [U/R] \to [U/R]$ is an equivalence of categories cofibered in groupoids, then φ is an isomorphism.*

Proof. A morphism $\varphi : [U/R] \to [U/R]$ is the same thing as a morphism $\varphi : (U, R, s, t, c) \to (U, R, s, t, c)$ of groupoids in functors over $\mathcal{C}_\Lambda$ as defined in Definition 21.1. Denote $\phi : U \to U$ and $\psi : R \to R$ the corresponding morphisms. Because the diagram

$$\begin{tikzcd} & \mathrm{Der}_\Lambda(k, k) \arrow[dl, "\gamma"'] \arrow[dr, "\gamma"] & \\ TU \arrow[rr, "d\phi"] \arrow[d] & & TU \arrow[d] \\ T[U/R] \arrow[rr, "d\varphi"] & & T[U/R] \end{tikzcd}$$

is commutative, since $d\varphi$ is bijective, and since we have the characterization of minimality in Lemma 27.2 we conclude that $d\phi$ is injective (hence bijective by dimension reasons). Thus $\phi : U \to U$ is an isomorphism by Lemma 27.3. We can use a similar argument, using the exact sequence

$$0 \to \mathrm{Inf}([U/R]) \to TR \to TU \oplus TU$$

of Lemma 26.2 to prove that $\psi : R \to R$ is an isomorphism. But is also a consequence of the fact that $R = U \times_{[U/R]} U$ and that φ and ϕ are isomorphisms. $\square$

06KR **Lemma 27.5.** *Let (U, R, s, t, c) and (U', R', s', t', c') be minimal smooth prorepresentable groupoids in functors on $\mathcal{C}_\Lambda$. If $\varphi : [U/R] \to [U'/R']$ is an equivalence of categories cofibered in groupoids, then φ is an isomorphism.*

Proof. Let $\psi : [U'/R'] \to [U/R]$ be a quasi-inverse to φ. Then $\psi \circ \varphi$ and $\varphi \circ \psi$ are isomorphisms by Lemma 27.4, hence φ and ψ are isomorphisms. $\square$

The following lemma summarizes some of the things we have seen earlier in this chapter.

06L2 **Lemma 27.6.** *Let $\mathcal{F}$ be a deformation category such that $\dim_k T\mathcal{F} < \infty$ and $\dim_k$ Inf($\mathcal{F}$) $< \infty$. Then there exists a minimal versal formal object ξ of $\mathcal{F}$. Say ξ lies over $R \in \mathrm{Ob}(\widehat{\mathcal{C}}_\Lambda)$. Let $U = \underline{R}|_{\mathcal{C}_\Lambda}$. Let $f = \underline{\xi} : U \to \mathcal{F}$ be the associated* *morphism. Let (U, R, s, t, c) be the groupoid in functors on $\mathcal{C}_\Lambda$ constructed from $f : U \to \mathcal{F}$ in Lemma 25.2. Then (U, R, s, t, c) is a minimal smooth prorepresentable groupoid in functors on $\mathcal{C}_\Lambda$ and there is an equivalence $[U/R] \to \mathcal{F}$.*

Proof. As $\mathcal{F}$ is a deformation category it satisfies (S1) and (S2), see Lemma 16.6. By Lemma 13.4 there exists a versal formal object. By Lemma 14.5 there exists a minimal versal formal object ξ/R as in the statement of the lemma. Setting $U = \underline{R}|_{\mathcal{C}_\Lambda}$ the associated map $\underline{\xi} : U \to \mathcal{F}$ is smooth (this is the definition of a versal formal object). Let (U, R, s, t, c) be the groupoid in functors constructed in Lemma 25.2 from the map $\underline{\xi}$. By Lemma 26.1 we see that (U, R, s, t, c) is a smooth groupoid in functors and that $[U/R] \to \mathcal{F}$ is an equivalence. By Lemma 26.3 we see that (U, R, s, t, c) is prorepresentable. Finally, (U, R, s, t, c) is minimal because $U \to [U/R] = \mathcal{F}$ corresponds to the minimal versal formal object ξ. □

Presentations by minimal prorepresentable groupoids in functors satisfy the following uniqueness property.

06L3 **Lemma 27.7.** *Let $\mathcal{F}$ be category cofibered in groupoids over $\mathcal{C}_\Lambda$. Assume there exist presentations of $\mathcal{F}$ by minimal smooth prorepresentable groupoids in functors (U, R, s, t, c) and (U', R', s', t', c'). Then (U, R, s, t, c) and (U', R', s', t', c') are isomorphic.*

Proof. Follows from Lemma 27.5 and the observation that a morphism $[U/R] \to [U'/R']$ is the same thing as a morphism of groupoids in functors (by our explicit construction of $[U/R]$ in Definition 21.9). □

In summary we have proved the following theorem.

06TE **Theorem 27.8.** *Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. Consider the following conditions*

(1) *$\mathcal{F}$ admits a presentation by a normalized smooth prorepresentable groupoid in functors on $\mathcal{C}_\Lambda$,*
(2) *$\mathcal{F}$ admits a presentation by a smooth prorepresentable groupoid in functors on $\mathcal{C}_\Lambda$,*
(3) *$\mathcal{F}$ admits a presentation by a minimal smooth prorepresentable groupoid in functors on $\mathcal{C}_\Lambda$, and*
(4) *$\mathcal{F}$ satisfies the following conditions*
    (a) *$\mathcal{F}$ is a deformation category.*
    (b) *$\dim_k T\mathcal{F}$ is finite.*
    (c) *$\dim_k Inf(\mathcal{F})$ is finite.*

Then (2), (3), (4) are equivalent and are implied by (1). If $k' \subset k$ is separable, then (1), (2), (3), (4) are all equivalent. Furthermore, the minimal smooth prorepresentable groupoids in functors which provide a presentation of $\mathcal{F}$ are unique up to isomorphism.

Proof. We see that (1) implies (3) and is equivalent to (3) if $k' \subset k$ is separable from Lemma 27.2. It is clear that (3) implies (2). We see that (2) implies (4) by Theorem 26.4. We see that (4) implies (3) by Lemma 27.6. This proves all the implications. The final uniqueness statement follows from Lemma 27.7. □

28. Uniqueness of versal rings

0DQA Given R, S in $\widehat{\mathcal{C}}_\Lambda$ we say maps $f, g : R \to S$ are *formally homotopic* if there exists an $r \geq 0$ and maps $h : R \to R[[t_1, \ldots, t_r]]$ and $k : R[[t_1, \ldots, t_r]] \to S$ in $\widehat{\mathcal{C}}_\Lambda$ such that for all $a \in R$ we have

(1) $h(a) \bmod (t_1, \ldots, t_r) = a$,
(2) $f(a) = k(a)$,
(3) $g(a) = k(h(a))$.

We will say (r, h, k) is a *formal homotopy* between f and g.

0DQB **Lemma 28.1.** *Being formally homotopic is an equivalence relation on sets of morphisms in $\widehat{\mathcal{C}}_\Lambda$.*

Proof. Suppose we have any $r \geq 1$ and two maps $h_1, h_2 : R \to R[[t_1, \ldots, t_r]]$ such that $h_1(a) \bmod (t_1, \ldots, t_r) = h_2(a) \bmod (t_1, \ldots, t_r) = a$ for all $a \in R$ and a map $k : R[[t_1, \ldots, t_r]] \to S$. Then we claim $k \circ h_1$ is formally homotopic to $k \circ h_2$. The symmetric inherent in this claim will show that our notion of formally homotopic is symmetric. Namely, the map

$$\Psi : R[[t_1, \ldots, t_r]] \longrightarrow R[[t_1, \ldots, t_r]], \quad \sum a_I t^I \longmapsto \sum h_1(a_I) t^I$$

is an isomorphism. Set $h(a) = \Psi^{-1}(h_2(a))$ for $a \in R$ and $k' = k \circ \Psi$, then we see that (r, h, k') is a formal homotopy between $k \circ h_1$ and $k \circ h_2$, proving the claim

Say we have three maps $f_1, f_2, f_3 : R \to S$ as above and a formal homotopy (r_1, h_1, k_1) between f_1 and f_2 and a formal homotopy (r_2, h_2, k_2) between f_3 and f_2 (!). After relabeling the coordinates we may assume $h_2 : R \to R[[t_{r_1+1}, \ldots, t_{r_1+r_2}]]$ and $k_2 : R[[t_{r_1+1}, \ldots, t_{r_1+r_2}]] \to S$. By choosing a suitable isomorphism

$$R[[t_1, \ldots, t_{r_1+r_2}]] \longrightarrow R[[t_{r_1+1}, \ldots, t_{r_1+r_2}]] \widehat{\otimes}_{h_2, R, h_1} R[[t_1, \ldots, t_{r_1}]]$$

we may fit these maps into a commutative diagram

$$\begin{tikzcd}
R \arrow[r, "h_1"'] \arrow[d, "h_2"] & R[[t_1, \ldots, t_{r_1}]] \arrow[d, "h_2'"] \\
R[[t_{r_1+1}, \ldots, t_{r_1+r_2}]] \arrow[r, "h_1'"] & R[[t_1, \ldots, t_{r_1+r_2}]]
\end{tikzcd}$$

with $h_2'(t_i) = t_i$ for $1 \leq i \leq r_1$ and $h_1'(t_i) = t_i$ for $r_1 + 1 \leq i \leq r_2$. Some details omitted. Since this diagram is a pushout in the category $\widehat{\mathcal{C}}_\Lambda$ (see proof of Lemma 4.3) and since $k_1 \circ h_1 = f_2 = k_2 \circ h_2$ we conclude there exists a map

$$k : R[[t_1, \ldots, t_{r_1+r_2}]] \to S$$

with $k_1 = k \circ h_2'$ and $k_2 = k \circ h_1'$. Denote $h = h_1' \circ h_2 = h_2' \circ h_1$. Then we have

(1) $k(h_1'(a)) = k_2(a) = f_3(a)$, and
(2) $k(h_2'(a)) = k_1(a) = f_1(a)$.

By the claim in the first paragraph of the proof this shows that f_1 and f_3 are formally homotopic. $\square$

0DQC **Lemma 28.2.** *In the category $\widehat{\mathcal{C}}_\Lambda$, if $f_1, f_2 : R \to S$ are formally homotopic and $g : S \to S'$ is a morphism, then $g \circ f_1$ and $g \circ f_2$ are formally homotopic.*

Proof. Namely, if (r, h, k) is a formal homotopy between f_1 and f_2, then $(r, h, g \circ k)$ is a formal homotopy between $g \circ f_1$ and $g \circ f_2$. $\square$

0DQD **Lemma 28.3.** *Let $\mathcal{F}$ be a deformation category over $\mathcal{C}_\Lambda$ with $\dim_k T\mathcal{F} < \infty$ and $\dim_k Inf(\mathcal{F}) < \infty$. Let ξ be a versal formal object lying over R. Let η be a formal object lying over S. Then any two maps*

$$f, g : R \to S$$

such that $f_*\xi \cong \eta \cong g_*\xi$ are formally homotopic.

Proof. By Theorem 26.4 and its proof, $\mathcal{F}$ has a presentation by a smooth prorepresentable groupoid

$$(\underline{R}, \underline{R_1}, s, t, c, e, i)|_{\mathcal{C}_\Lambda}$$

in functors on $\mathcal{C}_\lambda$ such that $\mathcal{F}$. Then the maps $s : R \to R_1$ and $t : R \to R_1$ are formally smooth ring maps and $e : R_1 \to R$ is a section. In particular, we can choose an isomorphism $R_1 = R[[t_1, \ldots, t_r]]$ for some $r \geq 0$ such that s is the embedding $R \subset R[[t_1, \ldots, t_r]]$ and t corresponds to a map $h : R \to R[[t_1, \ldots, t_r]]$ with $h(a) \bmod (t_1, \ldots, t_r) = a$ for all $a \in R$. The existence of the isomorphism $\alpha : f_*\xi \to g_*\xi$ means exactly that there is a map $k : R_1 \to S$ such that $f = k \circ s$ and $g = k \circ t$. This exactly means that (r, h, k) is a formal homotopy between f and g. $\square$

0DQE **Lemma 28.4.** *In the category $\widehat{\mathcal{C}}_\Lambda$, if $f_1, f_2 : R \to S$ are formally homotopic and $\mathfrak{p} \subset R$ is a minimal prime ideal, then $f_1(\mathfrak{p})S = f_2(\mathfrak{p})S$ as ideals.*

Proof. Suppose (r, h, k) is a formal homotopy between f_1 and f_2. We claim that $\mathfrak{p}R[[t_1, \ldots, t_r]] = h(\mathfrak{p})R[[t_1, \ldots, t_r]]$. The claim implies the lemma by further composing with k. To prove the claim, observe that the map $\mathfrak{p} \mapsto \mathfrak{p}R[[t_1, \ldots, t_r]]$ is a bijection between the minimal prime ideals of R and the minimal prime ideals of $R[[t_1, \ldots, t_r]]$. Finally, $h(\mathfrak{p})R[[t_1, \ldots, t_r]]$ is a minimal prime as h is flat, and hence of the form $\mathfrak{q}R[[t_1, \ldots, t_r]]$ for some minimal prime $\mathfrak{q} \subset R$ by what we just said. But since $h \bmod (t_1, \ldots, t_r) = \mathrm{id}_R$ by definition of a formal homotopy, we conclude that $\mathfrak{q} = \mathfrak{p}$ as desired. $\square$

29. Change of residue field

07W7 In this section we quickly discuss what happens if we replace the residue field k by a finite extension. Let Λ be a Noetherian ring and let $\Lambda \to k$ be a finite ring map where k is a field. Throughout this whole chapter we have used $\mathcal{C}_\Lambda$ to denote the category of Artinian local Λ-algebras whose residue field is identified with k, see Definition 3.1. However, since in this section we will discuss what happen when we change k we will instead use the notation $\mathcal{C}_{\Lambda,k}$ to indicate the dependence on k.

07W8 **Situation 29.1.** Let Λ be a Noetherian ring and let $\Lambda \to k \to l$ be a finite ring maps where k and l are fields. Thus l/k is a finite extensions of fields. A typical object of $\mathcal{C}_{\Lambda,l}$ will be denoted B and a typical object of $\mathcal{C}_{\Lambda,k}$ will be denoted A. We define

07W9 (29.1.1) $$\mathcal{C}_{\Lambda,l} \longrightarrow \mathcal{C}_{\Lambda,k}, \quad B \longmapsto B \times_l k$$

Given a category cofibred in groupoids $p : \mathcal{F} \to \mathcal{C}_{\Lambda,k}$ we obtain an associated category cofibred in groupoids

$$p_{l/k} : \mathcal{F}_{l/k} \longrightarrow \mathcal{C}_{\Lambda,l}$$

by setting $\mathcal{F}_{l/k}(B) = \mathcal{F}(B \times_l k)$.

The functor (29.1.1) makes sense: because $B \times_l k \subset B$ we have

$$\begin{aligned} [k : k'] \operatorname{length}_{B \times_l k}(B \times_l k) &= \operatorname{length}_\Lambda(B \times_l k) \\ &\leq \operatorname{length}_\Lambda(B) \\ &= [l : k'] \operatorname{length}_B(B) < \infty \end{aligned}$$

(see Lemma 3.4) hence $B \times_l k$ is Artinian (see Algebra, Lemma 53.6). Thus $B \times_l k$ is an Artinian local ring with residue field k. Note that (29.1.1) commutes with fibre products

$$(B_1 \times_B B_2) \times_l k = (B_1 \times_l k) \times_{(B \times_l k)} (B_2 \times_l k)$$

and transforms surjective ring maps into surjective ring maps. We use the "expensive" notation $\mathcal{F}_{l/k}$ to prevent confusion with the construction of Remark 6.4. Here are some elementary observations.

07WA **Lemma 29.2.** *With notation and assumptions as in Situation 29.1.*

1. *We have $\overline{\mathcal{F}_{l/k}} = (\overline{\mathcal{F}})_{l/k}$.*
2. *If $\mathcal{F}$ is a predeformation category, then $\mathcal{F}_{l/k}$ is a predeformation category.*
3. *If $\mathcal{F}$ satisfies (S1), then $\mathcal{F}_{l/k}$ satisfies (S1).*
4. *If $\mathcal{F}$ satisfies (S2), then $\mathcal{F}_{l/k}$ satisfies (S2).*
5. *If $\mathcal{F}$ satisfies (RS), then $\mathcal{F}_{l/k}$ satisfies (RS).*

Proof. Part (1) is immediate from the definitions.

Since $\mathcal{F}_{l/k}(l) = \mathcal{F}(k)$ part (2) follows from the definition, see Definition 6.2.

Part (3) follows as the functor (29.1.1) commutes with fibre products and transforms surjective maps into surjective maps, see Definition 10.1.

Part (4). To see this consider a diagram

$$\begin{tikzcd} & l[\epsilon] \arrow[d] \\ B \arrow[r] & l \end{tikzcd}$$

in $\mathcal{C}_{\Lambda,l}$ as in Definition 10.1. Applying the functor (29.1.1) we obtain

$$\begin{tikzcd} & k[l\epsilon] \arrow[d] \\ B \times_l k \arrow[r] & k \end{tikzcd}$$

where $l\epsilon$ denotes the finite dimensional k-vector space $l\epsilon \subset l[\epsilon]$. According to Lemma 10.4 the condition of (S2) for $\mathcal{F}$ also holds for this diagram. Hence (S2) holds for $\mathcal{F}_{l/k}$.

Part (5) follows from the characterization of (RS) in Lemma 16.4 part (2) and the fact that (29.1.1) commutes with fibre products. $\square$

The following lemma applies in particular when $\mathcal{F}$ satisfies (S2) and is a predeformation category, see Lemma 10.5.

07WB **Lemma 29.3.** *With notation and assumptions as in Situation 29.1. Assume $\mathcal{F}$ is a predeformation category and $\overline{\mathcal{F}}$ satisfies (S2). Then there is a canonical l-vector space isomorphism*

$$T\mathcal{F} \otimes_k l \longrightarrow T\mathcal{F}_{l/k}$$

of tangent spaces.

Proof. By Lemma 29.2 we may replace $\mathcal{F}$ by $\overline{\mathcal{F}}$. Moreover we see that $T\mathcal{F}$, resp. $T\mathcal{F}_{l/k}$ has a canonical k-vector space structure, resp. l-vector space structure, see Lemma 12.2. Then

$$T\mathcal{F}_{l/k} = \mathcal{F}_{l/k}(l[\epsilon]) = \mathcal{F}(k[l\epsilon]) = T\mathcal{F} \otimes_k l$$

the last equality by Lemma 12.2. More generally, given a finite dimensional l-vector space V we have

$$\mathcal{F}_{l/k}(l[V]) = \mathcal{F}(k[V_k]) = T\mathcal{F} \otimes_k V_k$$

where V_k denotes V seen as a k-vector space. We conclude that the functors $V \mapsto \mathcal{F}_{l/k}(l[V])$ and $V \mapsto T\mathcal{F} \otimes_k V_k$ are canonically identified as functors to the category of sets. By Lemma 11.4 we see there is at most one way to turn either functor into an l-linear functor. Hence the isomorphisms are compatible with the l-vector space structures and we win. □

07WC **Lemma 29.4.** *With notation and assumptions as in Situation 29.1. Assume $\mathcal{F}$ is a deformation category. Then there is a canonical l-vector space isomorphism*

$$\mathit{Inf}(\mathcal{F}) \otimes_k l \longrightarrow \mathit{Inf}(\mathcal{F}_{l/k})$$

of infinitesimal automorphism spaces.

Proof. Let $x_0 \in \mathrm{Ob}(\mathcal{F}(k))$ and denote $x_{l,0}$ the corresponding object of $\mathcal{F}_{l/k}$ over l. Recall that $\mathrm{Inf}(\mathcal{F}) = \mathrm{Inf}_{x_0}(\mathcal{F})$ and $\mathrm{Inf}(\mathcal{F}_{l/k}) = \mathrm{Inf}_{x_{l,0}}(\mathcal{F}_{l/k})$, see Remark 19.4. Recall that the vector space structure on $\mathrm{Inf}_{x_0}(\mathcal{F})$ comes from identifying it with the tangent space of the functor $Aut(x_0)$ which is defined on the category $\mathcal{C}_{k,k}$ of Artinian local k-algebras with residue field k. Similarly, $\mathrm{Inf}_{x_{l,0}}(\mathcal{F}_{l/k})$ is the tangent space of $Aut(x_{l,0})$ which is defined on the category $\mathcal{C}_{l,l}$ of Artinian local l-algebras with residue field l. Unwinding the definitions we see that $Aut(x_{l,0})$ is the restriction of $Aut(x_0)_{l/k}$ (which lives on $\mathcal{C}_{k,l}$) to $\mathcal{C}_{l,l}$. Since there is no difference between the tangent space of $Aut(x_0)_{l/k}$ seen as a functor on $\mathcal{C}_{k,l}$ or $\mathcal{C}_{l,l}$, the lemma follows from Lemma 29.3 and the fact that $Aut(x_0)$ satisfies (RS) by Lemma 19.6 (whence we have (S2) by Lemma 16.6). □

07WD **Lemma 29.5.** *With notation and assumptions as in Situation 29.1. If $\mathcal{F} \to \mathcal{G}$ is a smooth morphism of categories cofibred in groupoids over $\mathcal{C}_{\Lambda,k}$, then $\mathcal{F}_{l/k} \to \mathcal{G}_{l/k}$ is a smooth morphism of categories cofibred in groupoids over $\mathcal{C}_{\Lambda,l}$.*

Proof. This follows immediately from the definitions and the fact that (29.1.1) preserves surjections. □

There are many more things you can say about the relationship between $\mathcal{F}$ and $\mathcal{F}_{l/k}$ (in particular about the relationship between versal deformations) and we will add these here as needed.

0DQF **Lemma 29.6.** *With notation and assumptions as in Situation 29.1. Let ξ be a versal formal object for $\mathcal{F}$ lying over $R \in \mathrm{Ob}(\widehat{\mathcal{C}}_{\Lambda,k})$. Then there exist*

(1) *an $S \in \mathrm{Ob}(\widehat{\mathcal{C}}_{\Lambda,l})$ and a local Λ-algebra homomorphism $R \to S$ which is formally smooth in the $\mathfrak{m}_S$-adic topology and induces the given field extension l/k on residue fieds, and*
(2) *a versal formal object of $\mathcal{F}_{l/k}$ lying over S.*

Proof. Construction of S. Choose a surjection $R[x_1, \ldots, x_n] \to l$ of R-algebras. The kernel is a maximal ideal $\mathfrak{m}$. Set S equal to the $\mathfrak{m}$-adic completion of the Noetherian ring $R[x_1, \ldots, x_n]$. Then S is in $\widehat{\mathcal{C}}_{\Lambda,l}$ by Algebra, Lemma 97.6. The map $R \to S$ is formally smooth in the $\mathfrak{m}_S$-adic topology by More on Algebra, Lemmas 37.2 and 37.4 and the fact that $R \to R[x_1, \ldots, x_n]$ is formally smooth. (Compare with the proof Lemma 9.5.)

Since ξ is versal, the transformation $\underline{\xi} : \underline{R}|_{\mathcal{C}_{\Lambda,k}} \to \mathcal{F}$ is smooth. By Lemma 29.5 the induced map

$$(\underline{R}|_{\mathcal{C}_{\Lambda,k}})_{l/k} \longrightarrow \mathcal{F}_{l/k}$$

is smooth. Thus it suffices to construct a smooth morphism $\underline{S}|_{\mathcal{C}_{\Lambda,l}} \to (\underline{R}|_{\mathcal{C}_{\Lambda,k}})_{l/k}$. To give such a map means for every object B of $\mathcal{C}_{\Lambda,l}$ a map of sets

$$\mathrm{Mor}_{\widehat{\mathcal{C}}_{\Lambda,l}}(S, B) \longrightarrow \mathrm{Mor}_{\widehat{\mathcal{C}}_{\Lambda,k}}(R, B \times_l k)$$

functorial in B. Given an element $\varphi : S \to B$ on the left hand side we send it to the composition $R \to S \to B$ whose image is contained in the sub Λ-algebra $B \times_l k$. Smoothness of the map means that given a surjection $B' \to B$ and a commutative diagram

$$\begin{tikzcd} S \arrow[r] & B \arrow[r, equal] & B \\ R \arrow[u] \arrow[r] & B' \times_l k \arrow[u] \arrow[r] & B' \arrow[u] \end{tikzcd}$$

we have to find a ring map $S \to B'$ fitting into the outer rectangle. The existence of this map is guaranteed as we chose $R \to S$ to be formally smooth in the $\mathfrak{m}_S$-adic topology, see More on Algebra, Lemma 37.5. □

30. Other chapters

Preliminaries

Schemes

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