DERIVED CATEGORIES OF VARIETIES

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Contents

1. Introduction

0FY1 In this chapter we continue the discussion started in Derived Categories of Schemes, Section 1. We will discuss Fourier-Mukai transforms, first studied by Mukai in [Muk81]. We will prove Orlov's theorem on derived equivalences ([Orl97]). We also discuss the countability of derived equivalence classes proved by Anel and Toën in [AT09].

A good introduction to this material is the book [Huy06] by Daniel Huybrechts. Some other papers which helped popularize this topic are

(1) the paper by Bondal and Kapranov, see [BK89]
(2) the paper by Bondal and Orlov, see [BO01]
(3) the paper by Bondal and Van den Bergh, see [BV03]
(4) the papers by Beilinson, see [Bei78] and [Bei84]
(5) the paper by Orlov, see [Orl02]
(6) the paper by Orlov, see [Orl05]

(7) the paper by Rouquier, see [Rou08]
(8) there are many more we could mention here.

2. Conventions and notation

0FY2   Let k be a field. A k-linear triangulated category $\mathcal{T}$ is a triangulated category (Derived Categories, Section 3) which is endowed with a k-linear structure (Differential Graded Algebra, Section 24) such that the translation functors $[n] : \mathcal{T} \to \mathcal{T}$ are k-linear for all $n \in \mathbf{Z}$.

Let k be a field. We denote Vect_k the category of k-vector spaces. For a k-vector space V we denote $V^\vee$ the k-linear dual of V, i.e., $V^\vee = \mathrm{Hom}_k(V, k)$.

Let X be a scheme. We denote $D_{perf}(\mathcal{O}_X)$ the full subcategory of $D(\mathcal{O}_X)$ consisting of perfect complexes (Cohomology, Section 49). If X is Noetherian then $D_{perf}(\mathcal{O}_X) \subset D^b_{Coh}(\mathcal{O}_X)$, see Derived Categories of Schemes, Lemma 11.6. If X is Noetherian and regular, then $D_{perf}(\mathcal{O}_X) = D^b_{Coh}(\mathcal{O}_X)$, see Derived Categories of Schemes, Lemma 11.8.

Let k be a field. Let X and Y be schemes over k. In this situation we will write $X \times Y$ instead of $X \times_{\mathrm{Spec}(k)} Y$.

Let S be a scheme. Let X, Y be schemes over S. Let $\mathcal{F}$ be a $\mathcal{O}_X$-module and let $\mathcal{G}$ be a $\mathcal{O}_Y$-module. We set

$$\mathcal{F} \boxtimes \mathcal{G} = \mathrm{pr}_1^* \mathcal{F} \otimes_{\mathcal{O}_{X \times_S Y}} \mathrm{pr}_2^* \mathcal{G}$$

as $\mathcal{O}_{X \times_S Y}$-modules. If $K \in D(\mathcal{O}_X)$ and $M \in D(\mathcal{O}_Y)$ then we set

$$K \boxtimes M = L\mathrm{pr}_1^* K \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times_S Y}} L\mathrm{pr}_2^* M$$

as an object of $D(\mathcal{O}_{X \times_S Y})$. Thus our notation is potentially ambiguous, but context should make it clear which of the two is meant.

3. Serre functors

0FY3   The material in this section is taken from [BK89].

0FY4   **Lemma 3.1.** *Let k be a field. Let $\mathcal{T}$ be a k-linear triangulated category such that $\dim_k \mathrm{Hom}_\mathcal{T}(X, Y) < \infty$ for all $X, Y \in \mathrm{Ob}(\mathcal{T})$. The following are equivalent*

(1) *there exists a k-linear equivalence $S : \mathcal{T} \to \mathcal{T}$ and k-linear isomorphisms $c_{X,Y} : \mathrm{Hom}_\mathcal{T}(X, Y) \to \mathrm{Hom}_\mathcal{T}(Y, S(X))^\vee$ functorial in $X, Y \in \mathrm{Ob}(\mathcal{T})$,*
(2) *for every $X \in \mathrm{Ob}(\mathcal{T})$ the functor $Y \mapsto \mathrm{Hom}_\mathcal{T}(X, Y)^\vee$ is representable and the functor $Y \mapsto \mathrm{Hom}_\mathcal{T}(Y, X)^\vee$ is corepresentable.*

Proof. Condition (1) implies (2) since given (S, c) and $X \in \mathrm{Ob}(\mathcal{T})$ the object $S(X)$ represents the functor $Y \mapsto \mathrm{Hom}_\mathcal{T}(X, Y)^\vee$ and the object $S^{-1}(X)$ corepresents the functor $Y \mapsto \mathrm{Hom}_\mathcal{T}(Y, X)^\vee$.

Assume (2). We will repeatedly use the Yoneda lemma, see Categories, Lemma 3.5. For every X denote $S(X)$ the object representing the functor $Y \mapsto \mathrm{Hom}_\mathcal{T}(X, Y)^\vee$. Given $\varphi : X \to X'$, we obtain a unique arrow $S(\varphi) : S(X) \to S(X')$ determined by the corresponding transformation of functors $\mathrm{Hom}_\mathcal{T}(X, -)^\vee \to \mathrm{Hom}_\mathcal{T}(X', -)^\vee$. Thus S is a functor and we obtain the isomorphisms $c_{X,Y}$ by construction. It remains to show that S is an equivalence. For every X denote $S'(X)$ the object

corepresenting the functor $Y \mapsto \operatorname{Hom}_{\mathcal{T}}(Y, X)^{\vee}$. Arguing as above we find that S' is a functor. We claim that S' is quasi-inverse to S. To see this observe that

$$\operatorname{Hom}_{\mathcal{T}}(X, Y) = \operatorname{Hom}_{\mathcal{T}}(Y, S(X))^{\vee} = \operatorname{Hom}_{\mathcal{T}}(S'(S(X)), Y)$$

bifunctorially, i.e., we find $S' \circ S \cong \operatorname{id}_{\mathcal{T}}$. Similarly, we have

$$\operatorname{Hom}_{\mathcal{T}}(Y, X) = \operatorname{Hom}_{\mathcal{T}}(S'(X), Y)^{\vee} = \operatorname{Hom}_{\mathcal{T}}(Y, S(S'(X)))$$

and we find $S \circ S' \cong \operatorname{id}_{\mathcal{T}}$. $\square$

0FY5 **Definition 3.2.** Let k be a field. Let $\mathcal{T}$ be a k-linear triangulated category such that $\dim_k \operatorname{Hom}_{\mathcal{T}}(X, Y) < \infty$ for all $X, Y \in \operatorname{Ob}(\mathcal{T})$. We say *a Serre functor exists* if the equivalent conditions of Lemma 3.1 are satisfied. In this case a *Serre functor* is a k-linear equivalence $S : \mathcal{T} \to \mathcal{T}$ endowed with k-linear isomorphisms $c_{X,Y} : \operatorname{Hom}_{\mathcal{T}}(X, Y) \to \operatorname{Hom}_{\mathcal{T}}(Y, S(X))^{\vee}$ functorial in $X, Y \in \operatorname{Ob}(\mathcal{T})$.

0FY6 **Lemma 3.3.** *In the situation of Definition 3.2. If a Serre functor exists, then it is unique up to unique isomorphism and it is an exact functor of triangulated categories.*

Proof. Given a Serre functor S the object $S(X)$ represents the functor $Y \mapsto \operatorname{Hom}_{\mathcal{T}}(X, Y)^{\vee}$. Thus the object $S(X)$ together with the functorial identification $\operatorname{Hom}_{\mathcal{T}}(X, Y)^{\vee} = \operatorname{Hom}_{\mathcal{T}}(Y, S(X))$ is determined up to unique isomorphism by the Yoneda lemma (Categories, Lemma 3.5). Moreover, for $\varphi : X \to X'$, the arrow $S(\varphi) : S(X) \to S(X')$ is uniquely determined by the corresponding transformation of functors $\operatorname{Hom}_{\mathcal{T}}(X, -)^{\vee} \to \operatorname{Hom}_{\mathcal{T}}(X', -)^{\vee}$.

For objects X, Y of $\mathcal{T}$ we have

$$\begin{aligned} \operatorname{Hom}(Y, S(X)[1])^{\vee} &= \operatorname{Hom}(Y[-1], S(X))^{\vee} \\ &= \operatorname{Hom}(X, Y[-1]) \\ &= \operatorname{Hom}(X[1], Y) \\ &= \operatorname{Hom}(Y, S(X[1]))^{\vee} \end{aligned}$$

By the Yoneda lemma we conclude that there is a unique isomorphism $S(X[1]) \to S(X)[1]$ inducing the isomorphism from top left to bottom right. Since each of the isomorphisms above is functorial in both X and Y we find that this defines an isomorphism of functors $S \circ [1] \to [1] \circ S$.

Let (A, B, C, f, g, h) be a distinguished triangle in $\mathcal{T}$. We have to show that the triangle $(S(A), S(B), S(C), S(f), S(g), S(h))$ is distinguished. Here we use the canonical isomorphism $S(A[1]) \to S(A)[1]$ constructed above to identify the target $S(A[1])$ of $S(h)$ with $S(A)[1]$. We first observe that for any X in $\mathcal{T}$ the triangle $(S(A), S(B), S(C), S(f), S(g), S(h))$ induces a long exact sequence

$$\ldots \to \operatorname{Hom}(X, S(A)) \to \operatorname{Hom}(X, S(B)) \to \operatorname{Hom}(X, S(C)) \to \operatorname{Hom}(X, S(A)[1]) \to \ldots$$

of finite dimensional k-vector spaces. Namely, this sequence is k-linear dual of the sequence

$$\ldots \leftarrow \operatorname{Hom}(A, X) \leftarrow \operatorname{Hom}(B, X) \leftarrow \operatorname{Hom}(C, X) \leftarrow \operatorname{Hom}(A[1], X) \leftarrow \ldots$$

which is exact by Derived Categories, Lemma 4.2. Next, we choose a distinguished triangle $(S(A), E, S(C), i, p, S(h))$ which is possible by axioms TR1 and TR2. We

want to construct the dotted arrow making following diagram commute

$$\begin{array}{ccccccccc}
S(C)[-1] & \xrightarrow{S(h[-1])} & S(A) & \xrightarrow{S(f)} & S(B) & \xrightarrow{S(g)} & S(C) & \xrightarrow{S(h)} & S(A)[1] \\
\| & & \| & & \uparrow\varphi & & \| & & \| \\
S(C)[-1] & \xrightarrow{S(h[-1])} & S(A) & \xrightarrow{i} & E & \xrightarrow{p} & S(C) & \xrightarrow{S(h)} & S(A)[1]
\end{array}$$

Namely, if we have φ, then we claim for any X the resulting map $\mathrm{Hom}(X, E) \to \mathrm{Hom}(X, S(B))$ will be an isomorphism of k-vector spaces. Namely, we will obtain a commutative diagram

$$\begin{array}{ccccccccc}
\mathrm{Hom}(X, S(C)[-1]) & \longrightarrow & \mathrm{Hom}(X, S(A)) & \longrightarrow & \mathrm{Hom}(X, S(B)) & \longrightarrow & \mathrm{Hom}(X, S(C)) & \longrightarrow & \mathrm{Hom}(X, S(A)[1]) \\
\| & & \| & & \uparrow\varphi & & \| & & \| \\
\mathrm{Hom}(X, S(C)[-1]) & \longrightarrow & \mathrm{Hom}(X, S(A)) & \longrightarrow & \mathrm{Hom}(X, E) & \longrightarrow & \mathrm{Hom}(X, S(C)) & \longrightarrow & \mathrm{Hom}(X, S(A)[1])
\end{array}$$

with exact rows (see above) and we can apply the 5 lemma (Homology, Lemma 5.20) to see that the middle arrow is an isomorphism. By the Yoneda lemma we conclude that φ is an isomorphism. To find φ consider the following diagram

$$\begin{array}{ccc}
\mathrm{Hom}(E, S(C)) & \longrightarrow & \mathrm{Hom}(S(A), S(C)) \\
\uparrow & & \uparrow \\
\mathrm{Hom}(E, S(B)) & \longrightarrow & \mathrm{Hom}(S(A), S(B))
\end{array}$$

The elements p and $S(f)$ in positions $(0, 1)$ and $(1, 0)$ define a cohomology class ξ in the total complex of this double complex. The existence of φ is equivalent to whether ξ is zero. If we take k-linear duals of this and we use the defining property of S we obtain

$$\begin{array}{ccc}
\mathrm{Hom}(C, E) & \longleftarrow & \mathrm{Hom}(C, S(A)) \\
\downarrow & & \downarrow \\
\mathrm{Hom}(B, E) & \longleftarrow & \mathrm{Hom}(B, S(A))
\end{array}$$

Since both $A \to B \to C$ and $S(A) \to E \to S(C)$ are distinguished triangles, we know by TR3 that given elements $\alpha \in \mathrm{Hom}(C, E)$ and $\beta \in \mathrm{Hom}(B, S(A))$ mapping to the same element in $\mathrm{Hom}(B, E)$, there exists an element in $\mathrm{Hom}(C, S(A))$ mapping to both α and β. In other words, the cohomology of the total complex associated to this double complex is zero in degree 1, i.e., the degree corresponding to $\mathrm{Hom}(C, E) \oplus \mathrm{Hom}(B, S(A))$. Taking duals the same must be true for the previous one which concludes the proof. $\square$

4. Examples of Serre functors

0FY7   The lemma below is the standard example.

0FY8   **Lemma 4.1.** *Let k be a field. Let X be a proper scheme over k which is Gorenstein. Consider the complex $\omega_X^\bullet$ of Duality for Schemes, Lemmas 27.1. Then the functor*

$$S : D_{perf}(\mathcal{O}_X) \longrightarrow D_{perf}(\mathcal{O}_X), \quad K \longmapsto S(K) = \omega_X^\bullet \otimes_{\mathcal{O}_X}^{\mathbf{L}} K$$

is a Serre functor.

Proof. The statement make sense because $\dim \mathrm{Hom}_X(K, L) < \infty$ for $K, L \in D_{perf}(\mathcal{O}_X)$ by Derived Categories of Schemes, Lemma 11.7. Since X is Gorenstein the dualizing complex $\omega_X^\bullet$ is an invertible object of $D(\mathcal{O}_X)$, see Duality for Schemes, Lemma 24.4. In particular, locally on X the complex $\omega_X^\bullet$ has one nonzero cohomology sheaf which is an invertible module, see Cohomology, Lemma 52.2. Thus $S(K)$ lies in $D_{perf}(\mathcal{O}_X)$. On the other hand, the invertibility of $\omega_X^\bullet$ clearly implies that S is a self-equivalence of $D_{perf}(\mathcal{O}_X)$. Finally, we have to find an isomorphism

$$c_{K,L} : \mathrm{Hom}_X(K, L) \longrightarrow \mathrm{Hom}_X(L, \omega_X^\bullet \otimes_{\mathcal{O}_X}^{\mathbf{L}} K)^\vee$$

bifunctorially in K, L. To do this we use the canonical isomorphisms

$$\mathrm{Hom}_X(K, L) = H^0(X, L \otimes_{\mathcal{O}_X}^{\mathbf{L}} K^\vee)$$

and

$$\mathrm{Hom}_X(L, \omega_X^\bullet \otimes_{\mathcal{O}_X}^{\mathbf{L}} K) = H^0(X, \omega_X^\bullet \otimes_{\mathcal{O}_X}^{\mathbf{L}} K \otimes_{\mathcal{O}_X}^{\mathbf{L}} L^\vee)$$

given in Cohomology, Lemma 50.5. Since $(L \otimes_{\mathcal{O}_X}^{\mathbf{L}} K^\vee)^\vee = (K^\vee)^\vee \otimes_{\mathcal{O}_X}^{\mathbf{L}} L^\vee$ and since there is a canonical isomorphism $K \to (K^\vee)^\vee$ we find these k-vector spaces are canonically dual by Duality for Schemes, Lemma 27.4. This produces the isomorphisms $c_{K,L}$. We omit the proof that these isomorphisms are functorial. $\square$

5. Characterizing coherent modules

0FY9 This section is in some sense a continuation of the discussion in Derived Categories of Schemes, Section 34 and More on Morphisms, Section 69.

Before we can state the result we need some notation. Let k be a field. Let $n \geq 0$ be an integer. Let $S = k[X_0, \ldots, X_n]$. For an integer e denote $S_e \subset S$ the homogeneous polynomials of degree e. Consider the (noncommutative) k-algebra

$$R = \begin{pmatrix} S_0 & S_1 & S_2 & \ldots & \ldots \\ 0 & S_0 & S_1 & \ldots & \ldots \\ 0 & 0 & S_0 & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \ldots & \ldots & S_0 \end{pmatrix}$$

(with $n + 1$ rows and columns) with obvious multiplication and addition.

0FYA **Lemma 5.1.** *With k, n, and R as above, for an object K of $D(R)$ the following are equivalent*

(1) $\sum_{i \in \mathbf{Z}} \dim_k H^i(K) < \infty$, *and*
(2) K *is a compact object.*

Proof. If K is a compact object, then K can be represented by a complex $M^\bullet$ which is finite projective as a graded R-module, see Differential Graded Algebra, Lemma 36.6. Since $\dim_k R < \infty$ we conclude $\sum \dim_k M^i < \infty$ and a fortiori $\sum \dim_k H^i(M^\bullet) < \infty$. (One can also easily deduce this implication from the easier Differential Graded Algebra, Proposition 36.4.)

Assume K satisfies (1). Consider the distinguished triangle of trunctions $\tau_{\leq m} K \to K \to \tau_{\geq m+1} K$, see Derived Categories, Remark 12.4. It is clear that both $\tau_{\leq m} K$ and $\tau_{\geq m+1} K$ satisfy (1). If we can show both are compact, then so is K, see Derived Categories, Lemma 37.2. Hence, arguing on the number of nonzero cohomology modules of K we may assume $H^i(K)$ is nonzero only for one i. Shifting, we may

assume K is given by the complex consisting of a single finite dimensional R-module M sitting in degree 0.

Since $\dim_k(M) < \infty$ we see that M is Artinian as an R-module. Thus it suffices to show that every simple R-module represents a compact object of $D(R)$. Observe that

$$I = \begin{pmatrix} 0 & S_1 & S_2 & \ldots & \ldots \\ 0 & 0 & S_1 & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \ldots & \ldots & 0 \end{pmatrix}$$

is a nilpotent two sided ideal of R and that R/I is a commutative k-algebra isomorphic to a product of $n+1$ copies of k (placed along the diagonal in the matrix, i.e., R/I can be lifted to a k-subalgebra of R). It follows that R has exactly $n+1$ isomorphism classes of simple modules $M_0, \ldots, M_n$ (sitting along the diagonal). Consider the right R-module P_i of row vectors

$$P_i = \begin{pmatrix} 0 & \ldots & 0 & S_0 & \ldots & S_{i-1} & S_i \end{pmatrix}$$

with obvious multiplication $P_i \times R \to P_i$. Then we see that $R \cong P_0 \oplus \ldots \oplus P_n$ as a right R-module. Since clearly R is a compact object of $D(R)$, we conclude each P_i is a compact object of $D(R)$. (We of course also conclude each P_i is projective as an R-module, but this isn't what we have to show in this proof.) Clearly, $P_0 = M_0$ is the first of our simple R-modules. For P_1 we have a short exact sequence

$$0 \to P_0^{\oplus n+1} \to P_1 \to M_1 \to 0$$

which proves that M_1 fits into a distinguished triangle whose other members are compact objects and hence M_1 is a compact object of $D(R)$. More generally, there exists a short exact sequence

$$0 \to C_i \to P_i \to M_i \to 0$$

where C_i is a finite dimensional R-module whose simple constituents are isomorphic to M_j for $j < i$. By induction, we first conclude that C_i determines a compact object of $D(R)$ whereupon we conclude that M_i does too as desired. $\square$

0FYB **Lemma 5.2.** *Let k be a field. Let $n \geq 0$. Let $K \in D_{QCoh}(\mathcal{O}_{\mathbf{P}_k^n})$. The following are equivalent*

(1) *K is in $D_{Coh}^b(\mathcal{O}_{\mathbf{P}_k^n})$,*
(2) *$\sum_{i \in \mathbf{Z}} \dim_k H^i(\mathbf{P}_k^n, E \otimes^{\mathbf{L}} K) < \infty$ for each perfect object E of $D(\mathcal{O}_{\mathbf{P}_k^n})$,*
(3) *$\sum_{i \in \mathbf{Z}} \dim_k \operatorname{Ext}_{\mathbf{P}_k^n}^i(E, K) < \infty$ for each perfect object E of $D(\mathcal{O}_{\mathbf{P}_k^n})$,*
(4) *$\sum_{i \in \mathbf{Z}} \dim_k H^i(\mathbf{P}_k^n, K \otimes^{\mathbf{L}} \mathcal{O}_{\mathbf{P}_k^n}(d)) < \infty$ for $d = 0, 1, \ldots, n$.*

Proof. Parts (2) and (3) are equivalent by Cohomology, Lemma 50.5. If (1) is true, then for E perfect the derived tensor product $E \otimes^{\mathbf{L}} K$ is in $D_{Coh}^b(\mathcal{O}_{\mathbf{P}_k^n})$ and we see that (2) holds by Derived Categories of Schemes, Lemma 11.3. It is clear that (2) implies (4) as $\mathcal{O}_{\mathbf{P}_k^n}(d)$ can be viewed as a perfect object of the derived category of $\mathbf{P}_k^n$. Thus it suffices to prove that (4) implies (1).

Assume (4). Let R be as in Lemma 5.1. Let $P = \bigoplus_{d=0,\ldots,n} \mathcal{O}_{\mathbf{P}_k^n}(-d)$. Recall that $R = \operatorname{End}_{\mathbf{P}_k^n}(P)$ whereas all other self-Exts of P are zero and that P determines

an equivalence $- \otimes^{\mathbf{L}} P : D(R) \to D_{QCoh}(\mathcal{O}_{\mathbf{P}_k^n})$ by Derived Categories of Schemes, Lemma 20.1. Say K corresponds to L in $D(R)$. Then

$$
\begin{aligned}
H^i(L) &= \mathrm{Ext}^i_{D(R)}(R, L) \\
&= \mathrm{Ext}^i_{\mathbf{P}_k^n}(P, K) \\
&= H^i(\mathbf{P}_k^n, K \otimes P^\vee) \\
&= \bigoplus_{d=0,\ldots,n} H^i(\mathbf{P}_k^n, K \otimes \mathcal{O}(d))
\end{aligned}
$$

by Differential Graded Algebra, Lemma 35.4 (and the fact that $- \otimes^{\mathbf{L}} P$ is an equivalence) and Cohomology, Lemma 50.5. Thus our assumption (4) implies that L satisfies condition (2) of Lemma 5.1 and hence is a compact object of $D(R)$. Therefore K is a compact object of $D_{QCoh}(\mathcal{O}_{\mathbf{P}_k^n})$. Thus K is perfect by Derived Categories of Schemes, Proposition 17.1. Since $D_{perf}(\mathcal{O}_{\mathbf{P}_k^n}) = D^b_{Coh}(\mathcal{O}_{\mathbf{P}_k^n})$ by Derived Categories of Schemes, Lemma 11.8 we conclude (1) holds. $\square$

0FYC **Lemma 5.3.** *Let X be a scheme proper over a field k. Let $K \in D^b_{Coh}(\mathcal{O}_X)$ and let E in $D(\mathcal{O}_X)$ be perfect. Then $\sum_{i \in \mathbf{Z}} \dim_k \mathrm{Ext}^i_X(E, K) < \infty$.*

Proof. This follows for example by combining Derived Categories of Schemes, Lemmas 11.7 and 18.2. Alternative proof: combine Derived Categories of Schemes, Lemmas 11.6 and 11.3. $\square$

0FYD **Lemma 5.4.** *Let X be a proper scheme over a field k. Let $K \in \mathrm{Ob}(D_{QCoh}(\mathcal{O}_X))$. The following are equivalent*

(1) *$K \in D^b_{Coh}(\mathcal{O}_X)$, and*
(2) *$\sum_{i \in \mathbf{Z}} \dim_k \mathrm{Ext}^i_X(E, K) < \infty$ for all perfect E in $D(\mathcal{O}_X)$.*

In the projective case this is [Rou08, Lemma 7.46] and implicit in [BV03, Theorem A.1]

Proof. The implication (1) $\Rightarrow$ (2) follows from Lemma 5.3. The implication (2) $\Rightarrow$ (1) follows from More on Morphisms, Lemma 69.6 (see Derived Categories of Schemes, Example 35.2 for the meaning of a relatively perfect object over a field); the easier proof in the projective case is in the next paragraph.

Assume (2) and X projective over k. Choose a closed immersion $i : X \to \mathbf{P}_k^n$. It suffices to show that Ri_*K is in $D^b_{Coh}(\mathbf{P}_k^n)$ since a quasi-coherent module $\mathcal{F}$ on X is coherent, resp. zero if and only if $i_*\mathcal{F}$ is coherent, resp. zero. For a perfect object E of $D(\mathcal{O}_{\mathbf{P}_k^n})$, Li^*E is a perfect object of $D(\mathcal{O}_X)$ and

$$
\mathrm{Ext}^q_{\mathbf{P}_k^n}(E, Ri_*K) = \mathrm{Ext}^q_X(Li^*E, K)
$$

Hence by our assumption we see that $\sum_{q \in \mathbf{Z}} \dim_k \mathrm{Ext}^q_{\mathbf{P}_k^n}(E, Ri_*K) < \infty$. We conclude by Lemma 5.2. $\square$

6. A representability theorem

0FYE The material in this section is taken from [BV03].

Let $\mathcal{T}$ be a k-linear triangulated category. In this section we consider k-linear cohomological functors H from $\mathcal{T}$ to the category of k-vector spaces. This will mean H is a functor

$$
H : \mathcal{T}^{opp} \longrightarrow \mathrm{Vect}_k
$$

which is k-linear such that for any distinguished triangle $X \to Y \to Z$ in $\mathcal{T}$ the sequence $H(Z) \to H(Y) \to H(X)$ is an exact sequence of k-vector spaces. See Derived Categories, Definition 3.5 and Differential Graded Algebra, Section 24.

0FYF **Lemma 6.1.** *Let $\mathcal{D}$ be a triangulated category. Let $\mathcal{D}' \subset \mathcal{D}$ be a full triangulated subcategory. Let $X \in \mathrm{Ob}(\mathcal{D})$. The category of arrows $E \to X$ with $E \in \mathrm{Ob}(\mathcal{D}')$ is filtered.*

Proof. We check the conditions of Categories, Definition 19.1. The category is nonempty because it contains $0 \to X$. If $E_i \to X$, $i = 1, 2$ are objects, then $E_1 \oplus E_2 \to X$ is an object and there are morphisms $(E_i \to X) \to (E_1 \oplus E_2 \to X)$. Finally, suppose that $a, b : (E \to X) \to (E' \to X)$ are morphisms. Choose a distinguished triangle $E \xrightarrow{a-b} E' \to E''$ in $\mathcal{D}'$. By Axiom TR3 we obtain a morphism of triangles

$$
\begin{array}{ccccc}
E & \xrightarrow{a-b} & E' & \longrightarrow & E'' \\
\downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & X & \longrightarrow & X
\end{array}
$$

and we find that the resulting arrow $(E' \to X) \to (E'' \to X)$ equalizes a and b. $\square$

0FYG **Lemma 6.2.** *Let k be a field. Let $\mathcal{D}$ be a k-linear triangulated category which has direct sums and is compactly generated. Denote $\mathcal{D}_c$ the full subcategory of compact objects. Let $H : \mathcal{D}_c^{opp} \to \mathrm{Vect}_k$ be a k-linear cohomological functor such that $\dim_k H(X) < \infty$ for all $X \in \mathrm{Ob}(\mathcal{D}_c)$. Then H is isomorphic to the functor $X \mapsto \mathrm{Hom}(X, Y)$ for some $Y \in \mathrm{Ob}(\mathcal{D})$.* [CKN01, Lemma 2.14]

Proof. We will use Derived Categories, Lemma 37.2 without further mention. Denote $G : \mathcal{D}_c \to \mathrm{Vect}_k$ the k-linear homological functor which sends X to $H(X)^\vee$. For any object Y of $\mathcal{D}$ we set

$$G'(Y) = \mathrm{colim}_{X \to Y, X \in \mathrm{Ob}(\mathcal{D}_c)} G(X)$$

The colimit is filtered by Lemma 6.1. We claim that G' is a k-linear homological functor, the restriction of G' to $\mathcal{D}_c$ is G, and G' sends direct sums to direct sums.

Namely, suppose that $Y_1 \to Y_2 \to Y_3$ is a distinguished triangle. Let $\xi \in G'(Y_2)$ map to zero in $G'(Y_3)$. Since the colimit is filtered ξ is represented by some $X \to Y_2$ with $X \in \mathrm{Ob}(\mathcal{D}_c)$ and $g \in G(X)$. The fact that ξ maps to zero in $G'(Y_3)$ means the composition $X \to Y_2 \to Y_3$ factors as $X \to X' \to Y_3$ with $X' \in \mathcal{D}_c$ and g mapping to zero in $G(X')$. Choose a distinguished triangle $X'' \to X \to X'$. Then $X'' \in \mathrm{Ob}(\mathcal{D}_c)$. Since G is homological we find that g is the image of some $g'' \in G'(X'')$. By Axiom TR3 the maps $X \to Y_2$ and $X' \to Y_3$ fit into a morphism of distinguished triangles $(X'' \to X \to X') \to (Y_1 \to Y_2 \to Y_3)$ and we find that indeed ξ is the image of the element of $G'(Y_1)$ represented by $X'' \to Y_1$ and $g'' \in G(X'')$.

If $Y \in \mathrm{Ob}(\mathcal{D}_c)$, then $\mathrm{id} : Y \to Y$ is the final object in the category of arrows $X \to Y$ with $X \in \mathrm{Ob}(\mathcal{D}_c)$. Hence we see that $G'(Y) = G(Y)$ in this case and the statement on restriction holds. Let $Y = \bigoplus_{i \in I} Y_i$ be a direct sum. Let $a : X \to Y$ with $X \in \mathrm{Ob}(\mathcal{D}_c)$ and $g \in G(X)$ represent an element ξ of $G'(Y)$. The morphism $a : X \to Y$ can be uniquely written as a sum of morphisms $a_i : X \to Y_i$ almost all

zero as X is a compact object of $\mathcal{D}$. Let $I' = \{i \in I \mid a_i \neq 0\}$. Then we can factor a as the composition

$$X \xrightarrow{(1,...,1)} \bigoplus\nolimits_{i \in I'} X \xrightarrow{\bigoplus_{i \in I'} a_i} \bigoplus\nolimits_{i \in I} Y_i = Y$$

We conclude that $\xi = \sum_{i \in I'} \xi_i$ is the sum of the images of the elements $\xi_i \in G'(Y_i)$ corresponding to $a_i : X \to Y_i$ and $g \in G(X)$. Hence $\bigoplus G'(Y_i) \to G'(Y)$ is surjective. We omit the (trivial) verification that it is injective.

It follows that the functor $Y \mapsto G'(Y)^\vee$ is cohomological and sends direct sums to direct products. Hence by Brown representability, see Derived Categories, Proposition 38.2 we conclude that there exists a $Y \in \mathrm{Ob}(\mathcal{D})$ and an isomorphism $G'(Z)^\vee = \mathrm{Hom}(Z, Y)$ functorially in Z. For $X \in \mathrm{Ob}(\mathcal{D}_c)$ we have $G'(X)^\vee = G(X)^\vee = (H(X)^\vee)^\vee = H(X)$ because $\dim_k H(X) < \infty$ and the proof is complete. $\square$

0FYH **Theorem 6.3.** *Let X be a proper scheme over a field k. Let $F : D_{perf}(\mathcal{O}_X)^{opp} \to Vect_k$ be a k-linear cohomological functor such that*

$$\sum\nolimits_{n \in \mathbf{Z}} \dim_k F(E[n]) < \infty$$

In the projective case this is [BV03, Theorem A.1]

for all $E \in D_{perf}(\mathcal{O}_X)$. Then F is isomorphic to a functor of the form $E \mapsto \mathrm{Hom}_X(E, K)$ for some $K \in D^b_{Coh}(\mathcal{O}_X)$.

Proof. The derived category $D_{QCoh}(\mathcal{O}_X)$ has direct sums, is compactly generated, and $D_{perf}(\mathcal{O}_X)$ is the full subcategory of compact objects, see Derived Categories of Schemes, Lemma 3.1, Theorem 15.3, and Proposition 17.1. By Lemma 6.2 we may assume $F(E) = \mathrm{Hom}_X(E, K)$ for some $K \in \mathrm{Ob}(D_{QCoh}(\mathcal{O}_X))$. Then it follows that K is in $D^b_{Coh}(\mathcal{O}_X)$ by Lemma 5.4. $\square$

0H4A **Lemma 6.4.** *Let X be a proper scheme over a field k which is regular. Let $G : D_{perf}(\mathcal{O}_X) \to Vect_k$ be a k-linear homological functor such that*

$$\sum\nolimits_{n \in \mathbf{Z}} \dim_k G(E[n]) < \infty$$

for all $E \in D_{perf}(\mathcal{O}_X)$. Then G is isomorphic to a functor of the form $E \mapsto \mathrm{Hom}_X(K, E)$ for some $K \in D_{perf}(\mathcal{O}_X)$.

Proof. Consider the contravariant functor $E \mapsto E^\vee$ on $D_{perf}(\mathcal{O}_X)$, see Cohomology, Lemma 50.5. This functor is an exact anti-self-equivalence of $D_{perf}(\mathcal{O}_X)$. Hence we may apply Theorem 6.3 to the functor $F(E) = G(E^\vee)$ to find $K \in D_{perf}(\mathcal{O}_X)$ such that $G(E^\vee) = \mathrm{Hom}_X(E, K)$. It follows that $G(E) = \mathrm{Hom}_X(E^\vee, K) = \mathrm{Hom}_X(K^\vee, E)$ and we conclude that taking $K^\vee$ works. $\square$

7. Existence of adjoints

0FYM As a consequence of the results in the paper of Bondal and van den Bergh we get the following automatic existence of adjoints.

0FYN **Lemma 7.1.** *Let k be a field. Let X and Y be proper schemes over k. If X is regular, then any k-linear exact functor $F : D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_Y)$ has an exact right adjoint and an exact left adjoint.*

Proof. If an adjoint exists it is an exact functor by the very general Derived Categories, Lemma 7.1.

Let us prove the existence of a right adjoint. To see existence, it suffices to show that for $M \in D_{perf}(\mathcal{O}_Y)$ the contravariant functor $K \mapsto \mathrm{Hom}_Y(F(K), M)$ is representable. This functor is contravariant, k-linear, and cohomological. Hence by Theorem 6.3 it suffices to show that

$$\sum\nolimits_{i \in \mathbf{Z}} \dim_k \mathrm{Ext}^i_Y(F(K), M) < \infty$$

This follows from Lemma 5.3.

For the existence of the left adjoint we argue in the same manner using Lemma 6.4 in stead of Theorem 6.3. $\qquad\square$

8. Fourier-Mukai functors

0FYP   These functors were first introduced in [Muk81].

0FYQ   **Definition 8.1.** Let S be a scheme. Let X and Y be schemes over S. Let $K \in D(\mathcal{O}_{X \times_S Y})$. The exact functor

$$\Phi_K : D(\mathcal{O}_X) \longrightarrow D(\mathcal{O}_Y), \quad M \longmapsto R\mathrm{pr}_{2,*}(L\mathrm{pr}_1^* M \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times_S Y}} K)$$

of triangulated categories is called a *Fourier-Mukai functor* and K is called a *Fourier-Mukai kernel* for this functor. Moreover,
  (1) if Φ_K sends $D_{QCoh}(\mathcal{O}_X)$ into $D_{QCoh}(\mathcal{O}_Y)$ then the resulting exact functor $\Phi_K : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ is called a Fourier-Mukai functor,
  (2) if Φ_K sends $D_{perf}(\mathcal{O}_X)$ into $D_{perf}(\mathcal{O}_Y)$ then the resulting exact functor $\Phi_K : D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_Y)$ is called a Fourier-Mukai functor, and
  (3) if X and Y are Noetherian and Φ_K sends $D^b_{Coh}(\mathcal{O}_X)$ into $D^b_{Coh}(\mathcal{O}_Y)$ then the resulting exact functor $\Phi_K : D^b_{Coh}(\mathcal{O}_X) \to D^b_{Coh}(\mathcal{O}_Y)$ is called a Fourier-Mukai functor. Similarly for $D_{Coh}, D^+_{Coh}, D^-_{Coh}$.

0FYR   **Lemma 8.2.** *Let S be a scheme. Let X and Y be schemes over S. Let $K \in D(\mathcal{O}_{X \times_S Y})$. The corresponding Fourier-Mukai functor Φ_K sends $D_{QCoh}(\mathcal{O}_X)$ into $D_{QCoh}(\mathcal{O}_Y)$ if K is in $D_{QCoh}(\mathcal{O}_{X \times_S Y})$ and $X \to S$ is quasi-compact and quasi-separated.*

   Proof. This follows from the fact that derived pullback preserves D_{QCoh} (Derived Categories of Schemes, Lemma 3.8), derived tensor products preserve D_{QCoh} (Derived Categories of Schemes, Lemma 3.9), the projection $\mathrm{pr}_2 : X \times_S Y \to Y$ is quasi-compact and quasi-separated (Schemes, Lemmas 19.3 and 21.12), and total direct image along a quasi-separated and quasi-compact morphism preserves D_{QCoh} (Derived Categories of Schemes, Lemma 4.1). $\qquad\square$

0FYS   **Lemma 8.3.** *Let S be a scheme. Let X, Y, Z be schemes over S. Assume $X \to S$, $Y \to S$, and $Z \to S$ are quasi-compact and quasi-separated. Let $K \in D_{QCoh}(\mathcal{O}_{X \times_S Y})$. Let $K' \in D_{QCoh}(\mathcal{O}_{Y \times_S Z})$. Consider the Fourier-Mukai functors $\Phi_K : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ and $\Phi_{K'} : D_{QCoh}(\mathcal{O}_Y) \to D_{QCoh}(\mathcal{O}_Z)$. If X and Z are tor independent over S and $Y \to S$ is flat, then*

$$\Phi_{K'} \circ \Phi_K = \Phi_{K''} : D_{QCoh}(\mathcal{O}_X) \longrightarrow D_{QCoh}(\mathcal{O}_Z)$$

where

$$K'' = R pr_{13,*}(L pr_{12}^* K \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times_S Y \times_S Z}} L pr_{23}^* K')$$

in $D_{QCoh}(\mathcal{O}_{X\times_S Z})$.

Proof. The statement makes sense by Lemma 8.2. We are going to use Derived Categories of Schemes, Lemmas 3.8, 3.9, and 4.1 and Schemes, Lemmas 19.3 and 21.12 without further mention. By Derived Categories of Schemes, Lemma 22.4 we see that $X\times_S Y$ and $Y\times_S Z$ are tor independent over Y. This means that we have base change for the cartesian diagram

$$
\begin{array}{ccc}
X\times_S Y\times_S Z & \longrightarrow & Y\times_S Z \\
\downarrow & & \downarrow{\scriptstyle p_Y^{YZ}} \\
X\times_S Y & \xrightarrow{\;p_Y^{XY}\;} & Y
\end{array}
$$

for complexes with quasi-coherent cohomology sheaves, see Derived Categories of Schemes, Lemma 22.5. Abbreviating $p^* = Lp^*$, $p_* = Rp_*$ and $\otimes = \otimes^{\mathbf{L}}$ we have for $M \in D_{QCoh}(\mathcal{O}_X)$ the sequence of equalities

$$
\begin{aligned}
\Phi_{K'}(\Phi_K(M)) &= p_{Z,*}^{YZ}(p_Y^{YZ,*}p_{Y,*}^{XY}(p_X^{XY,*}M\otimes K)\otimes K') \\
&= p_{Z,*}^{YZ}(\mathrm{pr}_{23,*}\mathrm{pr}_{12}^*(p_X^{XY,*}M\otimes K)\otimes K') \\
&= p_{Z,*}^{YZ}(\mathrm{pr}_{23,*}(\mathrm{pr}_1^*M\otimes \mathrm{pr}_{12}^*K)\otimes K') \\
&= p_{Z,*}^{YZ}(\mathrm{pr}_{23,*}(\mathrm{pr}_1^*M\otimes \mathrm{pr}_{12}^*K\otimes \mathrm{pr}_{23}^*K')) \\
&= \mathrm{pr}_{3,*}(\mathrm{pr}_1^*M\otimes \mathrm{pr}_{12}^*K\otimes \mathrm{pr}_{23}^*K') \\
&= p_{Z,*}^{XZ}\mathrm{pr}_{13,*}(\mathrm{pr}_1^*M\otimes \mathrm{pr}_{12}^*K\otimes \mathrm{pr}_{23}^*K') \\
&= p_{Z,*}^{XZ}(p_X^{XZ,*}M\otimes \mathrm{pr}_{13,*}(\mathrm{pr}_{12}^*K\otimes \mathrm{pr}_{23}^*K'))
\end{aligned}
$$

as desired. Here we have used the remark on base change in the second equality and we have use Derived Categories of Schemes, Lemma 22.1 in the 4th and last equality. $\qquad\square$

0FYT **Lemma 8.4.** *Let S be a scheme. Let X and Y be schemes over S. Let $K \in D(\mathcal{O}_{X\times_S Y})$. The corresponding Fourier-Mukai functor Φ_K sends $D_{perf}(\mathcal{O}_X)$ into $D_{perf}(\mathcal{O}_Y)$ if at least one of the following conditions is satisfied:*

(1) *S is Noetherian, $X \to S$ and $Y \to S$ are of finite type, $K \in D_{Coh}^b(\mathcal{O}_{X\times_S Y})$, the support of $H^i(K)$ is proper over Y for all i, and K has finite tor dimension as an object of $D(pr_2^{-1}\mathcal{O}_Y)$,*

(2) *$X \to S$ is of finite presentation and K can be represented by a bounded complex $\mathcal{K}^\bullet$ of finitely presented $\mathcal{O}_{X\times_S Y}$-modules, flat over Y, with support proper over Y,*

(3) *$X \to S$ is a proper flat morphism of finite presentation and K is perfect,*

(4) *S is Noetherian, $X \to S$ is flat and proper, and K is perfect*

(5) *$X \to S$ is a proper flat morphism of finite presentation and K is Y-perfect,*

(6) *S is Noetherian, $X \to S$ is flat and proper, and K is Y-perfect.*

Proof. If M is perfect on X, then $L\mathrm{pr}_1^*M$ is perfect on $X\times_S Y$, see Cohomology, Lemma 49.6. We will use this without further mention below. We will also use that if $X \to S$ is of finite type, or proper, or flat, or of finite presentation, then the same thing is true for the base change $\mathrm{pr}_2 : X\times_S Y \to Y$, see Morphisms, Lemmas 15.4, 41.5, 25.8, and 21.4.

Part (1) follows from Derived Categories of Schemes, Lemma 27.1 combined with Derived Categories of Schemes, Lemma 11.6.

Part (2) follows from Derived Categories of Schemes, Lemma 30.1.

Part (3) follows from Derived Categories of Schemes, Lemma 30.4.

Part (4) follows from part (3) and the fact that a finite type morphism of Noetherian schemes is of finite presentation by Morphisms, Lemma 21.9.

Part (5) follows from Derived Categories of Schemes, Lemma 35.10 combined with Derived Categories of Schemes, Lemma 35.5.

Part (6) follows from part (5) in the same way that part (4) follows from part (3). $\square$

0FYU **Lemma 8.5.** *Let S be a Noetherian scheme. Let X and Y be schemes of finite type over S. Let $K \in D^b_{Coh}(\mathcal{O}_{X \times_S Y})$. The corresponding Fourier-Mukai functor Φ_K sends $D^b_{Coh}(\mathcal{O}_X)$ into $D^b_{Coh}(\mathcal{O}_Y)$ if at least one of the following conditions is satisfied:*

(1) *the support of $H^i(K)$ is proper over Y for all i, and K has finite tor dimension as an object of $D(pr_1^{-1}\mathcal{O}_X)$,*
(2) *K can be represented by a bounded complex $\mathcal{K}^\bullet$ of coherent $\mathcal{O}_{X \times_S Y}$-modules, flat over X, with support proper over Y,*
(3) *the support of $H^i(K)$ is proper over Y for all i and X is a regular scheme,*
(4) *K is perfect, the support of $H^i(K)$ is proper over Y for all i, and $Y \to S$ is flat.*

Furthermore in each case the support condition is automatic if $X \to S$ is proper.

Proof. Let M be an object of $D^b_{Coh}(\mathcal{O}_X)$. In each case we will use Derived Categories of Schemes, Lemma 11.3 to show that

$$\Phi_K(M) = R\mathrm{pr}_{2,*}(L\mathrm{pr}_1^* M \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times_S Y}} K)$$

is in $D^b_{Coh}(\mathcal{O}_Y)$. The derived tensor product $L\mathrm{pr}_1^* M \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times_S Y}} K$ is a pseudo-coherent object of $D(\mathcal{O}_{X \times_S Y})$ (by Cohomology, Lemma 47.3, Derived Categories of Schemes, Lemma 10.3, and Cohomology, Lemma 47.5) whence has coherent cohomology sheaves (by Derived Categories of Schemes, Lemma 10.3 again). In each case the supports of the cohomology sheaves $H^i(L\mathrm{pr}_1^* M \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times_S Y}} K)$ is proper over Y as these supports are contained in the union of the supports of the $H^i(K)$. Hence in each case it suffices to prove that this tensor product is bounded below.

Case (1). By Cohomology, Lemma 27.4 we have

$$L\mathrm{pr}_1^* M \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times_S Y}} K \cong \mathrm{pr}_1^{-1} M \otimes^{\mathbf{L}}_{\mathrm{pr}_1^{-1}\mathcal{O}_X} K$$

with obvious notation. Hence the assumption on tor dimension and the fact that M has only a finite number of nonzero cohomology sheaves, implies the bound we want.

Case (2) follows because here the assumption implies that K has finite tor dimension as an object of $D(\mathrm{pr}_1^{-1}\mathcal{O}_X)$ hence the argument in the previous paragraph applies.

In Case (3) it is also the case that K has finite tor dimension as an object of $D(\mathrm{pr}_1^{-1}\mathcal{O}_X)$. Namely, choose affine opens $U = \mathrm{Spec}(A)$ and $V = \mathrm{Spec}(B)$ of X and Y mapping into the affine open $W = \mathrm{Spec}(R)$ of S. Then $K|_{U \times V}$ is given by a

bounded complex of finite $A \otimes_R B$-modules $M^\bullet$. Since A is a regular ring of finite dimension we see that each M^i has finite projective dimension as an A-module (Algebra, Lemma 110.8) and hence finite tor dimension as an A-module. Thus $M^\bullet$ has finite tor dimension as a complex of A-modules (More on Algebra, Lemma 66.8). Since $X \times Y$ is quasi-compact we conclude there exist $[a, b]$ such that for every point $z \in X \times Y$ the stalk K_z has tor amplitude in $[a, b]$ over $\mathcal{O}_{X, \mathrm{pr}_1(z)}$. This implies K has bounded tor dimension as an object of $D(\mathrm{pr}_1^{-1}\mathcal{O}_X)$, see Cohomology, Lemma 48.5. We conclude as in the previous to paragraphs.

Case (4). With notation as above, the ring map $R \to B$ is flat. Hence the ring map $A \to A \otimes_R B$ is flat. Hence any projective $A \otimes_R B$-module is A-flat. Thus any perfect complex of $A \otimes_R B$-modules has finite tor dimension as a complex of A-modules and we conclude as before. $\square$

0FYV **Example 8.6.** Let $X \to S$ be a separated morphism of schemes. Then the diagonal $\Delta : X \to X \times_S X$ is a closed immersion and hence $\mathcal{O}_\Delta = \Delta_*\mathcal{O}_X = R\Delta_*\mathcal{O}_X$ is a quasi-coherent $\mathcal{O}_{X \times_S X}$-module of finite type which is flat over X (under either projection). The Fourier-Mukai functor $\Phi_{\mathcal{O}_\Delta}$ is equal to the identity in this case. Namely, for any $M \in D(\mathcal{O}_X)$ we have

$$
\begin{aligned}
L\mathrm{pr}_1^* M \otimes_{\mathcal{O}_{X \times_S X}}^{\mathbf{L}} \mathcal{O}_\Delta &= L\mathrm{pr}_1^* M \otimes_{\mathcal{O}_{X \times_S X}}^{\mathbf{L}} R\Delta_*\mathcal{O}_X \\
&= R\Delta_*(L\Delta^* L\mathrm{pr}_1^* M \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{O}_X) \\
&= R\Delta_*(M)
\end{aligned}
$$

The first equality we discussed above. The second equality is Cohomology, Lemma 54.4. The third because $\mathrm{pr}_1 \circ \Delta = \mathrm{id}_X$ and we have Cohomology, Lemma 27.2. If we push this to X using $R\mathrm{pr}_{2,*}$ we obtain M by Cohomology, Lemma 28.2 and the fact that $\mathrm{pr}_2 \circ \Delta = \mathrm{id}_X$.

0FYW **Lemma 8.7.** *Let $X \to S$ and $Y \to S$ be morphisms of quasi-compact and quasi-separated schemes. Let $\Phi : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ be a Fourier-Mukai functor with pseudo-coherent kernel $K \in D_{QCoh}(\mathcal{O}_{X \times_S Y})$. Let $a : D_{QCoh}(\mathcal{O}_Y) \to D_{QCoh}(\mathcal{O}_{X \times_S Y})$ be the right adjoint to $R\mathrm{pr}_{2,*}$, see Duality for Schemes, Lemma 3.1. Denote* Compare with discussion in [Riz17].

$$
K' = (Y \times_S X \to X \times_S Y)^* R\mathcal{H}om_{\mathcal{O}_{X \times_S Y}}(K, a(\mathcal{O}_Y)) \in D_{QCoh}(\mathcal{O}_{Y \times_S X})
$$

and denote $\Phi' : D_{QCoh}(\mathcal{O}_Y) \to D_{QCoh}(\mathcal{O}_X)$ the corresponding Fourier-Mukai transform. There is a canonical map

$$
\mathrm{Hom}_X(M, \Phi'(N)) \longrightarrow \mathrm{Hom}_Y(\Phi(M), N)
$$

functorial in M in $D_{QCoh}(\mathcal{O}_X)$ and N in $D_{QCoh}(\mathcal{O}_Y)$ which is an isomorphism if

(1) *N is perfect, or*
(2) *K is perfect and $X \to S$ is proper flat and of finite presentation.*

Proof. By Lemma 8.2 we obtain a functor Φ as in the statement. Observe that $a(\mathcal{O}_Y)$ is in $D_{QCoh}^+(\mathcal{O}_{X \times_S Y})$ by Duality for Schemes, Lemma 3.5. Hence for K pseudo-coherent we have $K' \in D_{QCoh}(\mathcal{O}_{Y \times_S X})$ by Derived Categories of Schemes, Lemma 10.8 we we obtain Φ' as indicated.

We abbreviate $\otimes^{\mathbf{L}} = \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times_S Y}}$ and $\mathcal{H}om = R\mathcal{H}om_{\mathcal{O}_{X \times_S Y}}$. Let M be in $D_{QCoh}(\mathcal{O}_X)$ and let N be in $D_{QCoh}(\mathcal{O}_Y)$. We have

$$
\begin{aligned}
\mathrm{Hom}_Y(\Phi(M), N) &= \mathrm{Hom}_Y(R\mathrm{pr}_{2,*}(L\mathrm{pr}_1^* M \otimes^{\mathbf{L}} K), N) \\
&= \mathrm{Hom}_{X \times_S Y}(L\mathrm{pr}_1^* M \otimes^{\mathbf{L}} K, a(N)) \\
&= \mathrm{Hom}_{X \times_S Y}(L\mathrm{pr}_1^* M, R\mathcal{H}om(K, a(N))) \\
&= \mathrm{Hom}_X(M, R\mathrm{pr}_{1,*} R\mathcal{H}om(K, a(N)))
\end{aligned}
$$

where we have used Cohomology, Lemmas 42.2 and 28.1. There are canonical maps

$$
L\mathrm{pr}_2^* N \otimes^{\mathbf{L}} R\mathcal{H}om(K, a(\mathcal{O}_Y)) \xrightarrow{\alpha} R\mathcal{H}om(K, L\mathrm{pr}_2^* N \otimes^{\mathbf{L}} a(\mathcal{O}_Y)) \xrightarrow{\beta} R\mathcal{H}om(K, a(N))
$$

Here α is Cohomology, Lemma 42.6 and β is Duality for Schemes, Equation (8.0.1). Combining all of these arrows we obtain the functorial displayed arrow in the statement of the lemma.

The arrow α is an isomorphism by Derived Categories of Schemes, Lemma 10.9 as soon as either K or N is perfect. The arrow β is an isomorphism if N is perfect by Duality for Schemes, Lemma 8.1 or in general if $X \to S$ is flat proper of finite presentation by Duality for Schemes, Lemma 12.3. $\qquad\square$

0FYX **Lemma 8.8.** *Let S be a Noetherian scheme. Let $Y \to S$ be a flat proper Gorenstein morphism and let $X \to S$ be a finite type morphism. Denote $\omega_{Y/S}^{\bullet}$ the relative dualizing complex of Y over S. Let $\Phi : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ be a Fourier-Mukai functor with perfect kernel $K \in D_{QCoh}(\mathcal{O}_{X \times_S Y})$. Denote*

Compare with discussion in [Riz17].

$$
K' = (Y \times_S X \to X \times_S Y)^*(K^{\vee} \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times_S Y}} L\mathrm{pr}_2^* \omega_{Y/S}^{\bullet}) \in D_{QCoh}(\mathcal{O}_{Y \times_S X})
$$

and denote $\Phi' : D_{QCoh}(\mathcal{O}_Y) \to D_{QCoh}(\mathcal{O}_X)$ the corresponding Fourier-Mukai transform. There is a canonical isomorphism

$$
\mathrm{Hom}_Y(N, \Phi(M)) \longrightarrow \mathrm{Hom}_X(\Phi'(N), M)
$$

functorial in M in $D_{QCoh}(\mathcal{O}_X)$ and N in $D_{QCoh}(\mathcal{O}_Y)$.

Proof. By Lemma 8.2 we obtain a functor Φ as in the statement.

Observe that formation of the relative dualizing complex commutes with base change in our setting, see Duality for Schemes, Remark 12.5. Thus $L\mathrm{pr}_2^* \omega_{Y/S}^{\bullet} = \omega_{X \times_S Y/X}^{\bullet}$. Moreover, we observe that $\omega_{Y/S}^{\bullet}$ is an invertible object of the derived category, see Duality for Schemes, Lemma 25.10, and a fortiori perfect.

To actually prove the lemma we're going to cheat. Namely, we will show that if we replace the roles of X and Y and K and K' then these are as in Lemma 8.7 and we get the result. It is clear that K' is perfect as a tensor product of perfect objects so that the discussion in Lemma 8.7 applies to it. To show that the procedure of Lemma 8.7 applied to K' on $Y \times_S X$ produces a complex isomorphic to K it suffices (details omitted) to show that

$$
R\mathcal{H}om(R\mathcal{H}om(K, \omega_{X \times_S Y/X}^{\bullet}), \omega_{X \times_S Y/X}^{\bullet}) = K
$$

This is clear because K is perfect and $\omega_{X \times_S Y/X}^{\bullet}$ is invertible; details omitted. Thus Lemma 8.7 produces a map

$$
\mathrm{Hom}_Y(N, \Phi(M)) \longrightarrow \mathrm{Hom}_X(\Phi'(N), M)
$$

functorial in M in $D_{QCoh}(\mathcal{O}_X)$ and N in $D_{QCoh}(\mathcal{O}_Y)$ which is an isomorphism because K' is perfect. This finishes the proof. $\square$

0FYY **Lemma 8.9.** *Let S be a Noetherian scheme.*

(1) *For X, Y proper and flat over S and K in $D_{perf}(\mathcal{O}_{X \times_S Y})$ we obtain a Fourier-Mukai functor $\Phi_K : D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_Y)$.*

(2) *For X, Y, Z proper and flat over S, $K \in D_{perf}(\mathcal{O}_{X \times_S Y})$, $K' \in D_{perf}(\mathcal{O}_{Y \times_S Z})$ the composition $\Phi_{K'} \circ \Phi_K : D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_Z)$ is equal to $\Phi_{K''}$ with $K'' \in D_{perf}(\mathcal{O}_{X \times_S Z})$ computed as in Lemma 8.3,*

(3) *For X, Y, K, Φ_K as in (1) if $X \to S$ is Gorenstein, then $\Phi_{K'} : D_{perf}(\mathcal{O}_Y) \to D_{perf}(\mathcal{O}_X)$ is a right adjoint to Φ_K where $K' \in D_{perf}(\mathcal{O}_{Y \times_S X})$ is the pullback of $L\mathrm{pr}_1^* \omega_{X/S}^\bullet \otimes_{\mathcal{O}_{X \times_S Y}}^{\mathbf{L}} K^\vee$ by $Y \times_S X \to X \times_S Y$.*

(4) *For X, Y, K, Φ_K as in (1) if $Y \to S$ is Gorenstein, then $\Phi_{K''} : D_{perf}(\mathcal{O}_Y) \to D_{perf}(\mathcal{O}_X)$ is a left adjoint to Φ_K where $K'' \in D_{perf}(\mathcal{O}_{Y \times_S X})$ is the pullback of $L\mathrm{pr}_2^* \omega_{Y/S}^\bullet \otimes_{\mathcal{O}_{X \times_S Y}}^{\mathbf{L}} K^\vee$ by $Y \times_S X \to X \times_S Y$.*

Proof. Part (1) is immediate from Lemma 8.4 part (4).

Part (2) follows from Lemma 8.3 and the fact that $K'' = R\mathrm{pr}_{13,*}(L\mathrm{pr}_{12}^* K \otimes_{\mathcal{O}_{X \times_S Y \times_S Z}}^{\mathbf{L}} L\mathrm{pr}_{23}^* K')$ is perfect for example by Derived Categories of Schemes, Lemma 27.4.

The adjointness in part (3) on all complexes with quasi-coherent cohomology sheaves follows from Lemma 8.7 with K' equal to the pullback of $R\mathcal{H}om_{\mathcal{O}_{X \times_S Y}}(K, a(\mathcal{O}_Y))$ by $Y \times_S X \to X \times_S Y$ where a is the right adjoint to $R\mathrm{pr}_{2,*} : D_{QCoh}(\mathcal{O}_{X \times_S Y}) \to D_{QCoh}(\mathcal{O}_Y)$. Denote $f : X \to S$ the structure morphism of X. Since f is proper the functor $f^! : D_{QCoh}^+(\mathcal{O}_S) \to D_{QCoh}^+(\mathcal{O}_X)$ is the restriction to $D_{QCoh}^+(\mathcal{O}_S)$ of the right adjoint to $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_S)$, see Duality for Schemes, Section 16. Hence the relative dualizing complex $\omega_{X/S}^\bullet$ as defined in Duality for Schemes, Remark 12.5 is equal to $\omega_{X/S}^\bullet = f^! \mathcal{O}_S$. Since formation of the relative dualizing complex commutes with base change (see Duality for Schemes, Remark 12.5) we see that $a(\mathcal{O}_Y) = L\mathrm{pr}_1^* \omega_{X/S}^\bullet$. Thus

$$R\mathcal{H}om_{\mathcal{O}_{X \times_S Y}}(K, a(\mathcal{O}_Y)) \cong L\mathrm{pr}_1^* \omega_{X/S}^\bullet \otimes_{\mathcal{O}_{X \times_S Y}}^{\mathbf{L}} K^\vee$$

by Cohomology, Lemma 50.5. Finally, since $X \to S$ is assumed Gorenstein the relative dualizing complex is invertible: this follows from Duality for Schemes, Lemma 25.10. We conclude that $\omega_{X/S}^\bullet$ is perfect (Cohomology, Lemma 52.2) and hence K' is perfect. Therefore $\Phi_{K'}$ does indeed map $D_{perf}(\mathcal{O}_Y)$ into $D_{perf}(\mathcal{O}_X)$ which finishes the proof of (3).

The proof of (4) is the same as the proof of (3) except one uses Lemma 8.8 instead of Lemma 8.7. $\square$

9. Resolutions and bounds

0FYZ The diagonal of a smooth proper scheme has a nice resolution.

0FZ0 **Lemma 9.1.** *Let R be a Noetherian ring. Let X, Y be finite type schemes over R having the resolution property. For any coherent $\mathcal{O}_{X \times_R Y}$-module $\mathcal{F}$ there exist a surjection $\mathcal{E} \boxtimes \mathcal{G} \to \mathcal{F}$ where $\mathcal{E}$ is a finite locally free $\mathcal{O}_X$-module and $\mathcal{G}$ is a finite locally free $\mathcal{O}_Y$-module.*

Proof. Let $U \subset X$ and $V \subset Y$ be affine open subschemes. Let $\mathcal{I} \subset \mathcal{O}_X$ be the ideal sheaf of the reduced induced closed subscheme structure on $X \setminus U$. Similarly, let $\mathcal{I}' \subset \mathcal{O}_Y$ be the ideal sheaf of the reduced induced closed subscheme structure on $Y \setminus V$. Then the ideal sheaf

$$\mathcal{J} = \mathrm{Im}(\mathrm{pr}_1^* \mathcal{I} \otimes_{\mathcal{O}_{X \times_R Y}} \mathrm{pr}_2^* \mathcal{I}' \to \mathcal{O}_{X \times_R Y})$$

satisfies $V(\mathcal{J}) = X \times_R Y \setminus U \times_R V$. For any section $s \in \mathcal{F}(U \times_R V)$ we can find an integer $n > 0$ and a map $\mathcal{J}^n \to \mathcal{F}$ whose restriction to $U \times_R V$ gives s, see Cohomology of Schemes, Lemma 10.5. By assumption we can choose surjections $\mathcal{E} \to \mathcal{I}$ and $\mathcal{G} \to \mathcal{I}'$. These produce corresponding surjections

$$\mathcal{E} \boxtimes \mathcal{G} \to \mathcal{J} \quad \text{and} \quad \mathcal{E}^{\otimes n} \boxtimes \mathcal{G}^{\otimes n} \to \mathcal{J}^n$$

and hence a map $\mathcal{E}^{\otimes n} \boxtimes \mathcal{G}^{\otimes n} \to \mathcal{F}$ whose image contains the section s over $U \times_R V$. Since we can cover $X \times_R Y$ by a finite number of affine opens of the form $U \times_R V$ and since $\mathcal{F}|_{U \times_R V}$ is generated by finitely many sections (Properties, Lemma 16.1) we conclude that there exists a surjection

$$\bigoplus_{j=1,\ldots,N} \mathcal{E}_j^{\otimes n_j} \boxtimes \mathcal{G}_j^{\otimes n_j} \to \mathcal{F}$$

where $\mathcal{E}_j$ is finite locally free on X and $\mathcal{G}_j$ is finite locally free on Y. Setting $\mathcal{E} = \bigoplus \mathcal{E}_j^{\otimes n_j}$ and $\mathcal{G} = \bigoplus \mathcal{G}_j^{\otimes n_j}$ we conclude that the lemma is true. $\square$

0FZ1 **Lemma 9.2.** *Let R be a ring. Let X, Y be quasi-compact and quasi-separated schemes over R having the resolution property. For any finite type quasi-coherent $\mathcal{O}_{X \times_R Y}$-module $\mathcal{F}$ there exist a surjection $\mathcal{E} \boxtimes \mathcal{G} \to \mathcal{F}$ where $\mathcal{E}$ is a finite locally free $\mathcal{O}_X$-module and $\mathcal{G}$ is a finite locally free $\mathcal{O}_Y$-module.*

Proof. Follows from Lemma 9.1 by a limit argument. We urge the reader to skip the proof. Since $X \times_R Y$ is a closed subscheme of $X \times_{\mathbf{Z}} Y$ it is harmless if we replace R by $\mathbf{Z}$. We can write $\mathcal{F}$ as the quotient of a finitely presented $\mathcal{O}_{X \times_R Y}$-module by Properties, Lemma 22.8. Hence we may assume $\mathcal{F}$ is of finite presentation. Next we can write $X = \lim X_i$ with X_i of finite presentation over $\mathbf{Z}$ and similarly $Y = \lim Y_j$, see Limits, Proposition 5.4. Then $\mathcal{F}$ will descend to $\mathcal{F}_{ij}$ on some $X_i \times_R Y_j$ (Limits, Lemma 10.2) and so does the property of having the resolution property (Derived Categories of Schemes, Lemma 36.9). Then we apply Lemma 9.1 to $\mathcal{F}_{ij}$ and we pullback. $\square$

0FZ2 **Lemma 9.3.** *Let R be a Noetherian ring. Let X be a separated finite type scheme over R which has the resolution property. Set $\mathcal{O}_\Delta = \Delta_*(\mathcal{O}_X)$ where $\Delta : X \to X \times_R X$ is the diagonal of X/k. There exists a resolution*

$$\ldots \to \mathcal{E}_2 \boxtimes \mathcal{G}_2 \to \mathcal{E}_1 \boxtimes \mathcal{G}_1 \to \mathcal{E}_0 \boxtimes \mathcal{G}_0 \to \mathcal{O}_\Delta \to 0$$

where each $\mathcal{E}_i$ and $\mathcal{G}_i$ is a finite locally free $\mathcal{O}_X$-module.

Proof. Since X is separated, the diagonal morphism Δ is a closed immersion and hence $\mathcal{O}_\Delta$ is a coherent $\mathcal{O}_{X \times_R X}$-module (Cohomology of Schemes, Lemma 9.8). Thus the lemma follows immediately from Lemma 9.1. $\square$

0FZ3 **Lemma 9.4.** *Let X be a regular Noetherian scheme of dimension $d < \infty$. Then*

   (1) *for $\mathcal{F}$, $\mathcal{G}$ coherent $\mathcal{O}_X$-modules we have $\mathrm{Ext}_X^n(\mathcal{F}, \mathcal{G}) = 0$ for $n > d$, and*
   (2) *for $K, L \in D^b_{Coh}(\mathcal{O}_X)$ and $a \in \mathbf{Z}$ if $H^i(K) = 0$ for $i < a + d$ and $H^i(L) = 0$ for $i \geq a$ then $\mathrm{Hom}_X(K, L) = 0$.*

Proof. To prove (1) we use the spectral sequence

$$H^p(X, \mathcal{E}xt^q(\mathcal{F}, \mathcal{G})) \Rightarrow \mathrm{Ext}_X^{p+q}(\mathcal{F}, \mathcal{G})$$

of Cohomology, Section 43. Let $x \in X$. We have

$$\mathcal{E}xt^q(\mathcal{F}, \mathcal{G})_x = \mathcal{E}xt^q_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x)$$

see Cohomology, Lemma 51.4 (this also uses that $\mathcal{F}$ is pseudo-coherent by Derived Categories of Schemes, Lemma 10.3). Set $d_x = \dim(\mathcal{O}_{X,x})$. Since $\mathcal{O}_{X,x}$ is regular the ring $\mathcal{O}_{X,x}$ has global dimension d_x, see Algebra, Proposition 110.1. Thus $\mathcal{E}xt^q_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x)$ is zero for $q > d_x$. It follows that the modules $\mathcal{E}xt^q(\mathcal{F}, \mathcal{G})$ have support of dimension at most $d - q$. Hence we have $H^p(X, \mathcal{E}xt^q(\mathcal{F}, \mathcal{G})) = 0$ for $p > d - q$ by Cohomology, Proposition 20.7. This proves (1).

Proof of (2). We may use induction on the number of nonzero cohomology sheaves of K and L. The case where these numbers are $0, 1$ follows from (1). If the number of nonzero cohomology sheaves of K is > 1, then we let $i \in \mathbf{Z}$ be minimal such that $H^i(K)$ is nonzero. We obtain a distinguished triangle

$$H^i(K)[-i] \to K \to \tau_{\geq i+1}K$$

(Derived Categories, Remark 12.4) and we get the vanishing of $\mathrm{Hom}(K, L)$ from the vanishing of $\mathrm{Hom}(H^i(K)[-i], L)$ and $\mathrm{Hom}(\tau_{\geq i+1}K, L)$ by Derived Categories, Lemma 4.2. Simlarly if L has more than one nonzero cohomology sheaf. $\square$

0FZ4 **Lemma 9.5.** *Let X be a regular Noetherian scheme of dimension $d < \infty$. Let $K \in D^b_{Coh}(\mathcal{O}_X)$ and $a \in \mathbf{Z}$. If $H^i(K) = 0$ for $a < i < a + d$, then $K = \tau_{\leq a}K \oplus \tau_{\geq a+d}K$.*

Proof. We have $\tau_{\leq a}K = \tau_{\leq a+d-1}K$ by the assumed vanishing of cohomology sheaves. By Derived Categories, Remark 12.4 we have a distinguished triangle

$$\tau_{\leq a}K \to K \to \tau_{\geq a+d}K \xrightarrow{\delta} (\tau_{\leq a}K)[1]$$

By Derived Categories, Lemma 4.11 it suffices to show that the morphism δ is zero. This follows from Lemma 9.4. $\square$

0FZ5 **Lemma 9.6.** *Let k be a field. Let X be a quasi-compact separated smooth scheme over k. There exist finite locally free $\mathcal{O}_X$-modules $\mathcal{E}$ and $\mathcal{G}$ such that*

$$\mathcal{O}_\Delta \in \langle \mathcal{E} \boxtimes \mathcal{G} \rangle$$

in $D(\mathcal{O}_{X \times X})$ where the notation is as in Derived Categories, Section 36.

Proof. Recall that X is regular by Varieties, Lemma 25.3. Hence X has the resolution property by Derived Categories of Schemes, Lemma 36.8. Hence we may choose a resolution as in Lemma 9.3. Say $\dim(X) = d$. Since $X \times X$ is smooth over k it is regular. Hence $X \times X$ is a regular Noetherian scheme with $\dim(X \times X) = 2d$. The object

$$K = (\mathcal{E}_{2d} \boxtimes \mathcal{G}_{2d} \to \ldots \to \mathcal{E}_0 \boxtimes \mathcal{G}_0)$$

of $D_{perf}(\mathcal{O}_{X \times X})$ has cohomology sheaves $\mathcal{O}_\Delta$ in degree 0 and $\mathrm{Ker}(\mathcal{E}_{2d} \boxtimes \mathcal{G}_{2d} \to \mathcal{E}_{2d-1} \boxtimes \mathcal{G}_{2d-1})$ in degree $-2d$ and zero in all other degrees. Hence by Lemma 9.5 we see that $\mathcal{O}_\Delta$ is a summand of K in $D_{perf}(\mathcal{O}_{X \times X})$. Clearly, the object K is in

$$\left\langle \bigoplus_{i=0,\ldots,2d} \mathcal{E}_i \boxtimes \mathcal{G}_i \right\rangle \subset \left\langle \left(\bigoplus_{i=0,\ldots,2d} \mathcal{E}_i \right) \boxtimes \left(\bigoplus_{i=0,\ldots,2d} \mathcal{G}_i \right) \right\rangle$$

which finishes the proof. (The reader may consult Derived Categories, Lemmas 36.1 and 35.7 to see that our object is contained in this category.) $\square$

0FZ6 **Lemma 9.7.** *Let k be a field. Let X be a scheme proper and smooth over k. Then $D_{perf}(\mathcal{O}_X)$ has a strong generator.*

Proof. Using Lemma 9.6 choose finite locally free $\mathcal{O}_X$-modules $\mathcal{E}$ and $\mathcal{G}$ such that $\mathcal{O}_\Delta \in \langle \mathcal{E} \boxtimes \mathcal{G} \rangle$ in $D(\mathcal{O}_{X \times X})$. We claim that $\mathcal{G}$ is a strong generator for $D_{perf}(\mathcal{O}_X)$. With notation as in Derived Categories, Section 35 choose $m, n \geq 1$ such that

$$\mathcal{O}_\Delta \in smd(add(\mathcal{E} \boxtimes \mathcal{G}[-m, m])^{\star n})$$

This is possible by Derived Categories, Lemma 36.2. Let K be an object of $D_{perf}(\mathcal{O}_X)$. Since $L\mathrm{pr}_1^* K \otimes_{\mathcal{O}_{X \times X}}^{\mathbf{L}} -$ is an exact functor and since

$$L\mathrm{pr}_1^* K \otimes_{\mathcal{O}_{X \times X}}^{\mathbf{L}} (\mathcal{E} \boxtimes \mathcal{G}) = (K \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{E}) \boxtimes \mathcal{G}$$

we conclude from Derived Categories, Remark 35.5 that

$$L\mathrm{pr}_1^* K \otimes_{\mathcal{O}_{X \times X}}^{\mathbf{L}} \mathcal{O}_\Delta \in smd(add((K \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{E}) \boxtimes \mathcal{G}[-m, m])^{\star n})$$

Applying the exact functor $R\mathrm{pr}_{2,*}$ and observing that

$$R\mathrm{pr}_{2,*}\left((K \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{E}) \boxtimes \mathcal{G}\right) = R\Gamma(X, K \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{E}) \otimes_k \mathcal{G}$$

by Derived Categories of Schemes, Lemma 22.1 we conclude that

$$K = R\mathrm{pr}_{2,*}(L\mathrm{pr}_1^* K \otimes_{\mathcal{O}_{X \times X}}^{\mathbf{L}} \mathcal{O}_\Delta) \in smd(add(R\Gamma(X, K \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{E}) \otimes_k \mathcal{G}[-m, m])^{\star n})$$

The equality follows from the discussion in Example 8.6. Since K is perfect, there exist $a \leq b$ such that $H^i(X, K)$ is nonzero only for $i \in [a, b]$. Since X is proper, each $H^i(X, K)$ is finite dimensional. We conclude that the right hand side is contained in $smd(add(\mathcal{G}[-m + a, m + b])^{\star n})$ which is itself contained in $\langle \mathcal{G} \rangle_n$ by one of the references given above. This finishes the proof. $\square$

0FZ7 **Lemma 9.8.** *Let k be a field. Let X be a proper smooth scheme over k. There exists integers $m, n \geq 1$ and a finite locally free $\mathcal{O}_X$-module $\mathcal{G}$ such that every coherent $\mathcal{O}_X$-module is contained in $smd(add(\mathcal{G}[-m, m])^{\star n})$ with notation as in Derived Categories, Section 35.*

Proof. In the proof of Lemma 9.7 we have shown that there exist $m', n \geq 1$ such that for any coherent $\mathcal{O}_X$-module $\mathcal{F}$,

$$\mathcal{F} \in smd(add(\mathcal{G}[-m' + a, m' + b])^{\star n})$$

for any $a \leq b$ such that $H^i(X, \mathcal{F})$ is nonzero only for $i \in [a, b]$. Thus we can take $a = 0$ and $b = \dim(X)$. Taking $m = \max(m', m' + b)$ finishes the proof. $\square$

The following lemma is the boundedness result referred to in the title of this section.

0FZ8 **Lemma 9.9.** *Let k be a field. Let X be a smooth proper scheme over k. Let $\mathcal{A}$ be an abelian category. Let $H : D_{perf}(\mathcal{O}_X) \to \mathcal{A}$ be a homological functor (Derived Categories, Definition 3.5) such that for all K in $D_{perf}(\mathcal{O}_X)$ the object $H^i(K)$ is nonzero for only a finite number of $i \in \mathbf{Z}$. Then there exists an integer $m \geq 1$ such that $H^i(\mathcal{F}) = 0$ for any coherent $\mathcal{O}_X$-module $\mathcal{F}$ and $i \notin [-m, m]$. Similarly for cohomological functors.*

Proof. Combine Lemma 9.8 with Derived Categories, Lemma 35.8. $\square$

0FZ9 **Lemma 9.10.** *Let k be a field. Let X, Y be finite type schemes over k. Let $K_0 \to K_1 \to K_2 \to \ldots$ be a system of objects of $D_{perf}(\mathcal{O}_{X \times Y})$ and $m \geq 0$ an integer such that*

(1) $H^q(K_i)$ is nonzero only for $q \le m$,
(2) for every coherent $\mathcal{O}_X$-module $\mathcal{F}$ with $\dim(Supp(\mathcal{F})) = 0$ the object

$$Rpr_{2,*}(pr_1^*\mathcal{F} \otimes_{\mathcal{O}_{X \times Y}}^{\mathbf{L}} K_n)$$

has vanishing cohomology sheaves in degrees outside $[-m, m] \cup [-m-n, m-n]$ and for $n > 2m$ the transition maps induce isomorphisms on cohomology sheaves in degrees in $[-m, m]$.

Then K_n has vanishing cohomology sheaves in degrees outside $[-m, m] \cup [-m-n, m-n]$ and for $n > 2m$ the transition maps induce isomorphisms on cohomology sheaves in degrees in $[-m, m]$. Moreover, if X and Y are smooth over k, then for n large enough we find $K_n = K \oplus C_n$ in $D_{perf}(\mathcal{O}_{X \times Y})$ where K has cohomology only in degrees $[-m, m]$ and C_n only in degrees $[-m-n, m-n]$ and the transition maps define isomorphisms between various copies of K.

Proof. Let Z be the scheme theoretic support of an $\mathcal{F}$ as in (2). Then $Z \to \mathrm{Spec}(k)$ is finite, hence $Z \times Y \to Y$ is finite. It follows that for an object M of $D_{QCoh}(\mathcal{O}_{X \times Y})$ with cohomology sheaves supported on $Z \times Y$ we have $H^i(Rpr_{2,*}(M)) = pr_{2,*}H^i(M)$ and the functor $pr_{2,*}$ is faithful on quasi-coherent modules supported on $Z \times Y$; details omitted. Hence we see that the objects

$$pr_1^*\mathcal{F} \otimes_{\mathcal{O}_{X \times Y}}^{\mathbf{L}} K_n$$

in $D_{perf}(\mathcal{O}_{X \times Y})$ have vanishing cohomology sheaves outside $[-m, m] \cup [-m-n, m-n]$ and for $n > 2m$ the transition maps induce isomorphisms on cohomology sheaves in $[-m, m]$. Let $z \in X \times Y$ be a closed point mapping to the closed point $x \in X$. Then we know that

$$K_{n,z} \otimes_{\mathcal{O}_{X \times Y,z}}^{\mathbf{L}} \mathcal{O}_{X \times Y,z}/\mathfrak{m}_x^t \mathcal{O}_{X \times Y,z}$$

has nonzero cohomology only in the intervals $[-m, m] \cup [-m-n, m-n]$. We conclude by More on Algebra, Lemma 100.2 that $K_{n,z}$ only has nonzero cohomology in degrees $[-m, m] \cup [-m-n, m-n]$. Since this holds for all closed points of $X \times Y$, we conclude K_n only has nonzero cohomology sheaves in degrees $[-m, m] \cup [-m-n, m-n]$. In exactly the same way we see that the maps $K_n \to K_{n+1}$ are isomorphisms on cohomology sheaves in degrees $[-m, m]$ for $n > 2m$.

If X and Y are smooth over k, then $X \times Y$ is smooth over k and hence regular by Varieties, Lemma 25.3. Thus we will obtain the direct sum decomposition of K_n as soon as $n > 2m + \dim(X \times Y)$ from Lemma 9.5. The final statement is clear from this. $\square$

10. Sibling functors

0FZS In this section we prove some categorical result on the following notion.

0FZT **Definition 10.1.** Let $\mathcal{A}$ be an abelian category. Let $\mathcal{D}$ be a triangulated category. We say two exact functors of triangulated categories

$$F, F' : D^b(\mathcal{A}) \longrightarrow \mathcal{D}$$

are *siblings*, or we say F' is a *sibling* of F, if the following two conditions are satisfied
    (1) the functors $F \circ i$ and $F' \circ i$ are isomorphic where $i : \mathcal{A} \to D^b(\mathcal{A})$ is the inclusion functor, and
    (2) $F(K) \cong F'(K)$ for any K in $D^b(\mathcal{A})$.

Sometimes the second condition is a consequence of the first.

0FZU **Lemma 10.2.** *Let $\mathcal{A}$ be an abelian category. Let $\mathcal{D}$ be a triangulated category. Let $F, F' : D^b(\mathcal{A}) \longrightarrow \mathcal{D}$ be exact functors of triangulated categories. Assume*

(1) *the functors $F \circ i$ and $F' \circ i$ are isomorphic where $i : \mathcal{A} \to D^b(\mathcal{A})$ is the inclusion functor, and*
(2) *for all $X, Y \in \mathrm{Ob}(\mathcal{A})$ we have $\mathrm{Ext}_{\mathcal{D}}^q(F(X), F(Y)) = 0$ for $q < 0$ (for example if F is fully faithful).*

Then F and F' are siblings.

Proof. Let $K \in D^b(\mathcal{A})$. We will show $F(K)$ is isomorphic to $F'(K)$. We can represent K by a bounded complex $A^\bullet$ of objects of $\mathcal{A}$. After replacing K by a translation we may assume $A^i = 0$ for $i > 0$. Choose $n \geq 0$ such that $A^{-i} = 0$ for $i > n$. The objects

$$M_i = (A^{-i} \to \ldots \to A^0)[-i], \quad i = 0, \ldots, n$$

form a Postnikov system in $D^b(\mathcal{A})$ for the complex $A^\bullet = A^{-n} \to \ldots \to A^0$ in $D^b(\mathcal{A})$. See Derived Categories, Example 41.2. Since both F and F' are exact functors of triangulated categories both

$$F(M_i) \quad \text{and} \quad F'(M_i)$$

form a Postnikov system in $\mathcal{D}$ for the complex

$$F(A^{-n}) \to \ldots \to F(A^0) = F'(A^{-n}) \to \ldots \to F'(A^0)$$

Since all negative Exts between these objects vanish by assumption we conclude by uniqueness of Postnikov systems (Derived Categories, Lemma 41.6) that $F(K) = F(M_n[n]) \cong F'(M_n[n]) = F'(K)$. $\square$

0FZV **Lemma 10.3.** *Let F and F' be siblings as in Definition 10.1. Then*

(1) *if F is essentially surjective, then F' is essentially surjective,*
(2) *if F is fully faithful, then F' is fully faithful.*

Proof. Part (1) is immediate from property (2) for siblings.

Assume F is fully faithful. Denote $\mathcal{D}' \subset \mathcal{D}$ the essential image of F so that $F : D^b(\mathcal{A}) \to \mathcal{D}'$ is an equivalence. Since the functor F' factors through $\mathcal{D}'$ by property (2) for siblings, we can consider the functor $H = F^{-1} \circ F' : D^b(\mathcal{A}) \to D^b(\mathcal{A})$. Observe that H is a sibling of the identity functor. Since it suffices to prove that H is fully faithful, we reduce to the problem discussed in the next paragraph.

Set $\mathcal{D} = D^b(\mathcal{A})$. We have to show a sibling $F : \mathcal{D} \to \mathcal{D}$ of the identity functor is fully faithful. Denote $a_X : X \to F(X)$ the functorial isomorphism for $X \in \mathrm{Ob}(\mathcal{A})$ given to us by Definition 10.1. For any K in $\mathcal{D}$ and distinguished triangle $K_1 \to K_2 \to K_3$ of $\mathcal{D}$ if the maps

$$F : \mathrm{Hom}(K, K_i[n]) \to \mathrm{Hom}(F(K), F(K_i[n]))$$

are isomorphisms for all $n \in \mathbf{Z}$ and $i = 1, 3$, then the same is true for $i = 2$ and all $n \in \mathbf{Z}$. This uses the 5-lemma Homology, Lemma 5.20 and Derived Categories, Lemma 4.2; details omitted. Similarly, if the maps

$$F : \mathrm{Hom}(K_i[n], K) \to \mathrm{Hom}(F(K_i[n]), F(K))$$

are isomorphisms for all $n \in \mathbf{Z}$ and $i = 1, 3$, then the same is true for $i = 2$ and all $n \in \mathbf{Z}$. Using the canonical truncations and induction on the number of nonzero cohomology objects, we see that it is enough to show

$$F : \text{Ext}^q(X, Y) \to \text{Ext}^q(F(X), F(Y))$$

is bijective for all $X, Y \in \text{Ob}(\mathcal{A})$ and all $q \in \mathbf{Z}$. Since F is a sibling of id we have $F(X) \cong X$ and $F(Y) \cong Y$ hence the right hand side is zero for $q < 0$. The case $q = 0$ is OK by our assumption that F is a sibling of the identity functor. It remains to prove the cases $q > 0$.

The case $q = 1$: Injectivity. An element ξ of $\text{Ext}^1(X, Y)$ gives rise to a distinguished triangle

$$Y \to E \to X \xrightarrow{\xi} Y[1]$$

Observe that $E \in \text{Ob}(\mathcal{A})$. Since F is a sibling of the identity functor we obtain a commutative diagram

$$
\begin{array}{ccc}
E & \longrightarrow & X \\
\downarrow & & \downarrow \\
F(E) & \longrightarrow & F(X)
\end{array}
$$

whose vertical arrows are the isomorphisms a_E and a_X. By TR3 the distinguished triangle associated to ξ we started with is isomorphic to the distinguished triangle

$$F(Y) \to F(E) \to F(X) \xrightarrow{F(\xi)} F(Y[1]) = F(Y)[1]$$

Thus $\xi = 0$ if and only if $F(\xi)$ is zero, i.e., we see that $F : \text{Ext}^1(X, Y) \to \text{Ext}^1(F(X), F(Y))$ is injective.

The case $q = 1$: Surjectivity. Let θ be an element of $\text{Ext}^1(F(X), F(Y))$. This defines an extension of $F(X)$ by $F(Y)$ in $\mathcal{A}$ which we may write as $F(E)$ as F is a sibling of the identity functor. We thus get a distinguished triangle

$$F(Y) \xrightarrow{F(\alpha)} F(E) \xrightarrow{F(\beta)} F(X) \xrightarrow{\theta} F(Y[1]) = F(Y)[1]$$

for some morphisms $\alpha : Y \to E$ and $\beta : E \to X$. Since F is a sibling of the identity functor, the sequence $0 \to Y \to E \to X \to 0$ is a short exact sequence in $\mathcal{A}$! Hence we obtain a distinguished triangle

$$Y \xrightarrow{\alpha} E \xrightarrow{\beta} X \xrightarrow{\delta} Y[1]$$

for some morphism $\delta : X \to Y[1]$. Applying the exact functor F we obtain the distinguished triangle

$$F(Y) \xrightarrow{F(\alpha)} F(E) \xrightarrow{F(\beta)} F(X) \xrightarrow{F(\delta)} F(Y)[1]$$

Arguing as above, we see that these triangles are isomorphic. Hence there exists a commutative diagram

$$
\begin{array}{ccc}
F(X) & \xrightarrow{F(\delta)} & F(Y[1]) \\
\downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \epsilon} \\
F(X) & \xrightarrow{\theta} & F(Y[1])
\end{array}
$$

for some isomorphisms γ, ϵ (we can say more but we won't need more information). We may write $\gamma = F(\gamma')$ and $\epsilon = F(\epsilon')$. Then we have $\theta = F(\epsilon' \circ \delta \circ (\gamma')^{-1})$ and we see the surjectivity holds.

The case $q > 1$: surjectivity. Using Yoneda extensions, see Derived Categories, Section 27, we find that for any element ξ in $\operatorname{Ext}^q(F(X), F(Y))$ we can find $F(X) = B_0, B_1, \ldots, B_{q-1}, B_q = F(Y) \in \operatorname{Ob}(\mathcal{A})$ and elements

$$\xi_i \in \operatorname{Ext}^1(B_{i-1}, B_i)$$

such that ξ is the composition $\xi_q \circ \ldots \circ \xi_1$. Write $B_i = F(A_i)$ (of course we have $A_i = B_i$ but we don't need to use this) so that

$$\xi_i = F(\eta_i) \in \operatorname{Ext}^1(F(A_{i-1}), F(A_i)) \quad \text{with} \quad \eta_i \in \operatorname{Ext}^1(A_{i-1}, A_i)$$

by surjectivity for $q = 1$. Then $\eta = \eta_q \circ \ldots \circ \eta_1$ is an element of $\operatorname{Ext}^q(X, Y)$ with $F(\eta) = \xi$.

The case $q > 1$: injectivity. An element ξ of $\operatorname{Ext}^q(X, Y)$ gives rise to a distinguished triangle

$$Y[q-1] \to E \to X \xrightarrow{\xi} Y[q]$$

Applying F we obtain a distinguished triangle

$$F(Y)[q-1] \to F(E) \to F(X) \xrightarrow{F(\xi)} F(Y)[q]$$

If $F(\xi) = 0$, then $F(E) \cong F(Y)[q-1] \oplus F(X)$ in $\mathcal{D}$, see Derived Categories, Lemma 4.11. Since F is a sibling of the identity functor we have $E \cong F(E)$ and hence

$$E \cong F(E) \cong F(Y)[q-1] \oplus F(X) \cong Y[q-1] \oplus X$$

In other words, E is isomorphic to the direct sum of its cohomology objects. This implies that the initial distinguished triangle is split, i.e., $\xi = 0$. $\qquad\square$

Let us make a nonstandard definition. Let $\mathcal{A}$ be an abelian category. Let us say $\mathcal{A}$ *has enough negative objects* if given any $X \in \operatorname{Ob}(\mathcal{A})$ there exists an object N such that

(1) there is a surjection $N \to X$ and
(2) $\operatorname{Hom}(X, N) = 0$.

Let us prove a couple of lemmas about this notion in order to help with the proof of Proposition 10.6.

0GWF **Lemma 10.4.** *Let $\mathcal{A}$ be an abelian category with enough negative objects. Let $X \in D^b(\mathcal{A})$. Let $b \in \mathbf{Z}$ with $H^i(X) = 0$ for $i > b$. Then there exists a map $N[-b] \to X$ such that the induced map $N \to H^b(X)$ is surjective and $\operatorname{Hom}(H^b(X), N) = 0$.*

Proof. Using the truncation functors we can represent X by a complex $A^a \to A^{a+1} \to \ldots \to A^b$ of objects of $\mathcal{A}$. Choose N in $\mathcal{A}$ such that there exists a surjection $t : N \to A^b$ and such that $\operatorname{Hom}(A^b, N) = 0$. Then the surjection t defines a map $N[-b] \to X$ as desired. $\qquad\square$

0GWG **Lemma 10.5.** *Let $\mathcal{A}$ be an abelian category with enough negative objects. Let $f : X \to X'$ be a morphism of $D^b(\mathcal{A})$. Let $b \in \mathbf{Z}$ such that $H^i(X) = 0$ for $i > b$ and $H^i(X') = 0$ for $i \geq b$. Then there exists a map $N[-b] \to X$ such that the induced map $N \to H^b(X)$ is surjective, such that $\operatorname{Hom}(H^b(X), N) = 0$, and such that the composition $N[-b] \to X \to X'$ is zero.*

Proof. We can represent f by a map $f^\bullet : A^\bullet \to B^\bullet$ of bounded complexes of objects of $\mathcal{A}$, see for example Derived Categories, Lemma 11.6. Consider the object

$$C = \mathrm{Ker}(A^b \to A^{b+1}) \times_{\mathrm{Ker}(B^b \to B^{b+1})} B^{b-1}$$

of $\mathcal{A}$. Since $H^b(B^\bullet) = 0$ we see that $C \to H^b(A^\bullet)$ is surjective. On the other hand, the map $C \to A^b \to B^b$ is the same as the map $C \to B^{b-1} \to B^b$ and hence the composition $C[-b] \to X \to X'$ is zero. Since $\mathcal{A}$ has enough negative objects, we can find an object N which has a surjection $N \to C \oplus H^b(X)$ such that $\mathrm{Hom}(C \oplus H^b(X), N) = 0$. Then N together with the map $N[-b] \to X$ is a solution to the problem posed by the lemma. $\square$

We encourage the reader to read the original [Orl97, Proposition 2.16] for the marvellous ideas that go into the proof of the following proposition.

0FZW **Proposition 10.6.** *Let F and F' be siblings as in Definition 10.1. Assume that F is fully faithful and that $\mathcal{A}$ has enough negative objects (see above). Then F and F' are isomorphic functors.*

[Orl97, Proposition 2.16]; the fact that we do not need to assume vanishing of $\mathrm{Ext}^q(N, X)$ for $q > 0$ in the definition of negative objects above is due to [CS14].

Proof. By part (2) of Definition 10.1 the image of the functor F' is contained in the essential image of the functor F. Hence the functor $H = F^{-1} \circ F'$ is a sibling of the identity functor. This reduces us to the case described in the next paragraph.

Let $\mathcal{D} = D^b(\mathcal{A})$. We have to show a sibling $F : \mathcal{D} \to \mathcal{D}$ of the identity functor is isomorphic to the identity functor. Given an object X of $\mathcal{D}$ let us say X has *width* $w = w(X)$ if $w \geq 0$ is minimal such that there exists an integer $a \in \mathbf{Z}$ with $H^i(X) = 0$ for $i \notin [a, a + w - 1]$. Since F is a sibling of the identity and since $F \circ [n] = [n] \circ F$ we are aready given isomorphisms

$$c_X : X \to F(X)$$

for $w(X) \leq 1$ compatible with shifts. Moreover, if $X = A[-a]$ and $X' = A'[-a]$ for some $A, A' \in \mathrm{Ob}(\mathcal{A})$ then for any morphism $f : X \to X'$ the diagram

0FZX (10.6.1)

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & X' \\ {\scriptstyle c_X}\downarrow & & \downarrow{\scriptstyle c_{X'}} \\ F(X) & \xrightarrow{F(f)} & F(X') \end{array}$$

is commutative.

Next, let us show that for any morphism $f : X \to X'$ with $w(X), w(X') \leq 1$ the diagram (10.6.1) commutes. If X or X' is zero, this is clear. If not then we can write $X = A[-a]$ and $X' = A'[-a']$ for unique A, A' in $\mathcal{A}$ and $a, a' \in \mathbf{Z}$. The case $a = a'$ was discussed above. If $a' > a$, then $f = 0$ (Derived Categories, Lemma 27.3) and the result is clear. If $a' < a$ then f corresponds to an element $\xi \in \mathrm{Ext}^q(A, A')$ with $q = a - a'$. Using Yoneda extensions, see Derived Categories, Section 27, we can find $A = A_0, A_1, \ldots, A_{q-1}, A_q = A' \in \mathrm{Ob}(\mathcal{A})$ and elements

$$\xi_i \in \mathrm{Ext}^1(A_{i-1}, A_i)$$

such that ξ is the composition $\xi_q \circ \ldots \circ \xi_1$. In other words, setting $X_i = A_i[-a + i]$ we obtain morphisms

$$X = X_0 \xrightarrow{f_1} X_1 \to \ldots \to X_{q-1} \xrightarrow{f_q} X_q = X'$$

whose compostion is f. Since the commutativity of (10.6.1) for $f_1, \ldots, f_q$ implies it for f, this reduces us to the case $q = 1$. In this case after shifting we may assume we have a distinguished triangle

$$A' \to E \to A \xrightarrow{f} A'[1]$$

Observe that E is an object of $\mathcal{A}$. Consider the following diagram

$$
\begin{array}{ccccccc}
E & \longrightarrow & A & \longrightarrow & A'[1] & \longrightarrow & E[1] \\
{\scriptstyle c_E}\downarrow & & {\scriptstyle c_A}\downarrow & {\scriptstyle f} & {\scriptstyle \gamma}\Big\downarrow {\scriptstyle c_{A'}[1]} & & \downarrow {\scriptstyle c_E[1]} \\
& & & {\scriptstyle \epsilon} & & & \\
F(E) & \longrightarrow & F(A) & \xrightarrow{F(f)} & F(A')[1] & \longrightarrow & F(E)[1]
\end{array}
$$

whose rows are distinguished triangles. The square on the right commutes already but we don't yet know that the middle square does. By the axioms of a triangulated category we can find a morphism γ which does make the diagram commute. Then $\gamma - c_{A'}[1]$ composed with $F(A')[1] \to F(E)[1]$ is zero hence we can find $\epsilon : A'[1] \to F(A)$ such that $\gamma - c_{A'}[1] = F(f) \circ \epsilon$. However, any arrow $A'[1] \to F(A)$ is zero as it is a negative ext class between objects of $\mathcal{A}$. Hence $\gamma = c_{A'}[1]$ and we conclude the middle square commutes too which is what we wanted to show.

To finish the proof we are going to argue by induction on w that there exist isomorphisms $c_X : X \to F(X)$ for all X with $w(X) \le w$ compatible with all morphisms between such objects. The base case $w = 1$ was shown above. Assume we know the result for some $w \ge 1$.

Let X be an object with $w(X) = w+1$. Pick $a \in \mathbf{Z}$ with $H^i(X) = 0$ for $i \notin [a, a+w]$. Set $b = a + w$ so that $H^b(X)$ is nonzero. Choose $N[-b] \to X$ as in Lemma 10.4. Choose a distinguished diagram

$$N[-b] \to X \to Y \to N[-b+1]$$

Computing the long exact cohomology sequence we find $w(Y) \le w$. Hence by induction we find the solid arrows in the following diagram

$$
\begin{array}{ccccccc}
N[-b] & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & N[-b+1] \\
{\scriptstyle c_N[-b]}\downarrow & & {\scriptstyle c_{N[-b]\to X}}\Big\downarrow & & \downarrow {\scriptstyle c_Y} & & \downarrow {\scriptstyle c_N[-b+1]} \\
F(N)[-b] & \longrightarrow & F(X) & \longrightarrow & F(Y) & \longrightarrow & F(N)[-b+1]
\end{array}
$$

We obtain the dotted arrow $c_{N[-b]\to X}$. By Derived Categories, Lemma 4.8 the dotted arrow is unique because $\operatorname{Hom}(X, F(N)[-b]) \cong \operatorname{Hom}(X, N[-b]) = 0$ by our choice of N. In fact, $c_{N[-b]\to X}$ is the unique dotted arrow making the square with vertices $X, Y, F(X), F(Y)$ commute.

Let $N'[-b] \to X$ be another map as in Lemma 10.4 and let us prove that $c_{N[-b]\to X} = c_{N'[-b]\to X}$. Observe that the map $(N \oplus N')[-b] \to X$ also satisfies the conditions of Lemma 10.4. Thus we may assume $N'[-b] \to X$ factors as $N'[-b] \to N[-b] \to X$ for some morphism $N' \to N$. Choose distinguished triangles $N[-b] \to X \to Y \to N[-b+1]$ and $N'[-b] \to X \to Y' \to N'[-b+1]$. By axiom TR3 we can find a morphism $g : Y' \to Y$ which joint with id_X and $N' \to N$ forms a morphism of triangles. Since we have (10.6.1) for g we conclude that

$$(F(X) \to F(Y)) \circ c_{N'[-b]\to X} = (F(X) \to F(Y)) \circ c_{N[-b]\to X}$$

The uniqueness of $c_{N[-b] \to X}$ pointed out in the construction above now shows that $c_{N'[-b] \to X} = c_{N[-b] \to X}$.

Thus we can now define for X of width $w + 1$ the isomorphism $c_X : X \to F(X)$ as the common value of the maps $c_{N[-b] \to X}$ where $N[-b] \to X$ is as in Lemma 10.4. To finish the proof, we have to show that the diagrams (10.6.1) commute for all morphisms $f : X \to X'$ between objects with $w(X) \le w + 1$ and $w(X') \le w + 1$. Choose $a \le b \le a + w$ such that $H^i(X) = 0$ for $i \notin [a, b]$ and $a' \le b' \le a' + w$ such that $H^i(X') = 0$ for $i \notin [a', b']$. We will use induction on $(b' - a') + (b - a)$ to show the claim. (The base case is when this number is zero which is OK because $w \ge 1$.) We distinguish two cases.

Case I: $b' < b$. In this case, by Lemma 10.5 we may choose $N[-b] \to X$ as in Lemma 10.4 such that the composition $N[-b] \to X \to X'$ is zero. Choose a distiuguished triangle $N[-b] \to X \to Y \to N[-b + 1]$. Since $N[-b] \to X'$ is zero, we find that f factors as $X \to Y \to X'$. Since $H^i(Y)$ is nonzero only for $i \in [a, b - 1]$ we see by induction that (10.6.1) commutes for $Y \to X'$. The diagram (10.6.1) commutes for $X \to Y$ by construction if $w(X) = w + 1$ and by our first induction hypothesis if $w(X) \le w$. Hence (10.6.1) commutes for f.

Case II: $b' \ge b$. In this case we choose $N'[-b'] \to X'$ as in Lemma 10.4. We may also assume that $\text{Hom}(H^{b'}(X), N') = 0$ (this is relevant only if $b' = b$), for example because we can replace N' by an object N'' which surjects onto $N' \oplus H^{b'}(X)$ and such that $\text{Hom}(N' \oplus H^{b'}(X), N'') = 0$. We choose a distinguished triangle $N'[-b'] \to X' \to Y' \to N'[-b' + 1]$. Since $\text{Hom}(X, X') \to \text{Hom}(X, Y')$ is injective by our choice of N' (details omitted) the same is true for $\text{Hom}(X, F(X')) \to \text{Hom}(X, F(Y'))$. Hence it suffices in this case to check that (10.6.1) commutes for the composition $X \to Y'$ of the morphisms $X \to X' \to Y'$. Since $H^i(Y')$ is nonzero only for $i \in [a', b' - 1]$ we conclude by induction hypothesis. $\square$

11. Deducing fully faithfulness

0G23   It will be useful for us to know when a functor is fully faithful we offer the following variant of [Orl97, Lemma 2.15].

0G24   **Lemma 11.1.** *Let $F : \mathcal{D} \to \mathcal{D}'$ be an exact functor of triangulated categories. Let $S \subset \text{Ob}(\mathcal{D})$ be a set of objects. Assume*   Variant of [Orl97, Lemma 2.15]

    (1) *F has both right and left adjoints,*
    (2) *for $K \in \mathcal{D}$ if $\text{Hom}(E, K[i]) = 0$ for all $E \in S$ and $i \in \mathbf{Z}$ then $K = 0$,*
    (3) *for $K \in \mathcal{D}$ if $\text{Hom}(K, E[i]) = 0$ for all $E \in S$ and $i \in \mathbf{Z}$ then $K = 0$,*
    (4) *the map $\text{Hom}(E, E'[i]) \to \text{Hom}(F(E), F(E')[i])$ induced by F is bijective for all $E, E' \in S$ and $i \in \mathbf{Z}$.*

Then F is fully faithful.

Proof. Denote F_r and F_l the right and left adjoints of F. For $E \in S$ choose a distinguished triangle

$$E \to F_r(F(E)) \to C \to E[1]$$

where the first arrow is the unit of the adjunction. For $E' \in S$ we have

$$\text{Hom}(E', F_r(F(E))[i]) = \text{Hom}(F(E'), F(E)[i]) = \text{Hom}(E', E[i])$$

The last equality holds by assumption (4). Hence applying the homological functor $\text{Hom}(E', -)$ (Derived Categories, Lemma 4.2) to the distinguished triangle above

we conclude that $\mathrm{Hom}(E', C[i]) = 0$ for all $i \in \mathbf{Z}$ and $E' \in S$. By assumption (2) we conclude that $C = 0$ and $E = F_r(F(E))$.

For $K \in \mathrm{Ob}(\mathcal{D})$ choose a distinguished triangle

$$F_l(F(K)) \to K \to C \to F_l(F(K))[1]$$

where the first arrow is the counit of the adjunction. For $E \in S$ we have

$$\mathrm{Hom}(F_l(F(K)), E[i]) = \mathrm{Hom}(F(K), F(E)[i]) = \mathrm{Hom}(K, F_r(F(E))[i]) = \mathrm{Hom}(K, E[i])$$

where the last equality holds by the result of the first paragraph. Thus we conclude as before that $\mathrm{Hom}(C, E[i]) = 0$ for all $E \in S$ and $i \in \mathbf{Z}$. Hence $C = 0$ by assumption (3). Thus F is fully faithful by Categories, Lemma 24.4. $\square$

0G02 **Lemma 11.2.** *Let k be a field. Let X be a scheme of finite type over k which is regular. Let $x \in X$ be a closed point. For a coherent $\mathcal{O}_X$-module $\mathcal{F}$ supported at x choose a coherent $\mathcal{O}_X$-module $\mathcal{F}'$ supported at x such that $\mathcal{F}_x$ and $\mathcal{F}'_x$ are Matlis dual. Then there is an isomorphism*

$$\mathrm{Hom}_X(\mathcal{F}, M) = H^0(X, M \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F}'[-d_x])$$

where $d_x = \dim(\mathcal{O}_{X,x})$ functorial in M in $D_{perf}(\mathcal{O}_X)$.

Proof. Since $\mathcal{F}$ is supported at x we have

$$\mathrm{Hom}_X(\mathcal{F}, M) = \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, M_x)$$

and similarly we have

$$H^0(X, M \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F}'[-d_x]) = \mathrm{Tor}^{\mathcal{O}_{X,x}}_{d_x}(M_x, \mathcal{F}'_x)$$

Thus it suffices to show that given a Noetherian regular local ring A of dimension d and a finite length A-module N, if N' is the Matlis dual to N, then there exists a functorial isomorphism

$$\mathrm{Hom}_A(N, K) = \mathrm{Tor}^A_d(K, N')$$

for K in $D_{perf}(A)$. We can write the left hand side as $H^0(R\,\mathrm{Hom}_A(N, A) \otimes^{\mathbf{L}}_A K)$ by More on Algebra, Lemma 74.15 and the fact that N determines a perfect object of $D(A)$. Hence the formula holds because

$$R\,\mathrm{Hom}_A(N, A) = R\,\mathrm{Hom}_A(N, A[d])[-d] = N'[-d]$$

by Dualizing Complexes, Lemma 16.4 and the fact that $A[d]$ is a normalized dualizing complex over A (A is Gorenstein by Dualizing Complexes, Lemma 21.3). $\square$

0G03 **Lemma 11.3.** *Let k be a field. Let X be a scheme of finite type over k which is regular. Let $x \in X$ be a closed point and denote $\mathcal{O}_x$ the skyscraper sheaf at x with value $\kappa(x)$. Let K in $D_{perf}(\mathcal{O}_X)$.*

(1) *If $\mathrm{Ext}^i_X(\mathcal{O}_x, K) = 0$ then there exists an open neighbourhood U of x such that $H^{i-d_x}(K)|_U = 0$ where $d_x = \dim(\mathcal{O}_{X,x})$.*
(2) *If $\mathrm{Hom}_X(\mathcal{O}_x, K[i]) = 0$ for all $i \in \mathbf{Z}$, then K is zero in an open neighbourhood of x.*
(3) *If $\mathrm{Ext}^i_X(K, \mathcal{O}_x) = 0$ then there exists an open neighbourhood U of x such that $H^i(K^\vee)|_U = 0$.*
(4) *If $\mathrm{Hom}_X(K, \mathcal{O}_x[i]) = 0$ for all $i \in \mathbf{Z}$, then K is zero in an open neighbourhood of x.*

(5) If $H^i(X, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{O}_x) = 0$ then there exists an open neighbourhood U of x such that $H^i(K)|_U = 0$.
(6) If $H^i(X, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{O}_x) = 0$ for $i \in \mathbf{Z}$ then K is zero in an open neighbourhood of x.

Proof. Observe that $H^i(X, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{O}_x)$ is equal to $K_x \otimes^{\mathbf{L}}_{\mathcal{O}_{X,x}} \kappa(x)$. Hence part (5) follows from More on Algebra, Lemma 76.4. Part (6) follows from part (5). Part (1) follows from part (5), Lemma 11.2, and the fact that the Matlis dual of $\kappa(x)$ is $\kappa(x)$. Part (2) follows from part (1). Part (3) follows from part (5) and the fact that $\operatorname{Ext}^i(K, \mathcal{O}_x) = H^i(X, K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{O}_x)$ by Cohomology, Lemma 50.5. Part (4) follows from part (3) and the fact that $K \cong (K^\vee)^\vee$ by the lemma just cited. $\square$

0GWZ **Lemma 11.4.** *Let X be a Noetherian scheme. Let $x \in X$ be a closed point and denote $\mathcal{O}_x$ the skyscraper sheaf at x with value $\kappa(x)$. Let K in $D^b_{Coh}(\mathcal{O}_X)$. Let $b \in \mathbf{Z}$. The following are equivalent*

(1) $H^i(K)_x = 0$ for all $i > b$ and
(2) $\operatorname{Hom}_X(K, \mathcal{O}_x[-i]) = 0$ for all $i > b$.

Proof. Consider the complex K_x in $D^b_{Coh}(\mathcal{O}_{X,x})$. There exist an integer $b_x \in \mathbf{Z}$ such that K_x can be represented by a bounded above complex

$$\ldots \to \mathcal{O}^{\oplus n_{b_x - 2}}_{X,x} \to \mathcal{O}^{\oplus n_{b_x - 1}}_{X,x} \to \mathcal{O}^{\oplus n_{b_x}}_{X,x} \to 0 \to \ldots$$

with $\mathcal{O}^{\oplus n_i}_{X,x}$ sitting in degree i where all the transition maps are given by matrices whose coefficients are in $\mathfrak{m}_x$. See More on Algebra, Lemma 75.5. The result follows easily from this (and the equivalent conditions hold if and only if $b \geq b_x$). $\square$

0G25 **Lemma 11.5.** *Let k be a field. Let X and Y be proper schemes over k. Assume X is regular. Then a k-linear exact functor $F : D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_Y)$ is fully faithful if and only if for any closed points $x, x' \in X$ the maps*

$$F : \operatorname{Ext}^i_X(\mathcal{O}_x, \mathcal{O}_{x'}) \longrightarrow \operatorname{Ext}^i_Y(F(\mathcal{O}_x), F(\mathcal{O}_{x'}))$$

are isomorphisms for all $i \in \mathbf{Z}$. Here $\mathcal{O}_x$ is the skyscraper sheaf at x with value $\kappa(x)$.

Proof. By Lemma 7.1 the functor F has both a left and a right adjoint. Thus we may apply the criterion of Lemma 11.1 because assumptions (2) and (3) of that lemma follow from Lemma 11.3. $\square$

0G26 **Lemma 11.6.** *Let k be a field. Let X be a proper scheme over k which is regular. Let $F : D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_X)$ be a k-linear exact functor. Assume for every coherent $\mathcal{O}_X$-module $\mathcal{F}$ with $\dim(Supp(\mathcal{F})) = 0$ there is an isomorphism $\mathcal{F} \cong F(\mathcal{F})$. Then F is fully faithful.* Email from Noah Olander of Jun 9, 2020

Proof. By Lemma 11.5 it suffices to show that the maps

$$F : \operatorname{Ext}^i_X(\mathcal{O}_x, \mathcal{O}_{x'}) \longrightarrow \operatorname{Ext}^i_X(F(\mathcal{O}_x), F(\mathcal{O}_{x'}))$$

are isomorphisms for all $i \in \mathbf{Z}$ and all closed points $x, x' \in X$. By assumption, the source and the target are isomorphic. If $x \neq x'$, then both sides are zero and the result is true. If $x = x'$, then it suffices to prove that the map is either injective or surjective. For $i < 0$ both sides are zero and the result is true. For $i = 0$ any nonzero map $\alpha : \mathcal{O}_x \to \mathcal{O}_x$ of $\mathcal{O}_X$-modules is an isomorphism. Hence $F(\alpha)$ is an

isomorphism too and so $F(\alpha)$ is nonzero. Thus the result for $i = 0$. For $i = 1$ a nonzero element ξ in $\mathrm{Ext}^1(\mathcal{O}_x, \mathcal{O}_x)$ corresponds to a nonsplit short exact sequence

$$0 \to \mathcal{O}_x \to \mathcal{F} \to \mathcal{O}_x \to 0$$

Since $F(\mathcal{F}) \cong \mathcal{F}$ we see that $F(\mathcal{F})$ is a nonsplit extension of $\mathcal{O}_x$ by $\mathcal{O}_x$ as well. Since $\mathcal{O}_x \cong F(\mathcal{O}_x)$ is a simple $\mathcal{O}_X$-module and $\mathcal{F} \cong F(\mathcal{F})$ has length 2, we see that in the distinguished triangle

$$F(\mathcal{O}_x) \to F(\mathcal{F}) \to F(\mathcal{O}_x) \xrightarrow{F(\xi)} F(\mathcal{O}_x)[1]$$

the first two arrows must form a short exact sequence which must be isomorphic to the above short exact sequence and hence is nonsplit. It follows that $F(\xi)$ is nonzero and we conclude for $i = 1$. For $i > 1$ composition of ext classes defines a surjection

$$\mathrm{Ext}^1(F(\mathcal{O}_x), F(\mathcal{O}_x)) \otimes \ldots \otimes \mathrm{Ext}^1(F(\mathcal{O}_x), F(\mathcal{O}_x)) \longrightarrow \mathrm{Ext}^i(F(\mathcal{O}_x), F(\mathcal{O}_x))$$

See Duality for Schemes, Lemma 15.4. Hence surjectivity in degree 1 implies surjectivity for $i > 0$. This finishes the proof. $\square$

12. Special functors

0FZY   In this section we prove some results on functors of a special type that we will use later in this chapter.

0FZZ   **Definition 12.1.** Let k be a field. Let X, Y be finite type schemes over k. Recall that $D^b_{Coh}(\mathcal{O}_X) = D^b(Coh(\mathcal{O}_X))$ by Derived Categories of Schemes, Proposition 11.2. We say two k-linear exact functors

$$F, F' : D^b_{Coh}(\mathcal{O}_X) = D^b(Coh(\mathcal{O}_X)) \longrightarrow D^b_{Coh}(\mathcal{O}_Y)$$

are *siblings*, or we say F' is a *sibling* of F if F and F' are siblings in the sense of Definition 10.1 with abelian category being $Coh(\mathcal{O}_X)$. If X is regular then $D_{perf}(\mathcal{O}_X) = D^b_{Coh}(\mathcal{O}_X)$ by Derived Categories of Schemes, Lemma 11.6 and we use the same terminology for k-linear exact functors $F, F' : D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_Y)$.

0G00   **Lemma 12.2.** *Let k be a field. Let X, Y be finite type schemes over k with X separated. Let $F : D^b_{Coh}(\mathcal{O}_X) \to D^b_{Coh}(\mathcal{O}_Y)$ be a k-linear exact functor sending $Coh(\mathcal{O}_X) \subset D^b_{Coh}(\mathcal{O}_X)$ into $Coh(\mathcal{O}_Y) \subset D^b_{Coh}(\mathcal{O}_Y)$. Then there exists a Fourier-Mukai functor $F' : D^b_{Coh}(\mathcal{O}_X) \to D^b_{Coh}(\mathcal{O}_Y)$ whose kernel is a coherent $\mathcal{O}_{X \times Y}$-module $\mathcal{K}$ flat over X and with support finite over Y which is a sibling of F.*

Proof. Denote $H : Coh(\mathcal{O}_X) \to Coh(\mathcal{O}_Y)$ the restriction of F. Since F is an exact functor of triangulated categories, we see that H is an exact functor of abelian categories. Of course H is k-linear as F is. By Functors and Morphisms, Lemma 7.5 we obtain a coherent $\mathcal{O}_{X \times Y}$-module $\mathcal{K}$ which is flat over X and has support finite over Y. Let F' be the Fourier-Mukai functor defined using $\mathcal{K}$ so that F' restricts to H on $Coh(\mathcal{O}_X)$. The functor F' sends $D^b_{Coh}(\mathcal{O}_X)$ into $D^b_{Coh}(\mathcal{O}_Y)$ by Lemma 8.5. Observe that F and F' satisfy the first and second condition of Lemma 10.2 and hence are siblings. $\square$

0G01   **Remark 12.3.** If $F, F' : D^b_{Coh}(\mathcal{O}_X) \to \mathcal{D}$ are siblings, F is fully faithful, and X is reduced and projective over k then $F \cong F'$; this follows from Proposition 10.6 via the argument given in the proof of Theorem 13.3. However, in general we do not know whether siblings are isomorphic. Even in the situation of Lemma 12.2 it

seems difficult to prove that the siblings F and F' are isomorphic functors. If X is smooth and proper over k and F is fully faithful, then $F \cong F'$ as is shown in [Ola20]. If you have a proof or a counter example in more general situations, please email stacks.project@gmail.com.

0GX0 **Lemma 12.4.** *Let k be a field. Let X, Y be proper schemes over k. Assume X is regular. Let $F, G : D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_Y)$ be k-linear exact functors such that*

(1) *$F(\mathcal{F}) \cong G(\mathcal{F})$ for any coherent $\mathcal{O}_X$-module $\mathcal{F}$ with $\dim(Supp(\mathcal{F})) = 0$,*
(2) *F is fully faithful.*

Then the essential image of G is contained in the essential image of F.

Proof. Recall that F and G have both adjoints, see Lemma 7.1. In particular the essential image $\mathcal{A} \subset D_{perf}(\mathcal{O}_Y)$ of F satisfies the equivalent conditions of Derived Categories, Lemma 40.7. We claim that G factors through $\mathcal{A}$. Since $\mathcal{A} = {}^{\perp}(\mathcal{A}^{\perp})$ by Derived Categories, Lemma 40.7 it suffices to show that $\mathrm{Hom}_Y(G(M), N) = 0$ for all M in $D_{perf}(\mathcal{O}_X)$ and $N \in \mathcal{A}^{\perp}$. We have

$$\mathrm{Hom}_Y(G(M), N) = \mathrm{Hom}_X(M, G_r(N))$$

where G_r is the right adjoint to G. Thus it suffices to prove that $G_r(N) = 0$. Since $G(\mathcal{F}) \cong F(\mathcal{F})$ for $\mathcal{F}$ as in (1) we see that

$$\mathrm{Hom}_X(\mathcal{F}, G_r(N)) = \mathrm{Hom}_Y(G(\mathcal{F}), N) = \mathrm{Hom}_Y(F(\mathcal{F}), N) = 0$$

as N is in the right orthogonal to the essential image $\mathcal{A}$ of F. Of course, the same vanishing holds for $\mathrm{Hom}_X(\mathcal{F}, G_r(N)[i])$ for any $i \in \mathbf{Z}$. Thus $G_r(N) = 0$ by Lemma 11.3 and we win. $\square$

0G27 **Lemma 12.5.** *Let k be a field. Let X be a proper scheme over k which is regular. Let $F : D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_X)$ be a k-linear exact functor. Assume for every coherent $\mathcal{O}_X$-module $\mathcal{F}$ with $\dim(Supp(\mathcal{F})) = 0$ there is an isomorphism $\mathcal{F} \cong F(\mathcal{F})$. Then there exists an automorphism $f : X \to X$ over k which induces the identity on the underlying topological space[1] and an invertible $\mathcal{O}_X$-module $\mathcal{L}$ such that F and $F'(M) = f^*M \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{L}$ are siblings.*

Proof. By Lemma 11.6 the functor F is fully faithful. By Lemma 12.4 the essential image of the identity functor is contained in the essential image of F, i.e., we see that F is essentially surjective. Thus F is an equivalence. Observe that the quasi-inverse F^{-1} satisfies the same assumptions as F.

Let $M \in D_{perf}(\mathcal{O}_X)$ and say $H^i(M) = 0$ for $i > b$. Since F is fully faithful, we see that

$$\mathrm{Hom}_X(M, \mathcal{O}_x[-i]) = \mathrm{Hom}_X(F(M), F(\mathcal{O}_x)[-i]) \cong \mathrm{Hom}_X(F(M), \mathcal{O}_x[-i])$$

for any $i \in \mathbf{Z}$ for any closed point x of X. Thus by Lemma 11.4 we see that $F(M)$ has vanishing cohomology sheaves in degrees $> b$.

Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. By the above $F(\mathcal{F})$ has nonzero cohomology sheaves only in degrees ≤ 0. Set $\mathcal{G} = H^0(F(\mathcal{F}))$. Choose a distinguished triangle

$$K \to F(\mathcal{F}) \to \mathcal{G} \to K[1]$$

---

[1]This often forces f to be the identity, see Varieties, Lemma 32.1.

Then K has nonvanishing cohomology sheaves only in degrees ≤ -1. Applying F^{-1} we obtain a distinguished triangle

$$F^{-1}(K) \to \mathcal{F} \to F^{-1}(\mathcal{G}) \to F^{-1}(K')[1]$$

Since $F^{-1}(K)$ has nonvanishing cohomology sheaves only in degrees ≤ -1 (by the previous paragraph applied to F^{-1}) we see that the arrow $F^{-1}(K) \to \mathcal{F}$ is zero (Derived Categories, Lemma 27.3). Hence $K \to F(\mathcal{F})$ is zero, which implies that $F(\mathcal{F}) = \mathcal{G}$ by our choice of the first distinguished triangle.

From the preceding paragraph, we deduce that F preserves $Coh(\mathcal{O}_X)$ and indeed defines an equivalence $H : Coh(\mathcal{O}_X) \to Coh(\mathcal{O}_X)$. By Functors and Morphisms, Lemma 7.8 we get an automorphism $f : X \to X$ over k and an invertible $\mathcal{O}_X$-module $\mathcal{L}$ such that $H(\mathcal{F}) = f^*\mathcal{F} \otimes \mathcal{L}$. Set $F'(M) = f^*M \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{L}$. Using Lemma 10.2 we see that F and F' are siblings. To see that f is the identity on the underlying topological space of X, we use that $F(\mathcal{O}_x) \cong \mathcal{O}_x$ and that the support of $\mathcal{O}_x$ is $\{x\}$. This finishes the proof. $\square$

0G06 **Lemma 12.6.** *Let k be a field. Let X, Y be proper schemes over k. Assume X regular. Let $F, G : D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_Y)$ be k-linear exact functors such that*

(1) *$F(\mathcal{F}) \cong G(\mathcal{F})$ for any coherent $\mathcal{O}_X$-module $\mathcal{F}$ with $\dim(Supp(\mathcal{F})) = 0$,*
(2) *F is fully faithful, and*
(3) *G is a Fourier-Mukai functor whose kernel is in $D_{perf}(\mathcal{O}_{X \times Y})$.*

Then there exists a Fourier-Mukai functor $F' : D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_Y)$ whose kernel is in $D_{perf}(\mathcal{O}_{X \times Y})$ such that F and F' are siblings.

Proof. The essential image of G is contained in the essential image of F by Lemma 12.4. Consider the functor $H = F^{-1} \circ G$ which makes sense as F is fully faithful. By Lemma 12.5 we obtain an automorphism $f : X \to X$ and an invertible $\mathcal{O}_X$-module $\mathcal{L}$ such that the functor $H' : K \mapsto f^*K \otimes \mathcal{L}$ is a sibling of H. In particular H is an auto-equivalence by Lemma 10.3 and H induces an auto-equivalence of $Coh(\mathcal{O}_X)$ (as this is true for its sibling functor H'). Thus the quasi-inverses H^{-1} and $(H')^{-1}$ exist, are siblings (small detail omitted), and $(H')^{-1}$ sends M to $(f^{-1})^*(M \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{L}^{\otimes-1})$ which is a Fourier-Mukai functor (details omitted). Then of course $F = G \circ H^{-1}$ is a sibling of $G \circ (H')^{-1}$. Since compositions of Fourier-Mukai functors are Fourier-Mukai by Lemma 8.3 we conclude. $\square$

13. Fully faithful functors

0G07 Our goal is to prove fully faithful functors between derived categories are siblings of Fourier-Mukai functors, following [Orl97] and [Bal08].

0G08 **Situation 13.1.** Here k is a field. We have proper smooth schemes X and Y over k. We have a k-linear, exact, fully faithful functor $F : D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_Y)$.

Before reading on, it makes sense to read at least some of Derived Categories, Section 41.

Recall that X is regular and hence has the resolution property (Varieties, Lemma 25.3 and Derived Categories of Schemes, Lemma 36.8). Thus on $X \times X$ we may choose a resolution

$$\ldots \to \mathcal{E}_2 \boxtimes \mathcal{G}_2 \to \mathcal{E}_1 \boxtimes \mathcal{G}_1 \to \mathcal{E}_0 \boxtimes \mathcal{G}_0 \to \mathcal{O}_\Delta \to 0$$

where each $\mathcal{E}_i$ and $\mathcal{G}_i$ is a finite locally free $\mathcal{O}_X$-module, see Lemma 9.3. Using the complex

0G09   (13.1.1) $\ldots \to \mathcal{E}_2 \boxtimes \mathcal{G}_2 \to \mathcal{E}_1 \boxtimes \mathcal{G}_1 \to \mathcal{E}_0 \boxtimes \mathcal{G}_0$

in $D_{perf}(\mathcal{O}_{X \times X})$ as in Derived Categories, Example 41.2 if for each n we denote

$$M_n = (\mathcal{E}_n \boxtimes \mathcal{G}_n \to \ldots \to \mathcal{E}_0 \boxtimes \mathcal{G}_0)[-n]$$

we obtain an infinite Postnikov system for the complex (13.1.1). This means the morphisms $M_0 \to M_1[1] \to M_2[2] \to \ldots$ and $M_n \to \mathcal{E}_n \boxtimes \mathcal{G}_n$ and $\mathcal{E}_n \boxtimes \mathcal{G}_n \to M_{n-1}$ satisfy certain conditions documented in Derived Categories, Definition 41.1. Set

$$\mathcal{F}_n = \mathrm{Ker}(\mathcal{E}_n \boxtimes \mathcal{G}_n \to \mathcal{E}_{n-1} \boxtimes \mathcal{G}_{n-1})$$

Observe that since $\mathcal{O}_\Delta$ is flat over X via pr_1 the same is true for $\mathcal{F}_n$ for all n (this is a convenient though not essential observation). We have

$$H^q(M_n[n]) = \begin{cases} \mathcal{O}_\Delta & \text{if} & q = 0 \\ \mathcal{F}_n & \text{if} & q = -n \\ 0 & \text{if} & q \neq 0, -n \end{cases}$$

Thus for $n \geq \dim(X \times X)$ we have

$$M_n[n] \cong \mathcal{O}_\Delta \oplus \mathcal{F}_n[n]$$

in $D_{perf}(\mathcal{O}_{X \times X})$ by Lemma 9.5.

We are interested in the complex

0G0A   (13.1.2) $\ldots \to \mathcal{E}_2 \boxtimes F(\mathcal{G}_2) \to \mathcal{E}_1 \boxtimes F(\mathcal{G}_1) \to \mathcal{E}_0 \boxtimes F(\mathcal{G}_0)$

in $D_{perf}(\mathcal{O}_{X \times Y})$ as the "totalization" of this complex should give us the kernel of the Fourier-Mukai functor we are trying to construct. For all $i, j \geq 0$ we have

$$\mathrm{Ext}^q_{X \times Y}(\mathcal{E}_i \boxtimes F(\mathcal{G}_i), \mathcal{E}_j \boxtimes F(\mathcal{G}_j)) = \bigoplus_p \mathrm{Ext}^{q+p}_X(\mathcal{E}_i, \mathcal{E}_j) \otimes_k \mathrm{Ext}^{-p}_Y(F(\mathcal{G}_i), F(\mathcal{G}_j))$$

$$= \bigoplus_p \mathrm{Ext}^{q+p}_X(\mathcal{E}_i, \mathcal{E}_j) \otimes_k \mathrm{Ext}^{-p}_X(\mathcal{G}_i, \mathcal{G}_j)$$

The second equality holds because F is fully faithful and the first by Derived Categories of Schemes, Lemma 25.1. We find these Ext^q are zero for $q < 0$. Hence by Derived Categories, Lemma 41.6 we can build an infinite Postnikov system $K_0, K_1, K_2, \ldots$ in $D_{perf}(\mathcal{O}_{X \times Y})$ for the complex (13.1.2). Parallel to what happens with $M_0, M_1, M_2, \ldots$ this means we obtain morphisms $K_0 \to K_1[1] \to K_2[2] \to \ldots$ and $K_n \to \mathcal{E}_n \boxtimes F(\mathcal{G}_n)$ and $\mathcal{E}_n \boxtimes F(\mathcal{G}_n) \to K_{n-1}$ in $D_{perf}(\mathcal{O}_{X \times Y})$ satisfying certain conditions documented in Derived Categories, Definition 41.1.

Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module whose support has a finite number of points, i.e., with $\dim(\mathrm{Supp}(\mathcal{F})) = 0$. Consider the exact functor of triangulated categories

$$D_{perf}(\mathcal{O}_{X \times Y}) \longrightarrow D_{perf}(\mathcal{O}_Y), \quad N \longmapsto R\mathrm{pr}_{2,*}(\mathrm{pr}_1^* \mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times Y}} N)$$

It follows that the objects $R\mathrm{pr}_{2,*}(\mathrm{pr}_1^* \mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times Y}} K_i)$ form a Postnikov system for the complex in $D_{perf}(\mathcal{O}_Y)$ with terms

$$R\mathrm{pr}_{2,*}((\mathcal{F} \otimes \mathcal{E}_i) \boxtimes F(\mathcal{G}_i)) = \Gamma(X, \mathcal{F} \otimes \mathcal{E}_i) \otimes_k F(\mathcal{G}_i) = F(\Gamma(X, \mathcal{F} \otimes \mathcal{E}_i) \otimes_k \mathcal{G}_i)$$

Here we have used that $\mathcal{F} \otimes \mathcal{E}_i$ has vanishing higher cohomology as its support has dimension 0. On the other hand, applying the exact functor

$$D_{perf}(\mathcal{O}_{X \times X}) \longrightarrow D_{perf}(\mathcal{O}_Y), \quad N \longmapsto F(R\mathrm{pr}_{2,*}(\mathrm{pr}_1^* \mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times X}} N))$$

we find that the objects $F(R\mathrm{pr}_{2,*}(\mathrm{pr}_1^*\mathcal{F} \otimes_{\mathcal{O}_{X\times X}}^{\mathbf{L}} M_n))$ form a second infinite Post-nikov system for the complex in $D_{perf}(\mathcal{O}_Y)$ with terms

$$F(R\mathrm{pr}_{2,*}((\mathcal{F} \otimes \mathcal{E}_i) \boxtimes \mathcal{G}_i)) = F(\Gamma(X, \mathcal{F} \otimes \mathcal{E}_i) \otimes_k \mathcal{G}_i)$$

This is the same as before! By uniqueness of Postnikov systems (Derived Categories, Lemma 41.6) which applies because

$$\mathrm{Ext}_Y^q(F(\Gamma(X, \mathcal{F} \otimes \mathcal{E}_i) \otimes_k \mathcal{G}_i), F(\Gamma(X, \mathcal{F} \otimes \mathcal{E}_j) \otimes_k \mathcal{G}_j)) = 0, \quad q < 0$$

as F is fully faithful, we find a system of isomorphisms

$$F(R\mathrm{pr}_{2,*}(\mathrm{pr}_1^*\mathcal{F} \otimes_{\mathcal{O}_{X\times X}}^{\mathbf{L}} M_n[n])) \cong R\mathrm{pr}_{2,*}(\mathrm{pr}_1^*\mathcal{F} \otimes_{\mathcal{O}_{X\times Y}}^{\mathbf{L}} K_n[n])$$

in $D_{perf}(\mathcal{O}_Y)$ compatible with the morphisms in $D_{perf}(\mathcal{O}_Y)$ induced by the morphisms

$$M_{n-1}[n-1] \to M_n[n] \quad \text{and} \quad K_{n-1}[n-1] \to K_n[n]$$
$$M_n \to \mathcal{E}_n \boxtimes \mathcal{G}_n \quad \text{and} \quad K_n \to \mathcal{E}_n \boxtimes F(\mathcal{G}_n)$$
$$\mathcal{E}_n \boxtimes \mathcal{G}_n \to M_{n-1} \quad \text{and} \quad \mathcal{E}_n \boxtimes F(\mathcal{G}_n) \to K_{n-1}$$

which are part of the structure of Postnikov systems. For n sufficiently large we obtain a direct sum decomposition

$$F(R\mathrm{pr}_{2,*}(\mathrm{pr}_1^*\mathcal{F} \otimes_{\mathcal{O}_{X\times X}}^{\mathbf{L}} M_n[n])) = F(\mathcal{F}) \oplus F(R\mathrm{pr}_{2,*}(\mathrm{pr}_1^*\mathcal{F} \otimes_{\mathcal{O}_{X\times Y}} \mathcal{F}_n))[n]$$

corresponding to the direct sum decomposition of M_n constructed above (we are using the flatness of $\mathcal{F}_n$ over X via pr_1 to write a usual tensor product in the formula above, but this isn't essential for the argument). By Lemma 9.9 we find there exists an integer $m \geq 0$ such that the first summand in this direct sum decomposition has nonzero cohomology sheaves only in the interval $[-m, m]$ and the second summand in this direct sum decomposition has nonzero cohomology sheaves only in the interval $[-m - n, m + \dim(X) - n]$. We conclude the system $K_0 \to K_1[1] \to K_2[2] \to \ldots$ in $D_{perf}(\mathcal{O}_{X\times Y})$ satisfies the assumptions of Lemma 9.10 after possibly replacing m by a larger integer. We conclude we can write

$$K_n[n] = K \oplus C_n$$

for $n \gg 0$ compatible with transition maps and with C_n having nonzero cohomology sheaves only in the range $[-m - n, m - n]$. Denote G the Fourier-Mukai functor corresponding to K. Putting everything together we find

$$G(\mathcal{F}) \oplus R\mathrm{pr}_{2,*}(\mathrm{pr}_1^*\mathcal{F} \otimes_{\mathcal{O}_{X\times Y}}^{\mathbf{L}} C_n) \cong$$
$$R\mathrm{pr}_{2,*}(\mathrm{pr}_1^*\mathcal{F} \otimes_{\mathcal{O}_{X\times Y}}^{\mathbf{L}} K_n[n]) \cong$$
$$F(R\mathrm{pr}_{2,*}(\mathrm{pr}_1^*\mathcal{F} \otimes_{\mathcal{O}_{X\times X}}^{\mathbf{L}} M_n[n])) \cong$$
$$F(\mathcal{F}) \oplus F(R\mathrm{pr}_{2,*}(\mathrm{pr}_1^*\mathcal{F} \otimes_{\mathcal{O}_{X\times Y}} \mathcal{F}_n))[n]$$

Looking at the degrees that objects live in we conclude that for $n \gg m$ we obtain an isomorphism

$$F(\mathcal{F}) \cong G(\mathcal{F})$$

Moreover, recall that this holds for every coherent $\mathcal{F}$ on X whose support has dimension 0.

0G0B **Lemma 13.2.** *Let k be a field. Let X and Y be smooth proper schemes over k. Given a k-linear, exact, fully faithful functor $F : D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_Y)$ there exists a Fourier-Mukai functor $F' : D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_Y)$ whose kernel is in $D_{perf}(\mathcal{O}_{X\times Y})$ which is a sibling to F.*

Proof. Apply Lemma 12.6 to F and the functor G constructed above. $\square$

The following theorem is also true without assuming X is projective, see [Ola20].

0G0C **Theorem 13.3** (Orlov)**.** *Let k be a field. Let X and Y be smooth proper schemes over k with X projective over k. Any k-linear fully faithful exact functor $F :$ $D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_Y)$ is a Fourier-Mukai functor for some kernel in $D_{perf}(\mathcal{O}_{X \times Y})$.*

[Orl97, Theorem 2.2]; this is shown in [Ola20] without the assumption that X be projective

Proof. Let F' be the Fourier-Mukai functor which is a sibling of F as in Lemma 13.2. By Proposition 10.6 we have $F \cong F'$ provided we can show that $Coh(\mathcal{O}_X)$ has enough negative objects. However, if $X = \mathrm{Spec}(k)$ for example, then this isn't true. Thus we first decompose $X = \coprod X_i$ into its connected (and irreducible) components and we argue that it suffices to prove the result for each of the (fully faithful) composition functors

$$F_i : D_{perf}(\mathcal{O}_{X_i}) \to D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_Y)$$

Details omitted. Thus we may assume X is irreducible.

The case $\dim(X) = 0$. Here X is the spectrum of a finite (separable) extension k'/k and hence $D_{perf}(\mathcal{O}_X)$ is equivalent to the category of graded k'-vector spaces such that $\mathcal{O}_X$ corresponds to the trivial 1-dimensional vector space in degree 0. It is straightforward to see that any two siblings $F, F' : D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_Y)$ are isomorphic. Namely, we are given an isomorphism $F(\mathcal{O}_X) \cong F'(\mathcal{O}_X)$ compatible the action of the k-algebra $k' = \mathrm{End}_{D_{perf}(\mathcal{O}_X)}(\mathcal{O}_X)$ which extends canonically to an isomorphism on any graded k'-vector space.

The case $\dim(X) > 0$. Here X is a projective smooth variety of dimension > 1. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. We have to show there exists a coherent module $\mathcal{N}$ such that

(1) there is a surjection $\mathcal{N} \to \mathcal{F}$ and
(2) $\mathrm{Hom}(\mathcal{F}, \mathcal{N}) = 0$.

Choose an ample invertible $\mathcal{O}_X$-module $\mathcal{L}$. We claim that $\mathcal{N} = (\mathcal{L}^{\otimes n})^{\oplus r}$ will work for $n \ll 0$ and r large enough. Condition (1) follows from Properties, Proposition 26.13. Finally, we have

$$\mathrm{Hom}(\mathcal{F}, \mathcal{L}^{\otimes n}) = H^0(X, \mathcal{H}om(\mathcal{F}, \mathcal{L}^{\otimes n})) = H^0(X, \mathcal{H}om(\mathcal{F}, \mathcal{O}_X) \otimes \mathcal{L}^{\otimes n})$$

Since the dual $\mathcal{H}om(\mathcal{F}, \mathcal{O}_X)$ is torsion free, this vanishes for $n \ll 0$ by Varieties, Lemma 48.1. This finishes the proof. $\square$

0G0D **Proposition 13.4.** *Let k be a field. Let X and Y be smooth proper schemes over k. If $F : D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_Y)$ is a k-linear exact equivalence of triangulated categories then there exists a Fourier-Mukai functor $F' : D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_Y)$ whose kernel is in $D_{perf}(\mathcal{O}_{X \times Y})$ which is an equivalence and a sibling of F.*

Proof. The functor F' of Lemma 13.2 is an equivalence by Lemma 10.3. $\square$

0G0E **Lemma 13.5.** *Let k be a field. Let X be a smooth proper scheme over k. Let $K \in D_{perf}(\mathcal{O}_{X \times X})$. If the Fourier-Mukai functor $\Phi_K : D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_X)$ is isomorphic to the identity functor, then $K \cong \Delta_* \mathcal{O}_X$ in $_{perf}(\mathcal{O}_{X \times X})$.*

Proof. Let i be the minimal integer such that the cohomology sheaf $H^i(K)$ is nonzero. Let $\mathcal{E}$ and $\mathcal{G}$ be finite locally free $\mathcal{O}_X$-modules. Then

$$H^i(X \times X, K \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times X}} (\mathcal{E} \boxtimes \mathcal{G})) = H^i(X, R\mathrm{pr}_{2,*}(K \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times X}} (\mathcal{E} \boxtimes \mathcal{G})))$$
$$= H^i(X, \Phi_K(\mathcal{E}) \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G})$$
$$\cong H^i(X, \mathcal{E} \otimes \mathcal{G})$$

which is zero if $i < 0$. On the other hand, we can choose $\mathcal{E}$ and $\mathcal{G}$ such that there is a surjection $\mathcal{E}^\vee \boxtimes \mathcal{G}^\vee \to H^i(K)$ by Lemma 9.1. In this case the left hand side of the equalities is nonzero. Hence we conclude that $H^i(K) = 0$ for $i < 0$.

Let i be the maximal integer such that $H^i(K)$ is nonzero. The same argument with $\mathcal{E}$ and $\mathcal{G}$ support of dimension 0 shows that $i \leq 0$. Hence we conclude that K is given by a single coherent $\mathcal{O}_{X \times X}$-module $\mathcal{K}$ sitting in degree 0.

Since $R\mathrm{pr}_{2,*}(\mathrm{pr}_1^* \mathcal{F} \otimes \mathcal{K})$ is $\mathcal{F}$, by taking $\mathcal{F}$ supported at closed points we see that the support of $\mathcal{K}$ is finite over X via pr_2. Since $R\mathrm{pr}_{2,*}(\mathcal{K}) \cong \mathcal{O}_X$ we conclude by Functors and Morphisms, Lemma 7.6 that $\mathcal{K} = s_* \mathcal{O}_X$ for some section $s : X \to X \times X$ of the second projection. Then $\Phi_K(M) = f^* M$ where $f = \mathrm{pr}_1 \circ s$ and this can happen only if s is the diagonal morphism as desired. $\square$

14. A category of Fourier-Mukai kernels

0G0F   Let S be a scheme. We claim there is a category with

(1) Objects are proper smooth schemes over S.
(2) Morphisms from X to Y are isomorphism classes of objects of $D_{perf}(\mathcal{O}_{X \times_S Y})$.
(3) Composition of the isomorphism class of $K \in D_{perf}(\mathcal{O}_{X \times_S Y})$ and the isomorphism class of K' in $D_{perf}(\mathcal{O}_{Y \times_S Z})$ is the isomorphism class of

$$R\mathrm{pr}_{13,*}(L\mathrm{pr}_{12}^* K \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times_S Y \times_S Z}} L\mathrm{pr}_{23}^* K')$$

which is in $D_{perf}(\mathcal{O}_{X \times_S Z})$ by Derived Categories of Schemes, Lemma 30.4.
(4) The identity morphism from X to X is the isomorphism class of $\Delta_{X/S,*} \mathcal{O}_X$ which is in $D_{perf}(\mathcal{O}_{X \times_S X})$ by More on Morphisms, Lemma 61.12 and the fact that $\Delta_{X/S}$ is a perfect morphism by Divisors, Lemma 22.11 and More on Morphisms, Lemma 61.7.

Let us check that associativity of composition of morphisms holds; we omit verifying that the identity morphisms are indeed identities. To see this suppose we have X, Y, Z, W and $c \in D_{perf}(\mathcal{O}_{X \times_S Y})$, $c' \in D_{perf}(\mathcal{O}_{Y \times_S Z})$, and $c'' \in D_{perf}(\mathcal{O}_{Z \times_S W})$. Then we have

$$c'' \circ (c' \circ c) \cong \mathrm{pr}^{134}_{14,*}(\mathrm{pr}^{134,*}_{13}\mathrm{pr}^{123}_{13,*}(\mathrm{pr}^{123,*}_{12} c \otimes \mathrm{pr}^{123,*}_{23} c') \otimes \mathrm{pr}^{134,*}_{34} c'')$$
$$\cong \mathrm{pr}^{134}_{14,*}(\mathrm{pr}^{1234}_{134,*}\mathrm{pr}^{1234,*}_{123}(\mathrm{pr}^{123,*}_{12} c \otimes \mathrm{pr}^{123,*}_{23} c') \otimes \mathrm{pr}^{134,*}_{34} c'')$$
$$\cong \mathrm{pr}^{134}_{14,*}(\mathrm{pr}^{1234}_{134,*}(\mathrm{pr}^{1234,*}_{12} c \otimes \mathrm{pr}^{1234,*}_{23} c') \otimes \mathrm{pr}^{134,*}_{34} c'')$$
$$\cong \mathrm{pr}^{134}_{14,*}\mathrm{pr}^{1234}_{134,*}((\mathrm{pr}^{1234,*}_{12} c \otimes \mathrm{pr}^{1234,*}_{23} c') \otimes \mathrm{pr}^{1234,*}_{34} c'')$$
$$\cong \mathrm{pr}^{1234}_{14,*}((\mathrm{pr}^{1234,*}_{12} c \otimes \mathrm{pr}^{1234,*}_{23} c') \otimes \mathrm{pr}^{1234,*}_{34} c'')$$

Here we use the notation

$$p^{1234}_{134} : X \times_S Y \times_S Z \times_S W \to X \times_S Z \times_S W \quad \text{and} \quad p^{134}_{14} : X \times_S Z \times_S W \to X \times_S W$$

the projections and similarly for other indices. We also write pr_* instead of $R\mathrm{pr}_*$ and pr^* instead of $L\mathrm{pr}^*$ and we drop all super and sub scripts on $\otimes$. The first equality is the definition of the composition. The second equality holds because $\mathrm{pr}_{13}^{134,*}\mathrm{pr}_{13,*}^{123} = \mathrm{pr}_{134,*}^{1234}\mathrm{pr}_{123}^{1234,*}$ by base change (Derived Categories of Schemes, Lemma 22.5). The third equality holds because pullbacks compose correctly and pass through tensor products, see Cohomology, Lemmas 27.2 and 27.3. The fourth equality follows from the "projection formula" for p_{134}^{1234}, see Derived Categories of Schemes, Lemma 22.1. The fifth equality is that proper pushforward is compatible with composition, see Cohomology, Lemma 28.2. Since tensor product is associative this concludes the proof of associativity of composition.

0G0G **Lemma 14.1.** *Let $S' \to S$ be a morphism of schemes. The rule which sends*

  (1) *a smooth proper scheme X over S to $X' = S' \times_S X$, and*
  (2) *the isomorphism class of an object K of $D_{perf}(\mathcal{O}_{X \times_S Y})$ to the isomorphism class of $L(X' \times_{S'} Y' \to X \times_S Y)^*K$ in $D_{perf}(\mathcal{O}_{X' \times_{S'} Y'})$*

is a functor from the category defined for S to the category defined for S'.

Proof. To see this suppose we have X, Y, Z and $K \in D_{perf}(\mathcal{O}_{X \times_S Y})$ and $M \in D_{perf}(\mathcal{O}_{Y \times_S Z})$. Denote $K' \in D_{perf}(\mathcal{O}_{X' \times_{S'} Y'})$ and $M' \in D_{perf}(\mathcal{O}_{Y' \times_{S'} Z'})$ their pullbacks as in the statement of the lemma. The diagram

$$
\begin{array}{ccc}
X' \times_{S'} Y' \times_{S'} Z' & \longrightarrow & X \times_S Y \times_S Z \\
\downarrow{\scriptstyle \mathrm{pr}'_{13}} & & \downarrow{\scriptstyle \mathrm{pr}_{13}} \\
X' \times_{S'} Z' & \longrightarrow & X \times_S Z
\end{array}
$$

is cartesian and pr_{13} is proper and smooth. By Derived Categories of Schemes, Lemma 30.4 we see that the derived pullback by the lower horizontal arrow of the composition

$$R\mathrm{pr}_{13,*}(L\mathrm{pr}_{12}^*K \otimes_{\mathcal{O}_{X \times_S Y \times_S Z}}^{\mathbf{L}} L\mathrm{pr}_{23}^*M)$$

indeed is (canonically) isomorphic to

$$R\mathrm{pr}'_{13,*}(L(\mathrm{pr}'_{12})^*K' \otimes_{\mathcal{O}_{X' \times_{S'} Y' \times_{S'} Z'}}^{\mathbf{L}} L(\mathrm{pr}'_{23})^*M')$$

as desired. Some details omitted. $\square$

15. Relative equivalences

0G0H In this section we prove some lemmas about the following concept.

0G0I **Definition 15.1.** Let S be a scheme. Let $X \to S$ and $Y \to S$ be smooth proper morphisms. An object $K \in D_{perf}(\mathcal{O}_{X \times_S Y})$ is said to be *the Fourier-Mukai kernel of a relative equivalence from X to Y over S* if there exist an object $K' \in D_{perf}(\mathcal{O}_{X \times_S Y})$ such that

$$\Delta_{X/S,*}\mathcal{O}_X \cong R\mathrm{pr}_{13,*}(L\mathrm{pr}_{12}^*K \otimes_{\mathcal{O}_{X \times_S Y \times_S X}}^{\mathbf{L}} L\mathrm{pr}_{23}^*K')$$

in $D(\mathcal{O}_{X \times_S X})$ and

$$\Delta_{Y/S,*}\mathcal{O}_Y \cong R\mathrm{pr}_{13,*}(L\mathrm{pr}_{12}^*K' \otimes_{\mathcal{O}_{Y \times_S X \times_S Y}}^{\mathbf{L}} L\mathrm{pr}_{23}^*K)$$

in $D(\mathcal{O}_{Y \times_S Y})$. In other words, the isomorphism class of K defines an invertible arrow in the category defined in Section 14.

The language is intentionally cumbersome.

0G0J   **Lemma 15.2.** *With notation as in Definition 15.1 let K be the Fourier-Mukai kernel of a relative equivalence from X to Y over S. Then the corresponding Fourier-Mukai functors $\Phi_K : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ (Lemma 8.2) and $\Phi_K : D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_Y)$ (Lemma 8.4) are equivalences.*

Proof. Immediate from Lemma 8.3 and Example 8.6. $\square$

0G0K   **Lemma 15.3.** *With notation as in Definition 15.1 let K be the Fourier-Mukai kernel of a relative equivalence from X to Y over S. Let $S_1 \to S$ be a morphism of schemes. Let $X_1 = S_1 \times_S X$ and $Y_1 = S_1 \times_S Y$. Then the pullback $K_1 = L(X_1 \times_{S_1} Y_1 \to X \times_S Y)^* K$ is the Fourier-Mukai kernel of a relative equivalence from X_1 to Y_1 over S_1.*

Proof. Let $K' \in D_{perf}(\mathcal{O}_{Y \times_S X})$ be the object assumed to exist in Definition 15.1. Denote K'_1 the pullback of K' by $Y_1 \times_{S_1} X_1 \to Y \times_S X$. Then it suffices to prove that we have

$$\Delta_{X_1/S_1,*}\mathcal{O}_X \cong R\mathrm{pr}_{13,*}(L\mathrm{pr}_{12}^* K_1 \otimes^{\mathbf{L}}_{\mathcal{O}_{X_1 \times_{S_1} Y_1 \times_{S_1} X_1}} L\mathrm{pr}_{23}^* K'_1)$$

in $D(\mathcal{O}_{X_1 \times_{S_1} X_1})$ and similarly for the other condition. Since

$$
\begin{array}{ccc}
X_1 \times_{S_1} Y_1 \times_{S_1} X_1 & \longrightarrow & X \times_S Y \times_S X \\
{\scriptstyle \mathrm{pr}_{13}}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_{13}} \\
X_1 \times_{S_1} X_1 & \longrightarrow & X \times_S X
\end{array}
$$

is cartesian it suffices by Derived Categories of Schemes, Lemma 30.4 to prove that

$$\Delta_{X_1/S_1,*}\mathcal{O}_{X_1} \cong L(X_1 \times_{S_1} X_1 \to X \times_S X)^* \Delta_{X/S,*}\mathcal{O}_X$$

This in turn will be true if X and $X_1 \times_{S_1} X_1$ are tor independent over $X \times_S X$, see Derived Categories of Schemes, Lemma 22.5. This tor independence can be seen directly but also follows from the more general More on Morphisms, Lemma 69.1 applied to the square with corners X, X, X, S and its base change by $S_1 \to S$. $\square$

0G0L   **Lemma 15.4.** *Let $S = \lim_{i \in I} S_i$ be a limit of a directed system of schemes with affine transition morphisms $g_{i'i} : S_{i'} \to S_i$. We assume that S_i is quasi-compact and quasi-separated for all $i \in I$. Let $0 \in I$. Let $X_0 \to S_0$ and $Y_0 \to S_0$ be smooth proper morphisms. We set $X_i = S_i \times_{S_0} X_0$ for $i \geq 0$ and $X = S \times_{S_0} X_0$ and similarly for Y_0. If K is the Fourier-Mukai kernel of a relative equivalence from X to Y over S then for some $i \geq 0$ there exists a Fourier-Mukai kernel of a relative equivalence from X_i to Y_i over S_i.*

Proof. Let $K' \in D_{perf}(\mathcal{O}_{Y \times_S X})$ be the object assumed to exist in Definition 15.1. Since $X \times_S Y = \lim X_i \times_{S_i} Y_i$ there exists an i and objects K_i and K'_i in $D_{perf}(\mathcal{O}_{Y_i \times_{S_i} X_i})$ whose pullbacks to $Y \times_S X$ give K and K'. See Derived Categories of Schemes, Lemma 29.3. By Derived Categories of Schemes, Lemma 30.4 the object

$$R\mathrm{pr}_{13,*}(L\mathrm{pr}_{12}^* K_i \otimes^{\mathbf{L}}_{\mathcal{O}_{X_i \times_{S_i} Y_i \times_{S_i} X_i}} L\mathrm{pr}_{23}^* K'_i)$$

is perfect and its pullback to $X \times_S X$ is equal to

$$R\mathrm{pr}_{13,*}(L\mathrm{pr}_{12}^* K \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times_S Y \times_S X}} L\mathrm{pr}_{23}^* K') \cong \Delta_{X/S,*}\mathcal{O}_X$$

See proof of Lemma 15.3. On the other hand, since $X_i \to S$ is smooth and separated the object

$$\Delta_{i,*}\mathcal{O}_{X_i}$$

of $D(\mathcal{O}_{X_i \times_{S_i} X_i})$ is also perfect (by More on Morphisms, Lemmas 62.18 and 61.13) and its pullback to $X \times_S X$ is equal to

$$\Delta_{X/S,*}\mathcal{O}_X$$

See proof of Lemma 15.3. Thus by Derived Categories of Schemes, Lemma 29.3 after increasing i we may assume that

$$\Delta_{i,*}\mathcal{O}_{X_i} \cong R\mathrm{pr}_{13,*}(L\mathrm{pr}_{12}^*K_i \otimes^{\mathbf{L}}_{\mathcal{O}_{X_i \times_{S_i} Y_i \times_{S_i} X_i}} L\mathrm{pr}_{23}^*K_i')$$

as desired. The same works for the roles of K and K' reversed. $\qquad\square$

16. No deformations

0G0M   The title of this section refers to Lemma 16.4

0G0N   **Lemma 16.1.** *Let $(R, \mathfrak{m}, \kappa) \to (A, \mathfrak{n}, \lambda)$ be a flat local ring homorphism of local rings which is essentially of finite presentation. Let $\overline{f}_1, \ldots, \overline{f}_r \in \mathfrak{n}/\mathfrak{m}A \subset A/\mathfrak{m}A$ be a regular sequence. Let $K \in D(A)$. Assume*

(1) *K is perfect,*
(2) *$K \otimes^{\mathbf{L}}_A A/\mathfrak{m}A$ is isomorphic in $D(A/\mathfrak{m}A)$ to the Koszul complex on $\overline{f}_1, \ldots, \overline{f}_r$.*

Then K is isomorphic in $D(A)$ to a Koszul complex on a regular sequence $f_1, \ldots, f_r \in A$ lifting the given elements $\overline{f}_1, \ldots, \overline{f}_r$. Moreover, $A/(f_1, \ldots, f_r)$ is flat over R.

Proof. Let us use chain complexes in the proof of this lemma. The Koszul complex $K_\bullet(\overline{f}_1, \ldots, \overline{f}_r)$ is defined in More on Algebra, Definition 28.2. By More on Algebra, Lemma 75.4 we can represent K by a complex

$$K_\bullet : A \to A^{\oplus r} \to \ldots \to A^{\oplus r} \to A$$

whose tensor product with $A/\mathfrak{m}A$ is equal (!) to $K_\bullet(\overline{f}_1, \ldots, \overline{f}_r)$. Denote $f_1, \ldots, f_r \in A$ the components of the arrow $A^{\oplus r} \to A$. These f_i are lifts of the $\overline{f}_i$. By Algebra, Lemma 128.6 $f_1, \ldots, f_r$ form a regular sequence in A and $A/(f_1, \ldots, f_r)$ is flat over R. Let $J = (f_1, \ldots, f_r) \subset A$. Consider the diagram

$$
\begin{array}{ccc}
K_\bullet & \cdots\cdots\xrightarrow{\varphi_\bullet}\cdots\cdots & K_\bullet(f_1, \ldots, f_r) \\
 & \searrow \qquad \swarrow & \\
 & A/J &
\end{array}
$$

Since $f_1, \ldots, f_r$ is a regular sequence the south-west arrow is a quasi-isomorphism (see More on Algebra, Lemma 30.2). Hence we can find the dotted arrow making the diagram commute for example by Algebra, Lemma 71.4. Reducing modulo $\mathfrak{m}$ we obtain a commutative diagram

$$
\begin{array}{ccc}
K_\bullet(\overline{f}_1, \ldots, \overline{f}_r) & \xrightarrow{\overline{\varphi}_\bullet} & K_\bullet(\overline{f}_1, \ldots, \overline{f}_r) \\
 & \searrow \qquad \swarrow & \\
 & (A/\mathfrak{m}A)/(\overline{f}_1, \ldots, \overline{f}_r) &
\end{array}
$$

by our choice of $K_\bullet$. Thus $\overline{\varphi}$ is an isomorphism in the derived category $D(A/\mathfrak{m}A)$. It follows that $\overline{\varphi} \otimes^{\mathbf{L}}_{A/\mathfrak{m}A} \lambda$ is an isomorphism. Since $\overline{f}_i \in \mathfrak{n}/\mathfrak{m}A$ we see that

$$\mathrm{Tor}_i^{A/\mathfrak{m}A}(K_\bullet(\overline{f}_1, \ldots, \overline{f}_r), \lambda) = K_i(\overline{f}_1, \ldots, \overline{f}_r) \otimes_{A/\mathfrak{m}A} \lambda$$

Hence φ_i mod $\mathfrak{n}$ is invertible. Since A is local this means that φ_i is an isomorphism and the proof is complete. $\square$

0G0P **Lemma 16.2.** *Let $R \to S$ be a finite type flat ring map of Noetherian rings. Let $\mathfrak{q} \subset S$ be a prime ideal lying over $\mathfrak{p} \subset R$. Let $K \in D(S)$ be perfect. Let $f_1, \ldots, f_r \in \mathfrak{q}S_{\mathfrak{q}}$ be a regular sequence such that $S_{\mathfrak{q}}/(f_1, \ldots, f_r)$ is flat over R and such that $K \otimes_S^{\mathbf{L}} S_{\mathfrak{q}}$ is isomorphic to the Koszul complex on $f_1, \ldots, f_r$. Then there exists a $g \in S$, $g \notin \mathfrak{q}$ such that*

(1) *$f_1, \ldots, f_r$ are the images of $f_1', \ldots, f_r' \in S_g$,*
(2) *$f_1', \ldots, f_r'$ form a regular sequence in S_g,*
(3) *$S_g/(f_1', \ldots, f_r')$ is flat over R,*
(4) *$K \otimes_S^{\mathbf{L}} S_g$ is isomorphic to the Koszul complex on $f_1, \ldots, f_r$.*

Proof. We can find $g \in S$, $g \notin \mathfrak{q}$ with property (1) by the definition of localizations. After replacing g by gg' for some $g' \in S$, $g' \notin \mathfrak{q}$ we may assume (2) holds, see Algebra, Lemma 68.6. By Algebra, Theorem 129.4 we find that $S_g/(f_1', \ldots, f_r')$ is flat over R in an open neighbourhood of $\mathfrak{q}$. Hence after once more replacing g by gg' for some $g' \in S$, $g' \notin \mathfrak{q}$ we may assume (3) holds as well. Finally, we get (4) for a further replacement by More on Algebra, Lemma 74.17. $\square$

For a generalization of the following lemma, please see More on Morphisms of Spaces, Lemma 49.6.

0G0Q **Lemma 16.3.** *Let S be a Noetherian scheme. Let $s \in S$. Let $p : X \to Y$ be a morphism of schemes over S. Assume*

(1) *$Y \to S$ and $X \to S$ proper,*
(2) *X is flat over S,*
(3) *$X_s \to Y_s$ an isomorphism.*

Then there exists an open neighbourhood $U \subset S$ of s such that the base change $X_U \to Y_U$ is an isomorphism.

Proof. The morphism p is proper by Morphisms, Lemma 41.6. By Cohomology of Schemes, Lemma 21.2 there is an open $Y_s \subset V \subset Y$ such that $p|_{p^{-1}(V)} : p^{-1}(V) \to V$ is finite. By More on Morphisms, Theorem 16.1 there is an open $X_s \subset U \subset X$ such that $p|_U : U \to Y$ is flat. After removing the images of $X \setminus U$ and $Y \setminus V$ (which are closed subsets not containing s) we may assume p is flat and finite. Then p is open (Morphisms, Lemma 25.10) and $Y_s \subset p(X) \subset Y$ hence after shrinking S we may assume p is surjective. As $p_s : X_s \to Y_s$ is an isomorphism, the map

$$p^{\sharp} : \mathcal{O}_Y \longrightarrow p_* \mathcal{O}_X$$

of coherent $\mathcal{O}_Y$-modules (p is finite) becomes an isomorphism after pullback by $i : Y_s \to Y$ (by Cohomology of Schemes, Lemma 5.1 for example). By Nakayama's lemma, this implies that $\mathcal{O}_{Y,y} \to (p_* \mathcal{O}_X)_y$ is surjective for all $y \in Y_s$. Hence there is an open $Y_s \subset V \subset Y$ such that $p^{\sharp}|_V$ is surjective (Modules, Lemma 9.4). Hence after shrinking S once more we may assume $p^{\sharp}$ is surjective which means that p is a closed immersion (as p is already finite). Thus now p is a surjective flat closed immersion of Noetherian schemes and hence an isomorphism, see Morphisms, Section 26. $\square$

0G0R **Lemma 16.4.** *Let k be a field. Let S be a finite type scheme over k with k-rational point s. Let $Y \to S$ be a smooth proper morphism. Let $X = Y_s \times S \to S$ be*

the constant family with fibre Y_s. Let K be the Fourier-Mukai kernel of a relative equivalence from X to Y over S. Assume the restriction

$$L(Y_s \times_S Y_s \to X \times_S Y)^* K \cong \Delta_{Y_s/k,*} \mathcal{O}_{Y_s}$$

in $D(\mathcal{O}_{Y_s \times Y_s})$. Then there is an open neighbourhood $s \in U \subset S$ such that $Y|_U$ is isomorphic to $Y_s \times U$ over U.

Proof. Denote $i : Y_s \times Y_s = X_s \times Y_s \to X \times_S Y$ the natural closed immersion. (We will write Y_s and not X_s for the fibre of X over s from now on.) Let $z \in Y_s \times Y_s = (X \times_S Y)_s \subset X \times_S Y$ be a closed point. As indicated we think of z both as a closed point of $Y_s \times Y_s$ as well as a closed point of $X \times_S Y$.

Case I: $z \notin \Delta_{Y_s/k}(Y_s)$. Denote $\mathcal{O}_z$ the coherent $\mathcal{O}_{Y_s \times Y_s}$-module supported at z whose value is $\kappa(z)$. Then $i_* \mathcal{O}_z$ is the coherent $\mathcal{O}_{X \times_S Y}$-module supported at z whose value is $\kappa(z)$. Our assumption means that

$$K \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times_S Y}} i_* \mathcal{O}_z = Li^* K \otimes^{\mathbf{L}}_{\mathcal{O}_{Y_s \times Y_s}} \mathcal{O}_z = 0$$

Hence by Lemma 11.3 we find an open neighbourhood $U(z) \subset X \times_S Y$ of z such that $K|_{U(z)} = 0$. In this case we set $Z(z) = \emptyset$ as closed subscheme of $U(z)$.

Case II: $z \in \Delta_{Y_s/k}(Y_s)$. Since Y_s is smooth over k we know that $\Delta_{Y_s/k} : Y_s \to Y_s \times Y_s$ is a regular immersion, see More on Morphisms, Lemma 62.18. Choose a regular sequence $\overline{f}_1, \ldots, \overline{f}_r \in \mathcal{O}_{Y_s \times Y_s, z}$ cutting out the ideal sheaf of $\Delta_{Y_s/k}(Y_s)$. Since a regular sequence is Koszul-regular (More on Algebra, Lemma 30.2) our assumption means that

$$K_z \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times_S Y, z}} \mathcal{O}_{Y_s \times Y_s, z} \in D(\mathcal{O}_{Y_s \times Y_s, z})$$

is represented by the Koszul complex on $\overline{f}_1, \ldots, \overline{f}_r$ over $\mathcal{O}_{Y_s \times Y_s, z}$. By Lemma 16.1 applied to $\mathcal{O}_{S,s} \to \mathcal{O}_{X \times_S Y, z}$ we conclude that $K_z \in D(\mathcal{O}_{X \times_S Y, z})$ is represented by the Koszul complex on a regular sequence $f_1, \ldots, f_r \in \mathcal{O}_{X \times_S Y, z}$ lifting the regular sequence $\overline{f}_1, \ldots, \overline{f}_r$ such that moreover $\mathcal{O}_{X \times_S Y}/(f_1, \ldots, f_r)$ is flat over $\mathcal{O}_{S,s}$. By some limit arguments (Lemma 16.2) we conclude that there exists an affine open neighbourhood $U(z) \subset X \times_S Y$ of z and a closed subscheme $Z(z) \subset U(z)$ such that

   (1) $Z(z) \to U(z)$ is a regular closed immersion,
   (2) $K|_{U(z)}$ is quasi-isomorphic to $\mathcal{O}_{Z(z)}$,
   (3) $Z(z) \to S$ is flat,
   (4) $Z(z)_s = \Delta_{Y_s/k}(Y_s) \cap U(z)_s$ as closed subschemes of $U(z)_s$.

By property (2), for $z, z' \in Y_s \times Y_s$, we find that $Z(z) \cap U(z') = Z(z') \cap U(z)$ as closed subschemes. Hence we obtain an open neighbourhood

$$U = \bigcup_{z \in Y_s \times Y_s \text{ closed}} U(z)$$

of $Y_s \times Y_s$ in $X \times_S Y$ and a closed subscheme $Z \subset U$ such that (1) $Z \to U$ is a regular closed immersion, (2) $Z \to S$ is flat, and (3) $Z_s = \Delta_{Y_s/k}(Y_s)$. Since $X \times_S Y \to S$ is proper, after replacing S by an open neighbourhood of s we may assume $U = X \times_S Y$. Since the projections $Z_s \to Y_s$ and $Z_s \to X_s$ are isomorphisms, we conclude that after shrinking S we may assume $Z \to Y$ and $Z \to X$ are isomorphisms, see Lemma 16.3. This finishes the proof. $\square$

0G0S   **Lemma 16.5.** *Let k be an algebraically closed field. Let X be a smooth proper scheme over k. Let $f : Y \to S$ be a smooth proper morphism with S of finite type over k. Let K be the Fourier-Mukai kernel of a relative equivalence from $X \times S$*

to Y over S. Then S can be covered by open subschemes U such that there is a U-isomorphism $f^{-1}(U) \cong Y_0 \times U$ for some Y_0 proper and smooth over k.

Proof. Choose a closed point $s \in S$. Since k is algebraically closed this is a k-rational point. Set $Y_0 = Y_s$. The restriction K_0 of K to $X \times Y_0$ is the Fourier-Mukai kernel of a relative equivalence from X to Y_0 over $\mathrm{Spec}(k)$ by Lemma 15.3. Let K_0' in $D_{perf}(\mathcal{O}_{Y_0 \times X})$ be the object assumed to exist in Definition 15.1. Then K_0' is the Fourier-Mukai kernel of a relative equivalence from Y_0 to X over $\mathrm{Spec}(k)$ by the symmetry inherent in Definition 15.1. Hence by Lemma 15.3 we see that the pullback

$$M = (Y_0 \times X \times S \to Y_0 \times X)^* K_0'$$

on $(Y_0 \times S) \times_S (X \times S) = Y_0 \times X \times S$ is the Fourier-Mukai kernel of a relative equivalence from $Y_0 \times S$ to $X \times S$ over S. Now consider the kernel

$$K_{new} = R\mathrm{pr}_{13,*}(L\mathrm{pr}_{12}^* M \otimes_{\mathcal{O}_{(Y_0 \times S) \times_S (X \times S) \times_S Y}}^{\mathbf{L}} L\mathrm{pr}_{23}^* K)$$

on $(Y_0 \times S) \times_S Y$. This is the Fourier-Mukai kernel of a relative equivalence from $Y_0 \times S$ to Y over S since it is the composition of two invertible arrows in the category constructed in Section 14. Moreover, this composition passes through base change (Lemma 14.1). Hence we see that the pullback of K_{new} to $((Y_0 \times S) \times_S Y)_s = Y_0 \times Y_0$ is equal to the composition of K_0 and K_0' and hence equal to the identity in this category. In other words, we have

$$L(Y_0 \times Y_0 \to (Y_0 \times S) \times_S Y)^* K_{new} \cong \Delta_{Y_0/k,*} \mathcal{O}_{Y_0}$$

Thus by Lemma 16.4 we conclude that $Y \to S$ is isomorphic to $Y_0 \times S$ in an open neighbourhood of s. This finishes the proof. $\square$

17. Countability

0G0T   In this section we prove some elementary lemmas about countability of certain sets. Let $\mathcal{C}$ be a category. In this section we will say that $\mathcal{C}$ is *countable* if

    (1) for any $X, Y \in \mathrm{Ob}(\mathcal{C})$ the set $\mathrm{Mor}_{\mathcal{C}}(X, Y)$ is countable, and
    (2) the set of isomorphism classes of objects of $\mathcal{C}$ is countable.

0G0U   **Lemma 17.1.** *Let R be a countable Noetherian ring. Then the category of schemes of finite type over R is countable.*

Proof. Omitted. $\square$

0G0V   **Lemma 17.2.** *Let $\mathcal{A}$ be a countable abelian category. Then $D^b(\mathcal{A})$ is countable.*

Proof. It suffices to prove the statement for $D(\mathcal{A})$ as the others are full subcategories of this one. Since every object in $D(\mathcal{A})$ is a complex of objects of $\mathcal{A}$ it is immediate that the set of isomorphism classes of objects of $D^b(\mathcal{A})$ is countable. Moreover, for bounded complexes $A^\bullet$ and $B^\bullet$ of $\mathcal{A}$ it is clear that $\mathrm{Hom}_{K^b(\mathcal{A})}(A^\bullet, B^\bullet)$ is countable. We have

$$\mathrm{Hom}_{D^b(\mathcal{A})}(A^\bullet, B^\bullet) = \mathrm{colim}_{s:(A')^\bullet \to A^\bullet \text{ qis and } (A')^\bullet \text{ bounded}} \mathrm{Hom}_{K^b(\mathcal{A})}((A')^\bullet, B^\bullet)$$

by Derived Categories, Lemma 11.6. Thus this is a countable set as a countable colimit of $\square$

0G0W   **Lemma 17.3.** *Let X be a scheme of finite type over a countable Noetherian ring. Then the categories $D_{perf}(\mathcal{O}_X)$ and $D^b_{Coh}(\mathcal{O}_X)$ are countable.*

Proof. Observe that X is Noetherian by Morphisms, Lemma 15.6. Hence $D_{perf}(\mathcal{O}_X)$ is a full subcategory of $D^b_{Coh}(\mathcal{O}_X)$ by Derived Categories of Schemes, Lemma 11.6. Thus it suffices to prove the result for $D^b_{Coh}(\mathcal{O}_X)$. Recall that $D^b_{Coh}(\mathcal{O}_X) = D^b(Coh(\mathcal{O}_X))$ by Derived Categories of Schemes, Proposition 11.2. Hence by Lemma 17.2 it suffices to prove that $Coh(\mathcal{O}_X)$ is countable. This we omit. $\qquad\square$

0G0X   **Lemma 17.4.** *Let K be an algebraically closed field. Let S be a finite type scheme over K. Let $X \to S$ and $Y \to S$ be finite type morphisms. There exists a countable set I and for $i \in I$ a pair $(S_i \to S, h_i)$ with the following properties*

(1) *$S_i \to S$ is a morphism of finite type, set $X_i = X \times_S S_i$ and $Y_i = Y \times_S S_i$,*
(2) *$h_i : X_i \to Y_i$ is an isomorphism over S_i, and*
(3) *for any closed point $s \in S(K)$ if $X_s \cong Y_s$ over $K = \kappa(s)$ then s is in the image of $S_i \to S$ for some i.*

Proof. The field K is the filtered union of its countable subfields. Dually, $\mathrm{Spec}(K)$ is the cofiltered limit of the spectra of the countable subfields of K. Hence Limits, Lemma 10.1 guarantees that we can find a countable subfield k and morphisms $X_0 \to S_0$ and $Y_0 \to S_0$ of schemes of finite type over k such that $X \to S$ and $Y \to S$ are the base changes of these.

By Lemma 17.1 there is a countable set I and pairs $(S_{0,i} \to S_0, h_{0,i})$ such that

(1) *$S_{0,i} \to S_0$ is a morphism of finite type, set $X_{0,i} = X_0 \times_{S_0} S_{0,i}$ and $Y_{0,i} = Y_0 \times_{S_0} S_{0,i}$,*
(2) *$h_{0,i} : X_{0,i} \to Y_{0,i}$ is an isomorphism over $S_{0,i}$.*

such that every pair $(T \to S_0, h_T)$ with $T \to S_0$ of finite type and $h_T : X_0 \times_{S_0} T \to Y_0 \times_{S_0} T$ an isomorphism is isomorphic to one of these. Denote $(S_i \to S, h_i)$ the base change of $(S_{0,i} \to S_0, h_{0,i})$ by $\mathrm{Spec}(K) \to \mathrm{Spec}(k)$. We claim this works.

Let $s \in S(K)$ and let $h_s : X_s \to Y_s$ be an isomorphism over $K = \kappa(s)$. We can write K as the filtered union of its finitely generated k-subalgebras. Hence by Limits, Proposition 6.1 and Lemma 10.1 we can find such a finitely generated k-subalgebra $K \supset A \supset k$ such that

(1) there is a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Spec}(K) & \longrightarrow & \mathrm{Spec}(A) \\
{\scriptstyle s}\downarrow & & \downarrow{\scriptstyle s'} \\
S & \longrightarrow & S_0
\end{array}
$$

for some morphism $s' : \mathrm{Spec}(A) \to S_0$ over k,
(2) h_s is the base change of an isomorphism $h_{s'} : X_0 \times_{S_0,s'} \mathrm{Spec}(A) \to X_0 \times_{S_0,s'} \mathrm{Spec}(A)$ over A.

Of course, then $(s' : \mathrm{Spec}(A) \to S_0, h_{s'})$ is isomorphic to the pair $(S_{0,i} \to S_0, h_{0,i})$ for some $i \in I$. This concludes the proof because the commutative diagram in (1) shows that s is in the image of the base change of s' to $\mathrm{Spec}(K)$. $\qquad\square$

0G0Y   **Lemma 17.5.** *Let K be an algebraically closed field. There exists a countable set I and for $i \in I$ a pair $(S_i/K, X_i \to S_i, Y_i \to S_i, M_i)$ with the following properties*

(1) *S_i is a scheme of finite type over K,*
(2) *$X_i \to S_i$ and $Y_i \to S_i$ are proper smooth morphisms of schemes,*

(3) $M_i \in D_{perf}(\mathcal{O}_{X_i \times_{S_i} Y_i})$ *is the Fourier-Mukai kernel of a relative equivalence from X_i to Y_i over S_i, and*

(4) *for any smooth proper schemes X and Y over K such that there is a K-linear exact equivalence $D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_Y)$ there exists an $i \in I$ and a $s \in S_i(K)$ such that $X \cong (X_i)_s$ and $Y \cong (Y_i)_s$.*

Proof. Choose a countable subfield $k \subset K$ for example the prime field. By Lemmas 17.1 and 17.3 there exists a countable set of isomorphism classes of systems over k satisfying parts (1), (2), (3) of the lemma. Thus we can choose a countable set I and for each $i \in I$ such a system

$$(S_{0,i}/k, X_{0,i} \to S_{0,i}, Y_{0,i} \to S_{0,i}, M_{0,i})$$

over k such that each isomorphism class occurs at least once. Denote $(S_i/K, X_i \to S_i, Y_i \to S_i, M_i)$ the base change of the displayed system to K. This system has properties (1), (2), (3), see Lemma 15.3. Let us prove property (4).

Consider smooth proper schemes X and Y over K such that there is a K-linear exact equivalence $F : D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_Y)$. By Proposition 13.4 we may assume that there exists an object $M \in D_{perf}(\mathcal{O}_{X \times Y})$ such that $F = \Phi_M$ is the corresponding Fourier-Mukai functor. By Lemma 8.9 there is an M' in $D_{perf}(\mathcal{O}_{Y \times X})$ such that $\Phi_{M'}$ is the right adjoint to Φ_M. Since Φ_M is an equivalence, this means that $\Phi_{M'}$ is the quasi-inverse to Φ_M. By Lemma 8.9 we see that the Fourier-Mukai functors defined by the objects

$$A = R\mathrm{pr}_{13,*}(L\mathrm{pr}_{12}^* M \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times Y \times X}} L\mathrm{pr}_{23}^* M')$$

in $D_{perf}(\mathcal{O}_{X \times X})$ and

$$B = R\mathrm{pr}_{13,*}(L\mathrm{pr}_{12}^* M' \otimes^{\mathbf{L}}_{\mathcal{O}_{Y \times X \times Y}} L\mathrm{pr}_{23}^* M)$$

in $D_{perf}(\mathcal{O}_{Y \times Y})$ are isomorphic to $\mathrm{id} : D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_X)$ and $\mathrm{id} : D_{perf}(\mathcal{O}_Y) \to D_{perf}(\mathcal{O}_Y)$ Hence $A \cong \Delta_{X/K,*} \mathcal{O}_X$ and $B \cong \Delta_{Y/K,*} \mathcal{O}_Y$ by Lemma 13.5. Hence we see that M is the Fourier-Mukai kernel of a relative equivalence from X to Y over K by definition.

We can write K as the filtered colimit of its finite type k-subalgebras $A \subset K$. By Limits, Lemma 10.1 we can find X_0, Y_0 of finite type over A whose base changes to K produces X and Y. By Limits, Lemmas 13.1 and 8.9 after enlarging A we may assume X_0 and Y_0 are smooth and proper over A. By Lemma 15.4 after enlarging A we may assume M is the pullback of some $M_0 \in D_{perf}(\mathcal{O}_{X_0 \times_{\mathrm{Spec}(A)} Y_0})$ which is the Fourier-Mukai kernel of a relative equivalence from X_0 to Y_0 over $\mathrm{Spec}(A)$. Thus we see that $(S_0/k, X_0 \to S_0, Y_0 \to S_0, M_0)$ is isomorphic to $(S_{0,i}/k, X_{0,i} \to S_{0,i}, Y_{0,i} \to S_{0,i}, M_{0,i})$ for some $i \in I$. Since $S_i = S_{0,i} \times_{\mathrm{Spec}(k)} \mathrm{Spec}(K)$ we conclude that (4) is true with $s : \mathrm{Spec}(K) \to S_i$ induced by the morphism $\mathrm{Spec}(K) \to \mathrm{Spec}(A) \cong S_{0,i}$ we get from $A \subset K$. $\square$

18. Countability of derived equivalent varieties

0G0Z   In this section we prove a result of Anel and Toën, see [AT09].

0G10   **Definition 18.1.** Let k be a field. Let X and Y be smooth projective schemes over k. We say X and Y are *derived equivalent* if there exists a k-linear exact equivalence $D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_Y)$.

Here is the result

0G11 **Theorem 18.2.** *Let K be an algebraically closed field. Let **X** be a smooth proper scheme over K. There are at most countably many isomorphism classes of smooth proper schemes **Y** over K which are derived equivalent to **X**.*

Slight improvement of [AT09]

Proof. Choose a countable set I and for $i \in I$ systems $(S_i/K, X_i \to S_i, Y_i \to S_i, M_i)$ satisfying properties (1), (2), (3), and (4) of Lemma 17.5. Pick $i \in I$ and set $S = S_i$, $X = X_i$, $Y = Y_i$, and $M = M_i$. Clearly it suffice to show that the set of isomorphism classes of fibres Y_s for $s \in S(K)$ such that $X_s \cong \mathbf{X}$ is countable. This we prove in the next paragraph.

Let S be a finite type scheme over K, let $X \to S$ and $Y \to S$ be proper smooth morphisms, and let $M \in D_{perf}(\mathcal{O}_{X \times_S Y})$ be the Fourier-Mukai kernel of a relative equivalence from X to Y over S. We will show the set of isomorphism classes of fibres Y_s for $s \in S(K)$ such that $X_s \cong \mathbf{X}$ is countable. By Lemma 17.4 applied to the families $\mathbf{X} \times S \to S$ and $X \to S$ there exists a countable set I and for $i \in I$ a pair $(S_i \to S, h_i)$ with the following properties

(1) $S_i \to S$ is a morphism of finite type, set $X_i = X \times_S S_i$,
(2) $h_i : \mathbf{X} \times S_i \to X_i$ is an isomorphism over S_i, and
(3) for any closed point $s \in S(K)$ if $\mathbf{X} \cong X_s$ over $K = \kappa(s)$ then s is in the image of $S_i \to S$ for some i.

Set $Y_i = Y \times_S S_i$. Denote $M_i \in D_{perf}(\mathcal{O}_{X_i \times_{S_i} Y_i})$ the pullback of M. By Lemma 15.3 M_i is the Fourier-Mukai kernel of a relative equivalence from X_i to Y_i over S_i. Since I is countable, by property (3) it suffices to prove that the set of isomorphism classes of fibres $Y_{i,s}$ for $s \in S_i(K)$ is countable. In fact, this number is finite by Lemma 16.5 and the proof is complete. $\qquad\square$

19. Other chapters

Preliminaries

(1) Introduction
(2) Conventions
(3) Set Theory
(4) Categories
(5) Topology
(6) Sheaves on Spaces
(7) Sites and Sheaves
(8) Stacks
(9) Fields
(10) Commutative Algebra
(11) Brauer Groups
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(16) Smoothing Ring Maps
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(18) Modules on Sites
(19) Injectives
(20) Cohomology of Sheaves

(21) Cohomology on Sites
(22) Differential Graded Algebra
(23) Divided Power Algebra
(24) Differential Graded Sheaves
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Schemes

(26) Schemes
(27) Constructions of Schemes
(28) Properties of Schemes
(29) Morphisms of Schemes
(30) Cohomology of Schemes
(31) Divisors
(32) Limits of Schemes
(33) Varieties
(34) Topologies on Schemes
(35) Descent
(36) Derived Categories of Schemes
(37) More on Morphisms
(38) More on Flatness
(39) Groupoid Schemes
(40) More on Groupoid Schemes

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