DUALIZING COMPLEXES

08XG

Contents

1. Introduction

08XH   In this chapter we discuss dualizing complexes in commutative algebra. A reference is [Har66].

We begin with a discussion of essential surjections and essential injections, projective covers, injective hulls, duality for Artinian rings, and study injective hulls of

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residue fields, leading quickly to a proof of Matlis duality. See Sections 2, 3, 4, 5, 6, and 7 and Proposition 7.8.

This is followed by three sections discussing local cohomology in great generality, see Sections 8, 9, and 10. We apply some of this to a discussion of depth in Section 11. In another application we show how, given a finitely generated ideal I of a ring A, the "I-complete" and "I-torsion" objects of the derived category of A are equivalent, see Section 12. To learn more about local cohomology, for example the finiteness theorem (which relies on local duality – see below) please visit Local Cohomology, Section 1.

The bulk of this chapter is devoted to duality for a ring map and dualizing complexes. See Sections 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, and 23. The key definition is that of a dualizing complex $\omega_A^\bullet$ over a Noetherian ring A as an object $\omega_A^\bullet \in D^+(A)$ whose cohomology modules $H^i(\omega_A^\bullet)$ are finite A-modules, which has finite injective dimension, and is such that the map

$$A \longrightarrow R\operatorname{Hom}_A(\omega_A^\bullet, \omega_A^\bullet)$$

is a quasi-isomorphism. After establishing some elementary properties of dualizing complexes, we show a dualizing complex gives rise to a dimension function. Next, we prove Grothendieck's local duality theorem. After briefly discussing dualizing modules and Cohen-Macaulay rings, we introduce Gorenstein rings and we show many familiar Noetherian rings have dualizing complexes. In a last section we apply the material to show there is a good theory of Noetherian local rings whose formal fibres are Gorenstein or local complete intersections.

In the last few sections, we describe an algebraic construction of the "upper shriek functors" used in algebraic geometry, for example in the book [Har66]. This topic is continued in the chapter on duality for schemes. See Duality for Schemes, Section 1.

2. Essential surjections and injections

08XI We will mostly work in categories of modules, but we may as well make the definition in general.

08XJ **Definition 2.1.** Let $\mathcal{A}$ be an abelian category.

    (1) An injection $A \subset B$ of $\mathcal{A}$ is *essential*, or we say that B is an *essential extension of A*, if every nonzero subobject $B' \subset B$ has nonzero intersection with A.

    (2) A surjection $f : A \to B$ of $\mathcal{A}$ is *essential* if for every proper subobject $A' \subset A$ we have $f(A') \neq B$.

Some lemmas about this notion.

08XK **Lemma 2.2.** *Let $\mathcal{A}$ be an abelian category.*

    (1) *If $A \subset B$ and $B \subset C$ are essential extensions, then $A \subset C$ is an essential extension.*

    (2) *If $A \subset B$ is an essential extension and $C \subset B$ is a subobject, then $A \cap C \subset C$ is an essential extension.*

    (3) *If $A \to B$ and $B \to C$ are essential surjections, then $A \to C$ is an essential surjection.*

(4) *Given an essential surjection $f : A \to B$ and a surjection $A \to C$ with kernel K, the morphism $C \to B/f(K)$ is an essential surjection.*

Proof. Omitted. □

08XL **Lemma 2.3.** *Let R be a ring. Let M be an R-module. Let $E = \operatorname{colim} E_i$ be a filtered colimit of R-modules. Suppose given a compatible system of essential injections $M \to E_i$ of R-modules. Then $M \to E$ is an essential injection.*

Proof. Immediate from the definitions and the fact that filtered colimits are exact (Algebra, Lemma 8.8). □

08XM **Lemma 2.4.** *Let R be a ring. Let $M \subset N$ be R-modules. The following are equivalent*

(1) *$M \subset N$ is an essential extension,*
(2) *for all $x \in N$ nonzero there exists an $f \in R$ such that $fx \in M$ and $fx \neq 0$.*

Proof. Assume (1) and let $x \in N$ be a nonzero element. By (1) we have $Rx \cap M \neq 0$. This implies (2).

Assume (2). Let $N' \subset N$ be a nonzero submodule. Pick $x \in N'$ nonzero. By (2) we can find $f \in R$ with $fx \in M$ and $fx \neq 0$. Thus $N' \cap M \neq 0$. □

3. Injective modules

08XN Some results about injective modules over rings.

08XP **Lemma 3.1.** *Let R be a ring. Any product of injective R-modules is injective.*

Proof. Special case of Homology, Lemma 27.3. □

08XQ **Lemma 3.2.** *Let $R \to S$ be a flat ring map. If E is an injective S-module, then E is injective as an R-module.*

Proof. This is true because $\operatorname{Hom}_R(M, E) = \operatorname{Hom}_S(M \otimes_R S, E)$ by Algebra, Lemma 14.3 and the fact that tensoring with S is exact. □

08YV **Lemma 3.3.** *Let $R \to S$ be an epimorphism of rings. Let E be an S-module. If E is injective as an R-module, then E is an injective S-module.*

Proof. This is true because $\operatorname{Hom}_R(N, E) = \operatorname{Hom}_S(N, E)$ for any S-module N, see Algebra, Lemma 107.14. □

08XR **Lemma 3.4.** *Let $R \to S$ be a ring map. If E is an injective R-module, then $\operatorname{Hom}_R(S, E)$ is an injective S-module.*

Proof. This is true because $\operatorname{Hom}_S(N, \operatorname{Hom}_R(S, E)) = \operatorname{Hom}_R(N, E)$ by Algebra, Lemma 14.4. □

08XS **Lemma 3.5.** *Let R be a ring. Let I be an injective R-module. Let $E \subset I$ be a submodule. The following are equivalent*

(1) *E is injective, and*
(2) *for all $E \subset E' \subset I$ with $E \subset E'$ essential we have $E = E'$.*

In particular, an R-module is injective if and only if every essential extension is trivial.

Proof. The final assertion follows from the first and the fact that the category of R-modules has enough injectives (More on Algebra, Section 55).

Assume (1). Let $E \subset E' \subset I$ as in (2). Then the map $\mathrm{id}_E : E \to E$ can be extended to a map $\alpha : E' \to E$. The kernel of α has to be zero because it intersects E trivially and E' is an essential extension. Hence $E = E'$.

Assume (2). Let $M \subset N$ be R-modules and let $\varphi : M \to E$ be an R-module map. In order to prove (1) we have to show that φ extends to a morphism $N \to E$. Consider the set $\mathcal{S}$ of pairs (M', φ') where $M \subset M' \subset N$ and $\varphi' : M' \to E$ is an R-module map agreeing with φ on M. We define an ordering on $\mathcal{S}$ by the rule $(M', \varphi') \leq (M'', \varphi'')$ if and only if $M' \subset M''$ and $\varphi''|_{M'} = \varphi'$. It is clear that we can take the maximum of a totally ordered subset of $\mathcal{S}$. Hence by Zorn's lemma we may assume (M, φ) is a maximal element.

Choose an extension $\psi : N \to I$ of φ composed with the inclusion $E \to I$. This is possible as I is injective. If $\psi(N) \subset E$, then ψ is the desired extension. If $\psi(N)$ is not contained in E, then by (2) the inclusion $E \subset E + \psi(N)$ is not essential. hence we can find a nonzero submodule $K \subset E + \psi(N)$ meeting E in 0. This means that $M' = \psi^{-1}(E + K)$ strictly contains M. Thus we can extend φ to M' using

$$M' \xrightarrow{\psi|_{M'}} E + K \to (E + K)/K = E$$

This contradicts the maximality of (M, φ). $\square$

08XT **Example 3.6.** Let R be a reduced ring. Let $\mathfrak{p} \subset R$ be a minimal prime so that $K = R_{\mathfrak{p}}$ is a field (Algebra, Lemma 25.1). Then K is an injective R-module. Namely, we have $\mathrm{Hom}_R(M, K) = \mathrm{Hom}_K(M_{\mathfrak{p}}, K)$ for any R-module M. Since localization is an exact functor and taking duals is an exact functor on K-vector spaces we conclude $\mathrm{Hom}_R(-, K)$ is an exact functor, i.e., K is an injective R-module.

08XV **Lemma 3.7.** *Let R be a Noetherian ring. A direct sum of injective modules is injective.*

Proof. Let E_i be a family of injective modules parametrized by a set I. Set $E = \bigoplus E_i$. To show that E is injective we use Injectives, Lemma 2.6. Thus let $\varphi : I \to E$ be a module map from an ideal of R into E. As I is a finite R-module (because R is Noetherian) we can find finitely many elements $i_1, \ldots, i_r \in I$ such that φ maps into $\bigoplus_{j=1,\ldots,r} E_{i_j}$. Then we can extend φ into $\bigoplus_{j=1,\ldots,r} E_{i_j}$ using the injectivity of the modules E_{i_j}. $\square$

0A6I **Lemma 3.8.** *Let R be a Noetherian ring. Let $S \subset R$ be a multiplicative subset. If E is an injective R-module, then $S^{-1}E$ is an injective $S^{-1}R$-module.*

Proof. Since $R \to S^{-1}R$ is an epimorphism of rings, it suffices to show that $S^{-1}E$ is injective as an R-module, see Lemma 3.3. To show this we use Injectives, Lemma 2.6. Thus let $I \subset R$ be an ideal and let $\varphi : I \to S^{-1}E$ be an R-module map. As I is a finitely presented R-module (because R is Noetherian) we can find an $f \in S$ and an R-module map $I \to E$ such that $f\varphi$ is the composition $I \to E \to S^{-1}E$ (Algebra, Lemma 10.2). Then we can extend $I \to E$ to a homomorphism $R \to E$. Then the composition

$$R \to E \to S^{-1}E \xrightarrow{f^{-1}} S^{-1}E$$

is the desired extension of φ to R. $\square$

08XW **Lemma 3.9.** *Let R be a Noetherian ring. Let I be an injective R-module.*
  (1) *Let $f \in R$. Then $E = \bigcup I[f^n] = I[f^\infty]$ is an injective submodule of I.*
  (2) *Let $J \subset R$ be an ideal. Then the J-power torsion submodule $I[J^\infty]$ is an injective submodule of I.*

Proof. We will use Lemma 3.5 to prove (1). Suppose that $E \subset E' \subset I$ and that E' is an essential extension of E. We will show that $E' = E$. If not, then we can find $x \in E'$ and $x \notin E$. Let $J = \{a \in R \mid ax \in E\}$. Since R is Noetherian, we may write $J = (g_1, \ldots, g_t)$ for some $g_i \in R$. By definition E is the set of elements of I annihilated by powers of f, so we may choose integers n_i so that $f^{n_i} g_i x = 0$. Set $n = \max\{n_i\}$. Then $x' = f^n x$ is an element of E' not in E and is annihilated by J. Set $J' = \{a \in R \mid ax' \in E\}$ so $J \subset J'$. Conversely, we have $a \in J'$ if and only if $ax' \in E$ if and only if $f^m ax' = 0$ for some $m \geq 0$. But then $f^m ax' = f^{m+n} ax$ implies $ax \in E$, i.e., $a \in J$. Hence $J = J'$. Thus $J = J' = \operatorname{Ann}(x')$, so $Rx' \cap E = 0$. Hence E' is not an essential extension of E, a contradiction.

To prove (2) write $J = (f_1, \ldots, f_t)$. Then $I[J^\infty]$ is equal to

$$(\ldots ((I[f_1^\infty])[f_2^\infty]) \ldots)[f_t^\infty]$$

and the result follows from (1) and induction. $\square$

0A6J **Lemma 3.10.** *Let A be a Noetherian ring. Let E be an injective A-module. Then $E \otimes_A A[x]$ has injective-amplitude $[0, 1]$ as an object of $D(A[x])$. In particular, $E \otimes_A A[x]$ has finite injective dimension as an $A[x]$-module.*

Proof. Let us write $E[x] = E \otimes_A A[x]$. Consider the short exact sequence of $A[x]$-modules

$$0 \to E[x] \to \operatorname{Hom}_A(A[x], E[x]) \to \operatorname{Hom}_A(A[x], E[x]) \to 0$$

where the first map sends $p \in E[x]$ to $f \mapsto fp$ and the second map sends φ to $f \mapsto \varphi(xf) - x\varphi(f)$. The second map is surjective because $\operatorname{Hom}_A(A[x], E[x]) = \prod_{n \geq 0} E[x]$ as an abelian group and the map sends (e_n) to $(e_{n+1} - xe_n)$ which is surjective. As an A-module we have $E[x] \cong \bigoplus_{n \geq 0} E$ which is injective by Lemma 3.7. Hence the $A[x]$-module $\operatorname{Hom}_A(A[x], E[x])$ is injective by Lemma 3.4 and the proof is complete. $\square$

4. Projective covers

08XX In this section we briefly discuss projective covers.

08XY **Definition 4.1.** Let R be a ring. A surjection $P \to M$ of R-modules is said to be a *projective cover*, or sometimes a *projective envelope*, if P is a projective R-module and $P \to M$ is an essential surjection.

Projective covers do not always exist. For example, if k is a field and $R = k[x]$ is the polynomial ring over k, then the module $M = R/(x)$ does not have a projective cover. Namely, for any surjection $f : P \to M$ with P projective over R, the proper submodule $(x - 1)P$ surjects onto M. Hence f is not essential.

08XZ **Lemma 4.2.** *Let R be a ring and let M be an R-module. If a projective cover of M exists, then it is unique up to isomorphism.*

Proof. Let $P \to M$ and $P' \to M$ be projective covers. Because P is a projective R-module and $P' \to M$ is surjective, we can find an R-module map $\alpha : P \to P'$ compatible with the maps to M. Since $P' \to M$ is essential, we see that α is surjective. As P' is a projective R-module we can choose a direct sum decomposition $P = \mathrm{Ker}(\alpha) \oplus P'$. Since $P' \to M$ is surjective and since $P \to M$ is essential we conclude that $\mathrm{Ker}(\alpha)$ is zero as desired. $\square$

Here is an example where projective covers exist.

08Y0   **Lemma 4.3.** *Let $(R, \mathfrak{m}, \kappa)$ be a local ring. Any finite R-module has a projective cover.*

Proof. Let M be a finite R-module. Let $r = \dim_\kappa(M/\mathfrak{m}M)$. Choose $x_1, \ldots, x_r \in M$ mapping to a basis of $M/\mathfrak{m}M$. Consider the map $f : R^{\oplus r} \to M$. By Nakayama's lemma this is a surjection (Algebra, Lemma 20.1). If $N \subset R^{\oplus r}$ is a proper submodule, then $N/\mathfrak{m}N \to \kappa^{\oplus r}$ is not surjective (by Nakayama's lemma again) hence $N/\mathfrak{m}N \to M/\mathfrak{m}M$ is not surjective. Thus f is an essential surjection. $\square$

5. Injective hulls

08Y1   In this section we briefly discuss injective hulls.

08Y2   **Definition 5.1.** Let R be a ring. A injection $M \to I$ of R-modules is said to be an *injective hull* if I is a injective R-module and $M \to I$ is an essential injection.

Injective hulls always exist.

08Y3   **Lemma 5.2.** *Let R be a ring. Any R-module has an injective hull.*

Proof. Let M be an R-module. By More on Algebra, Section 55 the category of R-modules has enough injectives. Choose an injection $M \to I$ with I an injective R-module. Consider the set $\mathcal{S}$ of submodules $M \subset E \subset I$ such that E is an essential extension of M. We order $\mathcal{S}$ by inclusion. If $\{E_\alpha\}$ is a totally ordered subset of $\mathcal{S}$, then $\bigcup E_\alpha$ is an essential extension of M too (Lemma 2.3). Thus we can apply Zorn's lemma and find a maximal element $E \in \mathcal{S}$. We claim $M \subset E$ is an injective hull, i.e., E is an injective R-module. This follows from Lemma 3.5. $\square$

08Y4   **Lemma 5.3.** *Let R be a ring. Let M, N be R-modules and let $M \to E$ and $N \to E'$ be injective hulls. Then*

    (1) *for any R-module map $\varphi : M \to N$ there exists an R-module map $\psi : E \to E'$ such that*

$$
\begin{array}{ccc}
M & \longrightarrow & E \\
\varphi \downarrow & & \downarrow \psi \\
N & \longrightarrow & E'
\end{array}
$$

    commutes,

    (2) *if φ is injective, then ψ is injective,*

    (3) *if φ is an essential injection, then ψ is an isomorphism,*

    (4) *if φ is an isomorphism, then ψ is an isomorphism,*

    (5) *if $M \to I$ is an embedding of M into an injective R-module, then there is an isomorphism $I \cong E \oplus I'$ compatible with the embeddings of M,*

In particular, the injective hull E of M is unique up to isomorphism.

Proof. Part (1) follows from the fact that E' is an injective R-module. Part (2) follows as $\text{Ker}(\psi) \cap M = 0$ and E is an essential extension of M. Assume φ is an essential injection. Then $E \cong \psi(E) \subset E'$ by (2) which implies $E' = \psi(E) \oplus E''$ because E is injective. Since E' is an essential extension of M (Lemma 2.2) we get $E'' = 0$. Part (4) is a special case of (3). Assume $M \to I$ as in (5). Choose a map $\alpha : E \to I$ extending the map $M \to I$. Arguing as before we see that α is injective. Thus as before $\alpha(E)$ splits off from I. This proves (5). $\square$

08Y5   **Example 5.4.** Let R be a domain with fraction field K. Then $R \subset K$ is an injective hull of R. Namely, by Example 3.6 we see that K is an injective R-module and by Lemma 2.4 we see that $R \subset K$ is an essential extension.

08Y6   **Definition 5.5.** An object X of an additive category is called *indecomposable* if it is nonzero and if $X = Y \oplus Z$, then either $Y = 0$ or $Z = 0$.

08Y7   **Lemma 5.6.** *Let R be a ring. Let E be an indecomposable injective R-module. Then*

(1) *E is the injective hull of any nonzero submodule of E,*
(2) *the intersection of any two nonzero submodules of E is nonzero,*
(3) *$\text{End}_R(E, E)$ is a noncommutative local ring with maximal ideal those $\varphi : E \to E$ whose kernel is nonzero, and*
(4) *the set of zerodivisors on E is a prime ideal $\mathfrak{p}$ of R and E is an injective $R_\mathfrak{p}$-module.*

Proof. Part (1) follows from Lemma 5.3. Part (2) follows from part (1) and the definition of injective hulls.

Proof of (3). Set $A = \text{End}_R(E, E)$ and $I = \{\varphi \in A \mid \text{Ker}(\varphi) \neq 0\}$. The statement means that I is a two sided ideal and that any $\varphi \in A$, $\varphi \notin I$ is invertible. Suppose φ and ψ are not injective. Then $\text{Ker}(\varphi) \cap \text{Ker}(\psi)$ is nonzero by (2). Hence $\varphi + \psi \in I$. It follows that I is a two sided ideal. If $\varphi \in A$, $\varphi \notin I$, then $E \cong \varphi(E) \subset E$ is an injective submodule, hence $E = \varphi(E)$ because E is indecomposable.

Proof of (4). Consider the ring map $R \to A$ and let $\mathfrak{p} \subset R$ be the inverse image of the maximal ideal I. Then it is clear that $\mathfrak{p}$ is a prime ideal and that $R \to A$ extends to $R_\mathfrak{p} \to A$. Thus E is an $R_\mathfrak{p}$-module. It follows from Lemma 3.3 that E is injective as an $R_\mathfrak{p}$-module. $\square$

08Y8   **Lemma 5.7.** *Let $\mathfrak{p} \subset R$ be a prime of a ring R. Let E be the injective hull of $R/\mathfrak{p}$. Then*

(1) *E is indecomposable,*
(2) *E is the injective hull of $\kappa(\mathfrak{p})$,*
(3) *E is the injective hull of $\kappa(\mathfrak{p})$ over the ring $R_\mathfrak{p}$.*

Proof. By Lemma 2.4 the inclusion $R/\mathfrak{p} \subset \kappa(\mathfrak{p})$ is an essential extension. Then Lemma 5.3 shows (2) holds. For $f \in R$, $f \notin \mathfrak{p}$ the map $f : \kappa(\mathfrak{p}) \to \kappa(\mathfrak{p})$ is an isomorphism hence the map $f : E \to E$ is an isomorphism, see Lemma 5.3. Thus E is an $R_\mathfrak{p}$-module. It is injective as an $R_\mathfrak{p}$-module by Lemma 3.3. Finally, let $E' \subset E$ be a nonzero injective R-submodule. Then $J = (R/\mathfrak{p}) \cap E'$ is nonzero. After shrinking E' we may assume that E' is the injective hull of J (see Lemma 5.3 for example). Observe that $R/\mathfrak{p}$ is an essential extension of J for example by Lemma 2.4. Hence $E' \to E$ is an isomorphism by Lemma 5.3 part (3). Hence E is indecomposable. $\square$

08Y9 **Lemma 5.8.** *Let R be a Noetherian ring. Let E be an indecomposable injective R-module. Then there exists a prime ideal $\mathfrak{p}$ of R such that E is the injective hull of $\kappa(\mathfrak{p})$.*

Proof. Let $\mathfrak{p}$ be the prime ideal found in Lemma 5.6. Say $\mathfrak{p} = (f_1, \ldots, f_r)$. Pick a nonzero element $x \in \bigcap \mathrm{Ker}(f_i : E \to E)$, see Lemma 5.6. Then $(R_\mathfrak{p})x$ is a module isomorphic to $\kappa(\mathfrak{p})$ inside E. We conclude by Lemma 5.6. $\qquad\square$

08YA **Proposition 5.9** (Structure of injective modules over Noetherian rings)**.** *Let R be a Noetherian ring. Every injective module is a direct sum of indecomposable injective modules. Every indecomposable injective module is the injective hull of the residue field at a prime.*

Proof. The second statement is Lemma 5.8. For the first statement, let I be an injective R-module. We will use transfinite recursion to construct $I_\alpha \subset I$ for ordinals α which are direct sums of indecomposable injective R-modules $E_{\beta+1}$ for $\beta < \alpha$. For $\alpha = 0$ we let $I_0 = 0$. Suppose given an ordinal α such that I_α has been constructed. Then I_α is an injective R-module by Lemma 3.7. Hence $I \cong I_\alpha \oplus I'$. If $I' = 0$ we are done. If not, then I' has an associated prime by Algebra, Lemma 63.7. Thus I' contains a copy of $R/\mathfrak{p}$ for some prime $\mathfrak{p}$. Hence I' contains an indecomposable submodule E by Lemmas 5.3 and 5.7. Set $I_{\alpha+1} = I_\alpha \oplus E_\alpha$. If α is a limit ordinal and I_β has been constructed for $\beta < \alpha$, then we set $I_\alpha = \bigcup_{\beta < \alpha} I_\beta$. Observe that $I_\alpha = \bigoplus_{\beta < \alpha} E_{\beta+1}$. This concludes the proof. $\qquad\square$

6. Duality over Artinian local rings

08YW Let $(R, \mathfrak{m}, \kappa)$ be an artinian local ring. Recall that this implies R is Noetherian and that R has finite length as an R-module. Moreover an R-module is finite if and only if it has finite length. We will use these facts without further mention in this section. Please see Algebra, Sections 52 and 53 and Algebra, Proposition 60.7 for more details.

08YX **Lemma 6.1.** *Let $(R, \mathfrak{m}, \kappa)$ be an artinian local ring. Let E be an injective hull of κ. For every finite R-module M we have*

$$length_R(M) = length_R(\mathrm{Hom}_R(M, E))$$

In particular, the injective hull E of κ is a finite R-module.

Proof. Because E is an essential extension of κ we have $\kappa = E[\mathfrak{m}]$ where $E[\mathfrak{m}]$ is the $\mathfrak{m}$-torsion in E (notation as in More on Algebra, Section 88). Hence $\mathrm{Hom}_R(\kappa, E) \cong \kappa$ and the equality of lengths holds for $M = \kappa$. We prove the displayed equality of the lemma by induction on the length of M. If M is nonzero there exists a surjection $M \to \kappa$ with kernel M'. Since the functor $M \mapsto \mathrm{Hom}_R(M, E)$ is exact we obtain a short exact sequence

$$0 \to \mathrm{Hom}_R(\kappa, E) \to \mathrm{Hom}_R(M, E) \to \mathrm{Hom}_R(M', E) \to 0.$$

Additivity of length for this sequence and the sequence $0 \to M' \to M \to \kappa \to 0$ and the equality for M' (induction hypothesis) and κ implies the equality for M. The final statement of the lemma follows as $E = \mathrm{Hom}_R(R, E)$. $\qquad\square$

08YY **Lemma 6.2.** *Let $(R, \mathfrak{m}, \kappa)$ be an artinian local ring. Let E be an injective hull of κ. For any finite R-module M the evaluation map*

$$M \longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_R(M, E), E)$$

is an isomorphism. In particular $R = \operatorname{Hom}_R(E, E)$.

Proof. Observe that the displayed arrow is injective. Namely, if $x \in M$ is a nonzero element, then there is a nonzero map $Rx \to \kappa$ which we can extend to a map $\varphi : M \to E$ that doesn't vanish on x. Since the source and target of the arrow have the same length by Lemma 6.1 we conclude it is an isomorphism. The final statement follows on taking $M = R$. $\square$

To state the next lemma, denote $\operatorname{Mod}_R^{fg}$ the category of finite R-modules over a ring R.

08YZ **Lemma 6.3.** *Let $(R, \mathfrak{m}, \kappa)$ be an artinian local ring. Let E be an injective hull of κ. The functor $D(-) = \operatorname{Hom}_R(-, E)$ induces an exact anti-equivalence $\operatorname{Mod}_R^{fg} \to \operatorname{Mod}_R^{fg}$ and $D \circ D \cong \mathrm{id}$.*

Proof. We have seen that $D \circ D = \mathrm{id}$ on $\operatorname{Mod}_R^{fg}$ in Lemma 6.2. It follows immediately that D is an anti-equivalence. $\square$

08Z0 **Lemma 6.4.** *Assumptions and notation as in Lemma 6.3. Let $I \subset R$ be an ideal and M a finite R-module. Then*

$$D(M[I]) = D(M)/ID(M) \quad and \quad D(M/IM) = D(M)[I]$$

Proof. Say $I = (f_1, \ldots, f_t)$. Consider the map

$$M^{\oplus t} \xrightarrow{f_1, \ldots, f_t} M$$

with cokernel M/IM. Applying the exact functor D we conclude that $D(M/IM)$ is $D(M)[I]$. The other case is proved in the same way. $\square$

7. Injective hull of the residue field

08Z1 Most of our results will be for Noetherian local rings in this section.

08Z2 **Lemma 7.1.** *Let $R \to S$ be a surjective map of local rings with kernel I. Let E be the injective hull of the residue field of R over R. Then $E[I]$ is the injective hull of the residue field of S over S.*

Proof. Observe that $E[I] = \operatorname{Hom}_R(S, E)$ as $S = R/I$. Hence $E[I]$ is an injective S-module by Lemma 3.4. Since E is an essential extension of $\kappa = R/\mathfrak{m}_R$ it follows that $E[I]$ is an essential extension of κ as well. The result follows. $\square$

08Z3 **Lemma 7.2.** *Let $(R, \mathfrak{m}, \kappa)$ be a local ring. Let E be the injective hull of κ. Let M be a $\mathfrak{m}$-power torsion R-module with $n = \dim_\kappa(M[\mathfrak{m}]) < \infty$. Then M is isomorphic to a submodule of $E^{\oplus n}$.*

Proof. Observe that $E^{\oplus n}$ is the injective hull of $\kappa^{\oplus n} = M[\mathfrak{m}]$. Thus there is an R-module map $M \to E^{\oplus n}$ which is injective on $M[\mathfrak{m}]$. Since M is $\mathfrak{m}$-power torsion the inclusion $M[\mathfrak{m}] \subset M$ is an essential extension (for example by Lemma 2.4) we conclude that the kernel of $M \to E^{\oplus n}$ is zero. $\square$

08Z4 **Lemma 7.3.** *Let $(R, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Let E be an injective hull of κ over R. Let E_n be an injective hull of κ over $R/\mathfrak{m}^n$. Then $E = \bigcup E_n$ and $E_n = E[\mathfrak{m}^n]$.*

Proof. We have $E_n = E[\mathfrak{m}^n]$ by Lemma 7.1. We have $E = \bigcup E_n$ because $\bigcup E_n = E[\mathfrak{m}^\infty]$ is an injective R-submodule which contains κ, see Lemma 3.9. $\square$

The following lemma tells us the injective hull of the residue field of a Noetherian local ring only depends on the completion.

08Z5 **Lemma 7.4.** *Let $R \to S$ be a flat local homomorphism of local Noetherian rings such that $R/\mathfrak{m}_R \cong S/\mathfrak{m}_R S$. Then the injective hull of the residue field of R is the injective hull of the residue field of S.*

Proof. Note that $\mathfrak{m}_R S = \mathfrak{m}_S$ as the quotient by the former is a field. Set $\kappa = R/\mathfrak{m}_R = S/\mathfrak{m}_S$. Let E_R be the injective hull of κ over R. Let E_S be the injective hull of κ over S. Observe that E_S is an injective R-module by Lemma 3.2. Choose an extension $E_R \to E_S$ of the identification of residue fields. This map is an isomorphism by Lemma 7.3 because $R \to S$ induces an isomorphism $R/\mathfrak{m}_R^n \to S/\mathfrak{m}_S^n$ for all n. $\square$

08Z6 **Lemma 7.5.** *Let $(R, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Let E be an injective hull of κ over R. Then $\mathrm{Hom}_R(E, E)$ is canonically isomorphic to the completion of R.*

Proof. Write $E = \bigcup E_n$ with $E_n = E[\mathfrak{m}^n]$ as in Lemma 7.3. Any endomorphism of E preserves this filtration. Hence

$$\mathrm{Hom}_R(E, E) = \lim \mathrm{Hom}_R(E_n, E_n)$$

The lemma follows as $\mathrm{Hom}_R(E_n, E_n) = \mathrm{Hom}_{R/\mathfrak{m}^n}(E_n, E_n) = R/\mathfrak{m}^n$ by Lemma 6.2. $\square$

08Z7 **Lemma 7.6.** *Let $(R, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Let E be an injective hull of κ over R. Then E satisfies the descending chain condition.*

Proof. If $E \supset M_1 \supset M_2 \supset \ldots$ is a sequence of submodules, then

$$\mathrm{Hom}_R(E, E) \to \mathrm{Hom}_R(M_1, E) \to \mathrm{Hom}_R(M_2, E) \to \ldots$$

is a sequence of surjections. By Lemma 7.5 each of these is a module over the completion $R^\wedge = \mathrm{Hom}_R(E, E)$. Since $R^\wedge$ is Noetherian (Algebra, Lemma 97.6) the sequence stabilizes: $\mathrm{Hom}_R(M_n, E) = \mathrm{Hom}_R(M_{n+1}, E) = \ldots$. Since E is injective, this can only happen if $\mathrm{Hom}_R(M_n/M_{n+1}, E)$ is zero. However, if M_n/M_{n+1} is nonzero, then it contains a nonzero element annihilated by $\mathfrak{m}$, because E is $\mathfrak{m}$-power torsion by Lemma 7.3. In this case M_n/M_{n+1} has a nonzero map into E, contradicting the assumed vanishing. This finishes the proof. $\square$

08Z8 **Lemma 7.7.** *Let $(R, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Let E be an injective hull of κ.*

(1) *For an R-module M the following are equivalent:*
   (a) *M satisfies the ascending chain condition,*
   (b) *M is a finite R-module, and*
   (c) *there exist n, m and an exact sequence $R^{\oplus m} \to R^{\oplus n} \to M \to 0$.*
(2) *For an R-module M the following are equivalent:*
   (a) *M satisfies the descending chain condition,*
   (b) *M is $\mathfrak{m}$-power torsion and $\dim_\kappa(M[\mathfrak{m}]) < \infty$, and*
   (c) *there exist n, m and an exact sequence $0 \to M \to E^{\oplus n} \to E^{\oplus m}$.*

Proof. We omit the proof of (1).

Let M be an R-module with the descending chain condition. Let $x \in M$. Then $\mathfrak{m}^n x$ is a descending chain of submodules, hence stabilizes. Thus $\mathfrak{m}^n x = \mathfrak{m}^{n+1} x$ for some n. By Nakayama's lemma (Algebra, Lemma 20.1) this implies $\mathfrak{m}^n x = 0$,

i.e., x is $\mathfrak{m}$-power torsion. Since $M[\mathfrak{m}]$ is a vector space over κ it has to be finite dimensional in order to have the descending chain condition.

Assume that M is $\mathfrak{m}$-power torsion and has a finite dimensional $\mathfrak{m}$-torsion submodule $M[\mathfrak{m}]$. By Lemma 7.2 we see that M is a submodule of $E^{\oplus n}$ for some n. Consider the quotient $N = E^{\oplus n}/M$. By Lemma 7.6 the module E has the descending chain condition hence so do $E^{\oplus n}$ and N. Therefore N satisfies (2)(a) which implies N satisfies (2)(b) by the second paragraph of the proof. Thus by Lemma 7.2 again we see that N is a submodule of $E^{\oplus m}$ for some m. Thus we have a short exact sequence $0 \to M \to E^{\oplus n} \to E^{\oplus m}$.

Assume we have a short exact sequence $0 \to M \to E^{\oplus n} \to E^{\oplus m}$. Since E satisfies the descending chain condition by Lemma 7.6 so does M. $\square$

08Z9 **Proposition 7.8** (Matlis duality). *Let $(R, \mathfrak{m}, \kappa)$ be a complete local Noetherian ring. Let E be an injective hull of κ over R. The functor $D(-) = \operatorname{Hom}_R(-, E)$ induces an anti-equivalence*

$$\left\{ \begin{array}{c} \text{R-modules with the} \\ \text{descending chain condition} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{R-modules with the} \\ \text{ascending chain condition} \end{array} \right\}$$

and we have $D \circ D = \mathrm{id}$ on either side of the equivalence.

Proof. By Lemma 7.5 we have $R = \operatorname{Hom}_R(E, E) = D(E)$. Of course we have $E = \operatorname{Hom}_R(R, E) = D(R)$. Since E is injective the functor D is exact. The result now follows immediately from the description of the categories in Lemma 7.7. $\square$

0EGL **Remark 7.9.** Let $(R, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Let E be an injective hull of κ over R. Here is an addendum to Matlis duality: If N is an $\mathfrak{m}$-power torsion module and $M = \operatorname{Hom}_R(N, E)$ is a finite module over the completion of R, then N satisfies the descending chain condition. Namely, for any submodules $N'' \subset N' \subset N$ with $N'' \neq N'$, we can find an embedding $\kappa \subset N''/N'$ and hence a nonzero map $N' \to E$ annihilating N'' which we can extend to a map $N \to E$ annihilating N''. Thus $N \supset N' \mapsto M' = \operatorname{Hom}_R(N/N', E) \subset M$ is an inclusion preserving map from submodules of N to submodules of M, whence the conclusion.

8. Deriving torsion

0BJA Let A be a ring and let $I \subset A$ be a finitely generated ideal (if I is not finitely generated perhaps a different definition should be used). Let $Z = V(I) \subset \operatorname{Spec}(A)$. Recall that the category I^∞-torsion of I-power torsion modules only depends on the closed subset Z and not on the choice of the finitely generated ideal I such that $Z = V(I)$, see More on Algebra, Lemma 88.6. In this section we will consider the functor

$$H_I^0 : \operatorname{Mod}_A \longrightarrow I^\infty\text{-torsion}, \quad M \longmapsto M[I^\infty] = \bigcup M[I^n]$$

which sends M to the submodule of I-power torsion.

Let A be a ring and let I be a finitely generated ideal. Note that I^∞-torsion is a Grothendieck abelian category (direct sums exist, filtered colimits are exact, and $\bigoplus A/I^n$ is a generator by More on Algebra, Lemma 88.2). Hence the derived category $D(I^\infty\text{-torsion})$ exists, see Injectives, Remark 13.3. Our functor H_I^0 is left exact and has a derived extension which we will denote

$$R\Gamma_I : D(A) \longrightarrow D(I^\infty\text{-torsion}).$$

Warning: this functor does not deserve the name local cohomology unless the ring A is Noetherian. The functors H_I^0, $R\Gamma_I$, and the satellites H_I^p only depend on the closed subset $Z \subset \mathrm{Spec}(A)$ and not on the choice of the finitely generated ideal I such that $V(I) = Z$. However, we insist on using the subscript I for the functors above as the notation $R\Gamma_Z$ is going to be used for a different functor, see (9.0.1), which agrees with the functor $R\Gamma_I$ only (as far as we know) in case A is Noetherian (see Lemma 10.1).

0A6L **Lemma 8.1.** *Let A be a ring and let $I \subset A$ be a finitely generated ideal. The functor $R\Gamma_I$ is right adjoint to the functor $D(I^\infty\text{-torsion}) \to D(A)$.*

Proof. This follows from the fact that taking I-power torsion submodules is the right adjoint to the inclusion functor I^∞-torsion $\to \mathrm{Mod}_A$. See Derived Categories, Lemma 30.3. $\square$

0954 **Lemma 8.2.** *Let A be a ring and let $I \subset A$ be a finitely generated ideal. For any object K of $D(A)$ we have*

$$R\Gamma_I(K) = \mathrm{hocolim}\, R\mathrm{Hom}_A(A/I^n, K)$$

in $D(A)$ and

$$R^q\Gamma_I(K) = \mathrm{colim}_n \mathrm{Ext}_A^q(A/I^n, K)$$

as modules for all $q \in \mathbf{Z}$.

Proof. Let $J^\bullet$ be a K-injective complex representing K. Then

$$R\Gamma_I(K) = J^\bullet[I^\infty] = \mathrm{colim}\, J^\bullet[I^n] = \mathrm{colim}\,\mathrm{Hom}_A(A/I^n, J^\bullet)$$

where the first equality is the definition of $R\Gamma_I(K)$. By Derived Categories, Lemma 33.7 we obtain the first displayed equality in the statement of the lemma. The second displayed equality in the statement of the lemma then follows because $H^q(\mathrm{Hom}_A(A/I^n, J^\bullet)) = \mathrm{Ext}_A^q(A/I^n, K)$ and because filtered colimits are exact in the category of abelian groups. $\square$

0A6M **Lemma 8.3.** *Let A be a ring and let $I \subset A$ be a finitely generated ideal. Let $K^\bullet$ be a complex of A-modules such that $f : K^\bullet \to K^\bullet$ is an isomorphism for some $f \in I$, i.e., $K^\bullet$ is a complex of A_f-modules. Then $R\Gamma_I(K^\bullet) = 0$.*

Proof. Namely, in this case the cohomology modules of $R\Gamma_I(K^\bullet)$ are both f-power torsion and f acts by automorphisms. Hence the cohomology modules are zero and hence the object is zero. $\square$

Let A be a ring and $I \subset A$ a finitely generated ideal. By More on Algebra, Lemma 88.5 the category of I-power torsion modules is a Serre subcategory of the category of all A-modules, hence there is a functor

0A6N (8.3.1) $$D(I^\infty\text{-torsion}) \to D_{I^\infty\text{-torsion}}(A)$$

see Derived Categories, Section 17.

0A6P **Lemma 8.4.** *Let A be a ring and let I be a finitely generated ideal. Let M and N be I-power torsion modules.*

(1) $\mathrm{Hom}_{D(A)}(M, N) = \mathrm{Hom}_{D(I^\infty\text{-}torsion)}(M, N)$,
(2) $\mathrm{Ext}_{D(A)}^1(M, N) = \mathrm{Ext}_{D(I^\infty\text{-}torsion)}^1(M, N)$,
(3) $\mathrm{Ext}_{D(I^\infty\text{-}torsion)}^2(M, N) \to \mathrm{Ext}_{D(A)}^2(M, N)$ *is not surjective in general,*
(4) *(8.3.1) is not an equivalence in general.*

Proof. Parts (1) and (2) follow immediately from the fact that I-power torsion forms a Serre subcategory of Mod_A. Part (4) follows from part (3).

For part (3) let A be a ring with an element $f \in A$ such that $A[f]$ contains a nonzero element x annihilated by f and A contains elements x_n with $f^n x_n = x$. Such a ring A exists because we can take

$$A = \mathbf{Z}[f, x, x_n]/(fx, f^n x_n - x)$$

Given A set $I = (f)$. Then the exact sequence

$$0 \to A[f] \to A \xrightarrow{f} A \to A/fA \to 0$$

defines an element in $\text{Ext}^2_A(A/fA, A[f])$. We claim this element does not come from an element of $\text{Ext}^2_{D(f^\infty\text{-torsion})}(A/fA, A[f])$. Namely, if it did, then there would be an exact sequence

$$0 \to A[f] \to M \to N \to A/fA \to 0$$

where M and N are f-power torsion modules defining the same 2 extension class. Since $A \to A$ is a complex of free modules and since the 2 extension classes are the same we would be able to find a map

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A[f] & \longrightarrow & A & \longrightarrow & A & \longrightarrow & A/fA & \longrightarrow & 0 \\
& & \downarrow & & \varphi\downarrow & & \psi\downarrow & & \downarrow & & \\
0 & \longrightarrow & A[f] & \longrightarrow & M & \longrightarrow & N & \longrightarrow & A/fA & \longrightarrow & 0
\end{array}
$$

(some details omitted). Then we could replace M by the image of φ and N by the image of ψ. Then M would be a cyclic module, hence $f^n M = 0$ for some n. Considering $\varphi(x_{n+1})$ we get a contradiction with the fact that $f^{n+1} x_n = x$ is nonzero in $A[f]$. $\square$

9. Local cohomology

0952   Let A be a ring and let $I \subset A$ be a finitely generated ideal. Set $Z = V(I) \subset \text{Spec}(A)$. We will construct a functor

0A6Q   (9.0.1) $$R\Gamma_Z : D(A) \longrightarrow D_{I^\infty\text{-torsion}}(A).$$

which is right adjoint to the inclusion functor. For notation see Section 8. The cohomology modules of $R\Gamma_Z(K)$ are the *local cohomology groups of K with respect to Z*. By Lemma 8.4 this functor will in general **not** be equal to $R\Gamma_I(-)$ even viewed as functors into $D(A)$. In Section 10 we will show that if A is Noetherian, then the two agree.

We will continue the discussion of local cohomology in the chapter on local cohomology, see Local Cohomology, Section 1. For example, there we will show that $R\Gamma_Z$ computes cohomology with support in Z for the associated complex of quasi-coherent sheaves on $\text{Spec}(A)$. See Local Cohomology, Lemma 2.1.

0A6R   **Lemma 9.1.** *Let A be a ring and let $I \subset A$ be a finitely generated ideal. There exists a right adjoint $R\Gamma_Z$ (9.0.1) to the inclusion functor $D_{I^\infty\text{-torsion}}(A) \to D(A)$. In fact, if I is generated by $f_1, \ldots, f_r \in A$, then we have*

$$R\Gamma_Z(K) = (A \to \prod_{i_0} A_{f_{i_0}} \to \prod_{i_0 < i_1} A_{f_{i_0} f_{i_1}} \to \ldots \to A_{f_1 \ldots f_r}) \otimes^{\mathbf{L}}_A K$$

functorially in $K \in D(A)$.

Proof. Say $I = (f_1, \ldots, f_r)$ is an ideal. Let $K^\bullet$ be a complex of A-modules. There is a canonical map of complexes

$$(A \to \prod_{i_0} A_{f_{i_0}} \to \prod_{i_0 < i_1} A_{f_{i_0} f_{i_1}} \to \ldots \to A_{f_1 \ldots f_r}) \longrightarrow A.$$

from the extended Čech complex to A. Tensoring with $K^\bullet$, taking associated total complex, we get a map

$$\mathrm{Tot}\left(K^\bullet \otimes_A (A \to \prod_{i_0} A_{f_{i_0}} \to \prod_{i_0 < i_1} A_{f_{i_0} f_{i_1}} \to \ldots \to A_{f_1 \ldots f_r})\right) \longrightarrow K^\bullet$$

in $D(A)$. We claim the cohomology modules of the complex on the left are I-power torsion, i.e., the LHS is an object of $D_{I^\infty\text{-torsion}}(A)$. Namely, we have

$$(A \to \prod_{i_0} A_{f_{i_0}} \to \prod_{i_0 < i_1} A_{f_{i_0} f_{i_1}} \to \ldots \to A_{f_1 \ldots f_r}) = \mathrm{colim}\, K(A, f_1^n, \ldots, f_r^n)$$

by More on Algebra, Lemma 29.6. Moreover, multiplication by f_i^n on the complex $K(A, f_1^n, \ldots, f_r^n)$ is homotopic to zero by More on Algebra, Lemma 28.6. Since

$$H^q(LHS) = \mathrm{colim}\, H^q(\mathrm{Tot}(K^\bullet \otimes_A K(A, f_1^n, \ldots, f_r^n)))$$

we obtain our claim. On the other hand, if $K^\bullet$ is an object of $D_{I^\infty\text{-torsion}}(A)$, then the complexes $K^\bullet \otimes_A A_{f_{i_0} \ldots f_{i_p}}$ have vanishing cohomology. Hence in this case the map $LHS \to K^\bullet$ is an isomorphism in $D(A)$. The construction

$$R\Gamma_Z(K^\bullet) = \mathrm{Tot}\left(K^\bullet \otimes_A (A \to \prod_{i_0} A_{f_{i_0}} \to \prod_{i_0 < i_1} A_{f_{i_0} f_{i_1}} \to \ldots \to A_{f_1 \ldots f_r})\right)$$

is functorial in $K^\bullet$ and defines an exact functor $D(A) \to D_{I^\infty\text{-torsion}}(A)$ between triangulated categories. It follows formally from the existence of the natural transformation $R\Gamma_Z \to \mathrm{id}$ given above and the fact that this evaluates to an isomorphism on $K^\bullet$ in the subcategory, that $R\Gamma_Z$ is the desired right adjoint. $\square$

0BJB **Lemma 9.2.** *Let $A \to B$ be a ring homomorphism and let $I \subset A$ be a finitely generated ideal. Set $J = IB$. Set $Z = V(I)$ and $Y = V(J)$. Then*

$$R\Gamma_Z(M_A) = R\Gamma_Y(M)_A$$

functorially in $M \in D(B)$. Here $(-)_A$ denotes the restriction functors $D(B) \to D(A)$ and $D_{J^\infty\text{-torsion}}(B) \to D_{I^\infty\text{-torsion}}(A)$.

Proof. This follows from uniqueness of adjoint functors as both $R\Gamma_Z((-)_A)$ and $R\Gamma_Y(-)_A$ are right adjoint to the functor $D_{I^\infty\text{-torsion}}(A) \to D(B)$, $K \mapsto K \otimes_A^{\mathbf{L}} B$. Alternatively, one can use the description of $R\Gamma_Z$ and $R\Gamma_Y$ in terms of alternating Čech complexes (Lemma 9.1). Namely, if $I = (f_1, \ldots, f_r)$ then J is generated by the images $g_1, \ldots, g_r \in B$ of $f_1, \ldots, f_r$. Then the statement of the lemma follows from the existence of a canonical isomorphism

$$M_A \otimes_A (A \to \prod_{i_0} A_{f_{i_0}} \to \prod_{i_0 < i_1} A_{f_{i_0} f_{i_1}} \to \ldots \to A_{f_1 \ldots f_r})$$
$$= M \otimes_B (B \to \prod_{i_0} B_{g_{i_0}} \to \prod_{i_0 < i_1} B_{g_{i_0} g_{i_1}} \to \ldots \to B_{g_1 \ldots g_r})$$

for any B-module M. $\square$

0ALZ **Lemma 9.3.** *Let $A \to B$ be a ring homomorphism and let $I \subset A$ be a finitely generated ideal. Set $J = IB$. Let $Z = V(I)$ and $Y = V(J)$. Then*

$$R\Gamma_Z(K) \otimes_A^{\mathbf{L}} B = R\Gamma_Y(K \otimes_A^{\mathbf{L}} B)$$

functorially in $K \in D(A)$.

Proof. Write $I = (f_1, \ldots, f_r)$. Then J is generated by the images $g_1, \ldots, g_r \in B$ of $f_1, \ldots, f_r$. Then we have

$$(A \to \prod A_{f_{i_0}} \to \ldots \to A_{f_1 \ldots f_r}) \otimes_A B = (B \to \prod B_{g_{i_0}} \to \ldots \to B_{g_1 \ldots g_r})$$

as complexes of B-modules. Represent K by a K-flat complex $K^\bullet$ of A-modules. Since the total complexes associated to

$$K^\bullet \otimes_A (A \to \prod A_{f_{i_0}} \to \ldots \to A_{f_1 \ldots f_r}) \otimes_A B$$

and

$$K^\bullet \otimes_A B \otimes_B (B \to \prod B_{g_{i_0}} \to \ldots \to B_{g_1 \ldots g_r})$$

represent the left and right hand side of the displayed formula of the lemma (see Lemma 9.1) we conclude. $\square$

0A6S **Lemma 9.4.** *Let A be a ring and let $I \subset A$ be a finitely generated ideal. Let $K^\bullet$ be a complex of A-modules such that $f : K^\bullet \to K^\bullet$ is an isomorphism for some $f \in I$, i.e., $K^\bullet$ is a complex of A_f-modules. Then $R\Gamma_Z(K^\bullet) = 0$.*

Proof. Namely, in this case the cohomology modules of $R\Gamma_Z(K^\bullet)$ are both f-power torsion and f acts by automorphisms. Hence the cohomology modules are zero and hence the object is zero. $\square$

0ALY **Lemma 9.5.** *Let A be a ring and let $I \subset A$ be a finitely generated ideal. For $K, L \in D(A)$ we have*

$$R\Gamma_Z(K \otimes_A^{\mathbf{L}} L) = K \otimes_A^{\mathbf{L}} R\Gamma_Z(L) = R\Gamma_Z(K) \otimes_A^{\mathbf{L}} L = R\Gamma_Z(K) \otimes_A^{\mathbf{L}} R\Gamma_Z(L)$$

If K or L is in $D_{I^\infty\text{-torsion}}(A)$ then so is $K \otimes_A^{\mathbf{L}} L$.

Proof. By Lemma 9.1 we know that $R\Gamma_Z$ is given by $C \otimes^{\mathbf{L}} -$ for some $C \in D(A)$. Hence, for $K, L \in D(A)$ general we have

$$R\Gamma_Z(K \otimes_A^{\mathbf{L}} L) = K \otimes^{\mathbf{L}} L \otimes_A^{\mathbf{L}} C = K \otimes_A^{\mathbf{L}} R\Gamma_Z(L)$$

The other equalities follow formally from this one. This also implies the last statement of the lemma. $\square$

0BJC **Lemma 9.6.** *Let A be a ring and let $I, J \subset A$ be finitely generated ideals. Set $Z = V(I)$ and $Y = V(J)$. Then $Z \cap Y = V(I + J)$ and $R\Gamma_Y \circ R\Gamma_Z = R\Gamma_{Y \cap Z}$ as functors $D(A) \to D_{(I+J)^\infty\text{-torsion}}(A)$. For $K \in D^+(A)$ there is a spectral sequence*

$$E_2^{p,q} = H_Y^p(H_Z^q(K)) \Rightarrow H_{Y \cap Z}^{p+q}(K)$$

as in Derived Categories, Lemma 22.2.

Proof. There is a bit of abuse of notation in the lemma as strictly speaking we cannot compose $R\Gamma_Y$ and $R\Gamma_Z$. The meaning of the statement is simply that we are composing $R\Gamma_Z$ with the inclusion $D_{I^\infty\text{-torsion}}(A) \to D(A)$ and then with $R\Gamma_Y$. Then the equality $R\Gamma_Y \circ R\Gamma_Z = R\Gamma_{Y \cap Z}$ follows from the fact that

$$D_{I^\infty\text{-torsion}}(A) \to D(A) \xrightarrow{R\Gamma_Y} D_{(I+J)^\infty\text{-torsion}}(A)$$

is right adjoint to the inclusion $D_{(I+J)^\infty\text{-torsion}}(A) \to D_{I^\infty\text{-torsion}}(A)$. Alternatively one can prove the formula using Lemma 9.1 and the fact that the tensor product of extended Čech complexes on $f_1, \ldots, f_r$ and $g_1, \ldots, g_m$ is the extended Čech complex on $f_1, \ldots, f_n.g_1, \ldots, g_m$. The final assertion follows from this and the cited lemma. $\square$

The following lemma is the analogue of More on Algebra, Lemma 91.24 for complexes with torsion cohomologies.

0AM0 **Lemma 9.7.** *Let $A \to B$ be a flat ring map and let $I \subset A$ be a finitely generated ideal such that $A/I = B/IB$. Then base change and restriction induce quasi-inverse equivalences $D_{I^\infty\text{-}torsion}(A) = D_{(IB)^\infty\text{-}torsion}(B)$.*

Proof. More precisely the functors are $K \mapsto K \otimes_A^{\mathbf{L}} B$ for K in $D_{I^\infty\text{-torsion}}(A)$ and $M \mapsto M_A$ for M in $D_{(IB)^\infty\text{-torsion}}(B)$. The reason this works is that $H^i(K \otimes_A^{\mathbf{L}} B) = H^i(K) \otimes_A B = H^i(K)$. The first equality holds as $A \to B$ is flat and the second by More on Algebra, Lemma 89.2. $\square$

The following lemma was shown for Hom and Ext^1 of modules in More on Algebra, Lemmas 89.3 and 89.8.

05EH **Lemma 9.8.** *Let $A \to B$ be a flat ring map and let $I \subset A$ be a finitely generated ideal such that $A/I \to B/IB$ is an isomorphism. For $K \in D_{I^\infty\text{-}torsion}(A)$ and $L \in D(A)$ the map*

$$R\operatorname{Hom}_A(K, L) \longrightarrow R\operatorname{Hom}_B(K \otimes_A B, L \otimes_A B)$$

is a quasi-isomorphism. In particular, if M, N are A-modules and M is I-power torsion, then the canonical map

$$\operatorname{Ext}_A^i(M, N) \longrightarrow \operatorname{Ext}_B^i(M \otimes_A B, N \otimes_A B)$$

is an isomorphism for all i.

Proof. Let $Z = V(I) \subset \operatorname{Spec}(A)$ and $Y = V(IB) \subset \operatorname{Spec}(B)$. Since the cohomology modules of K are I power torsion, the canonical map $R\Gamma_Z(L) \to L$ induces an isomorphism

$$R\operatorname{Hom}_A(K, R\Gamma_Z(L)) \to R\operatorname{Hom}_A(K, L)$$

in $D(A)$. Similarly, the cohomology modules of $K \otimes_A B$ are IB power torsion and we have an isomorphism

$$R\operatorname{Hom}_B(K \otimes_A B, R\Gamma_Y(L \otimes_A B)) \to R\operatorname{Hom}_B(K \otimes_A B, L \otimes_A B)$$

in $D(B)$. By Lemma 9.3 we have $R\Gamma_Z(L) \otimes_A B = R\Gamma_Y(L \otimes_A B)$. Hence it suffices to show that the map

$$R\operatorname{Hom}_A(K, R\Gamma_Z(L)) \to R\operatorname{Hom}_B(K \otimes_A B, R\Gamma_Z(L) \otimes_A B)$$

is a quasi-isomorphism. This follows from Lemma 9.7. $\square$

10. Local cohomology for Noetherian rings

0BJD Let A be a ring and let $I \subset A$ be a finitely generated ideal. Set $Z = V(I) \subset \operatorname{Spec}(A)$. Recall that (8.3.1) is the functor

$$D(I^\infty\text{-torsion}) \to D_{I^\infty\text{-torsion}}(A)$$

In fact, there is a natural transformation of functors

0A6U (10.0.1) $\qquad\qquad (8.3.1) \circ R\Gamma_I(-) \longrightarrow R\Gamma_Z(-)$

Namely, given a complex of A-modules $K^\bullet$ the canonical map $R\Gamma_I(K^\bullet) \to K^\bullet$ in $D(A)$ factors (uniquely) through $R\Gamma_Z(K^\bullet)$ as $R\Gamma_I(K^\bullet)$ has I-power torsion cohomology modules (see Lemma 8.1). In general this map is not an isomorphism (we've seen this in Lemma 8.4).

0955 **Lemma 10.1.** *Let A be a Noetherian ring and let $I \subset A$ be an ideal.*

    (1) *the adjunction $R\Gamma_I(K) \to K$ is an isomorphism for $K \in D_{I^\infty\text{-}torsion}(A)$,*

    (2) *the functor (8.3.1) $D(I^\infty\text{-}torsion) \to D_{I^\infty\text{-}torsion}(A)$ is an equivalence,*

    (3) *the transformation of functors (10.0.1) is an isomorphism, in other words $R\Gamma_I(K) = R\Gamma_Z(K)$ for $K \in D(A)$.*

Proof. A formal argument, which we omit, shows that it suffices to prove (1).

Let M be an I-power torsion A-module. Choose an embedding $M \to J$ into an injective A-module. Then $J[I^\infty]$ is an injective A-module, see Lemma 3.9, and we obtain an embedding $M \to J[I^\infty]$. Thus every I-power torsion module has an injective resolution $M \to J^\bullet$ with J^n also I-power torsion. It follows that $R\Gamma_I(M) = M$ (this is not a triviality and this is not true in general if A is not Noetherian). Next, suppose that $K \in D^+_{I^\infty\text{-}torsion}(A)$. Then the spectral sequence

$$R^q\Gamma_I(H^p(K)) \Rightarrow R^{p+q}\Gamma_I(K)$$

(Derived Categories, Lemma 21.3) converges and above we have seen that only the terms with $q = 0$ are nonzero. Thus we see that $R\Gamma_I(K) \to K$ is an isomorphism.

Suppose K is an arbitrary object of $D_{I^\infty\text{-}torsion}(A)$. We have

$$R^q\Gamma_I(K) = \operatorname{colim} \operatorname{Ext}^q_A(A/I^n, K)$$

by Lemma 8.2. Choose $f_1, \ldots, f_r \in A$ generating I. Let $K_n^\bullet = K(A, f_1^n, \ldots, f_r^n)$ be the Koszul complex with terms in degrees $-r, \ldots, 0$. Since the pro-objects $\{A/I^n\}$ and $\{K_n^\bullet\}$ in $D(A)$ are the same by More on Algebra, Lemma 94.1, we see that

$$R^q\Gamma_I(K) = \operatorname{colim} \operatorname{Ext}^q_A(K_n^\bullet, K)$$

Pick any complex $K^\bullet$ of A-modules representing K. Since $K_n^\bullet$ is a finite complex of finite free modules we see that

$$\operatorname{Ext}^q_A(K_n, K) = H^q(\operatorname{Tot}((K_n^\bullet)^\vee \otimes_A K^\bullet))$$

where $(K_n^\bullet)^\vee$ is the dual of the complex $K_n^\bullet$. See More on Algebra, Lemma 73.2. As $(K_n^\bullet)^\vee$ is a complex of finite free A-modules sitting in degrees $0, \ldots, r$ we see that the terms of the complex $\operatorname{Tot}((K_n^\bullet)^\vee \otimes_A K^\bullet)$ are the same as the terms of the complex $\operatorname{Tot}((K_n^\bullet)^\vee \otimes_A \tau_{\geq q-r-2} K^\bullet)$ in degrees $q-1$ and higher. Hence we see that

$$\operatorname{Ext}^q_A(K_n, K) = \operatorname{Ext}^q_A(K_n, \tau_{\geq q-r-2} K)$$

for all n. It follows that

$$R^q\Gamma_I(K) = R^q\Gamma_I(\tau_{\geq q-r-2}K) = H^q(\tau_{\geq q-r-2}K) = H^q(K)$$

Thus we see that the map $R\Gamma_I(K) \to K$ is an isomorphism. $\square$

0956 **Lemma 10.2.** *Let A be a Noetherian ring and let $I = (f_1, \ldots, f_r)$ be an ideal of A. Set $Z = V(I) \subset \operatorname{Spec}(A)$. There are canonical isomorphisms*

$$R\Gamma_I(A) \to (A \to \prod_{i_0} A_{f_{i_0}} \to \prod_{i_0 < i_1} A_{f_{i_0} f_{i_1}} \to \ldots \to A_{f_1 \ldots f_r}) \to R\Gamma_Z(A)$$

in $D(A)$. If M is an A-module, then we have similarly

$$R\Gamma_I(M) \cong (M \to \prod_{i_0} M_{f_{i_0}} \to \prod_{i_0 < i_1} M_{f_{i_0} f_{i_1}} \to \ldots \to M_{f_1 \ldots f_r}) \cong R\Gamma_Z(M)$$

in $D(A)$.

Proof. This follows from Lemma 10.1 and the computation of the functor $R\Gamma_Z$ in Lemma 9.1. $\square$

0957   **Lemma 10.3.** *If $A \to B$ is a homomorphism of Noetherian rings and $I \subset A$ is an ideal, then in $D(B)$ we have*

$$R\Gamma_I(A) \otimes_A^{\mathbf{L}} B = R\Gamma_Z(A) \otimes_A^{\mathbf{L}} B = R\Gamma_Y(B) = R\Gamma_{IB}(B)$$

where $Y = V(IB) \subset \mathrm{Spec}(B)$.

Proof. Combine Lemmas 10.2 and 9.3.   □

11. Depth

0AVY   In this section we revisit the notion of depth introduced in Algebra, Section 72.

0AVZ   **Lemma 11.1.** *Let A be a Noetherian ring, let $I \subset A$ be an ideal, and let M be a finite A-module such that $IM \neq M$. Then the following integers are equal:*

(1) *$\mathrm{depth}_I(M)$,*
(2) *the smallest integer i such that $\mathrm{Ext}_A^i(A/I, M)$ is nonzero, and*
(3) *the smallest integer i such that $H_I^i(M)$ is nonzero.*

Moreover, we have $\mathrm{Ext}_A^i(N, M) = 0$ for $i < \mathrm{depth}_I(M)$ for any finite A-module N annihilated by a power of I.

Proof. We prove the equality of (1) and (2) by induction on $\mathrm{depth}_I(M)$ which is allowed by Algebra, Lemma 72.4.

Base case. If $\mathrm{depth}_I(M) = 0$, then I is contained in the union of the associated primes of M (Algebra, Lemma 63.9). By prime avoidance (Algebra, Lemma 15.2) we see that $I \subset \mathfrak{p}$ for some associated prime $\mathfrak{p}$. Hence $\mathrm{Hom}_A(A/I, M)$ is nonzero. Thus equality holds in this case.

Assume that $\mathrm{depth}_I(M) > 0$. Let $f \in I$ be a nonzerodivisor on M such that $\mathrm{depth}_I(M/fM) = \mathrm{depth}_I(M) - 1$. Consider the short exact sequence

$$0 \to M \to M \to M/fM \to 0$$

and the associated long exact sequence for $\mathrm{Ext}_A^*(A/I, -)$. Note that $\mathrm{Ext}_A^i(A/I, M)$ is a finite A/I-module (Algebra, Lemmas 71.9 and 71.8). Hence we obtain

$$\mathrm{Hom}_A(A/I, M/fM) = \mathrm{Ext}_A^1(A/I, M)$$

and short exact sequences

$$0 \to \mathrm{Ext}_A^i(A/I, M) \to \mathrm{Ext}_A^i(A/I, M/fM) \to \mathrm{Ext}_A^{i+1}(A/I, M) \to 0$$

Thus the equality of (1) and (2) by induction.

Observe that $\mathrm{depth}_I(M) = \mathrm{depth}_{I^n}(M)$ for all $n \geq 1$ for example by Algebra, Lemma 68.9. Hence by the equality of (1) and (2) we see that $\mathrm{Ext}_A^i(A/I^n, M) = 0$ for all n and $i < \mathrm{depth}_I(M)$. Let N be a finite A-module annihilated by a power of I. Then we can choose a short exact sequence

$$0 \to N' \to (A/I^n)^{\oplus m} \to N \to 0$$

for some $n, m \geq 0$. Then $\mathrm{Hom}_A(N, M) \subset \mathrm{Hom}_A((A/I^n)^{\oplus m}, M)$ and $\mathrm{Ext}_A^i(N, M) \subset \mathrm{Ext}_A^{i-1}(N', M)$ for $i < \mathrm{depth}_I(M)$. Thus a simply induction argument shows that the final statement of the lemma holds.

Finally, we prove that (3) is equal to (1) and (2). We have $H_I^p(M) = \mathrm{colim}\, \mathrm{Ext}_A^p(A/I^n, M)$ by Lemma 8.2. Thus we see that $H_I^i(M) = 0$ for $i < \mathrm{depth}_I(M)$. For $i =$

$\text{depth}_I(M)$, using the vanishing of $\text{Ext}_A^{i-1}(I/I^n, M)$ we see that the map $\text{Ext}_A^i(A/I, M) \to$ $H_I^i(M)$ is injective which proves nonvanishing in the correct degree. $\square$

0BUV **Lemma 11.2.** *Let A be a Noetherian ring. Let $0 \to N' \to N \to N'' \to 0$ be a short exact sequence of finite A-modules. Let $I \subset A$ be an ideal.*

(1) *$depth_I(N) \geq \min\{depth_I(N'), depth_I(N'')\}$*
(2) *$depth_I(N'') \geq \min\{depth_I(N), depth_I(N') - 1\}$*
(3) *$depth_I(N') \geq \min\{depth_I(N), depth_I(N'') + 1\}$*

Proof. Assume $IN \neq N$, $IN' \neq N'$, and $IN'' \neq N''$. Then we can use the characterization of depth using the Ext groups $\text{Ext}^i(A/I, N)$, see Lemma 11.1, and use the long exact cohomology sequence

$$0 \to \text{Hom}_A(A/I, N') \to \text{Hom}_A(A/I, N) \to \text{Hom}_A(A/I, N'')$$
$$\to \text{Ext}_A^1(A/I, N') \to \text{Ext}_A^1(A/I, N) \to \text{Ext}_A^1(A/I, N'') \to \dots$$

from Algebra, Lemma 71.6. This argument also works if $IN = N$ because in this case $\text{Ext}_A^i(A/I, N) = 0$ for all i. Similarly in case $IN' \neq N'$ and/or $IN'' \neq N''$. $\square$

0BUW **Lemma 11.3.** *Let A be a Noetherian ring, let $I \subset A$ be an ideal, and let M a finite A-module with $IM \neq M$.*

(1) *If $x \in I$ is a nonzerodivisor on M, then $depth_I(M/xM) = depth_I(M) - 1$.*
(2) *Any M-regular sequence $x_1, \dots, x_r$ in I can be extended to an M-regular sequence in I of length $depth_I(M)$.*

Proof. Part (2) is a formal consequence of part (1). Let $x \in I$ be as in (1). By the short exact sequence $0 \to M \to M \to M/xM \to 0$ and Lemma 11.2 we see that $\text{depth}_I(M/xM) \geq \text{depth}_I(M) - 1$. On the other hand, if $x_1, \dots, x_r \in I$ is a regular sequence for M/xM, then $x, x_1, \dots, x_r$ is a regular sequence for M. Hence (1) holds. $\square$

0BUX **Lemma 11.4.** *Let R be a Noetherian local ring. If M is a finite Cohen-Macaulay R-module and $I \subset R$ a nontrivial ideal. Then*

$$depth_I(M) = \dim(Supp(M)) - \dim(Supp(M/IM)).$$

Proof. We will prove this by induction on $\text{depth}_I(M)$.

If $\text{depth}_I(M) = 0$, then I is contained in one of the associated primes $\mathfrak{p}$ of M (Algebra, Lemma 63.18). Then $\mathfrak{p} \in \text{Supp}(M/IM)$, hence $\dim(\text{Supp}(M/IM)) \geq \dim(R/\mathfrak{p}) = \dim(\text{Supp}(M))$ where equality holds by Algebra, Lemma 103.7. Thus the lemma holds in this case.

If $\text{depth}_I(M) > 0$, we pick $x \in I$ which is a nonzerodivisor on M. Note that $(M/xM)/I(M/xM) = M/IM$. On the other hand we have $\text{depth}_I(M/xM) = \text{depth}_I(M) - 1$ by Lemma 11.3 and $\dim(\text{Supp}(M/xM)) = \dim(\text{Supp}(M)) - 1$ by Algebra, Lemma 63.10. Thus the result by induction hypothesis. $\square$

0BUY **Lemma 11.5.** *Let $R \to S$ be a flat local ring homomorphism of Noetherian local rings. Denote $\mathfrak{m} \subset R$ the maximal ideal. Let $I \subset S$ be an ideal. If $S/\mathfrak{m}S$ is Cohen-Macaulay, then*

$$depth_I(S) \geq \dim(S/\mathfrak{m}S) - \dim(S/\mathfrak{m}S + I)$$

Proof. By Algebra, Lemma 99.3 any sequence in S which maps to a regular sequence in $S/\mathfrak{m}S$ is a regular sequence in S. Thus it suffices to prove the lemma in case R is a field. This is a special case of Lemma 11.4. $\square$

0AW0   **Lemma 11.6.** *Let A be a ring and let $I \subset A$ be a finitely generated ideal. Let M be an A-module. Let $Z = V(I)$. Then $H_I^0(M) = H_Z^0(M)$. Let N be the common value and set $M' = M/N$. Then*

(1) *$H_I^0(M') = 0$ and $H_I^p(M) = H_I^p(M')$ and $H_I^p(N) = 0$ for all $p > 0$,*
(2) *$H_Z^0(M') = 0$ and $H_Z^p(M) = H_Z^p(M')$ and $H_Z^p(N) = 0$ for all $p > 0$.*

Proof. By definition $H_I^0(M) = M[I^\infty]$ is I-power torsion. By Lemma 9.1 we see that

$$H_Z^0(M) = \mathrm{Ker}(M \longrightarrow M_{f_1} \times \ldots \times M_{f_r})$$

if $I = (f_1, \ldots, f_r)$. Thus $H_I^0(M) \subset H_Z^0(M)$ and conversely, if $x \in H_Z^0(M)$, then it is annihilated by a $f_i^{e_i}$ for some $e_i \geq 1$ hence annihilated by some power of I. This proves the first equality and moreover N is I-power torsion. By Lemma 8.1 we see that $R\Gamma_I(N) = N$. By Lemma 9.1 we see that $R\Gamma_Z(N) = N$. This proves the higher vanishing of $H_I^p(N)$ and $H_Z^p(N)$ in (1) and (2). The vanishing of $H_I^0(M')$ and $H_Z^0(M')$ follow from the preceding remarks and the fact that M' is I-power torsion free by More on Algebra, Lemma 88.4. The equality of higher cohomologies for M and M' follow immediately from the long exact cohomology sequence. $\square$

12. Torsion versus complete modules

0A6V   Let A be a ring and let I be a finitely generated ideal. In this case we can consider the derived category $D_{I^\infty\text{-torsion}}(A)$ of complexes with I-power torsion cohomology modules (Section 9) and the derived category $D_{comp}(A, I)$ of derived complete complexes (More on Algebra, Section 91). In this section we show these categories are equivalent. A more general statement can be found in [DG02].

0A6W   **Lemma 12.1.** *Let A be a ring and let I be a finitely generated ideal. Let $R\Gamma_Z$ be as in Lemma 9.1. Let $^\wedge$ denote derived completion as in More on Algebra, Lemma 91.10. For an object K in $D(A)$ we have*

$$R\Gamma_Z(K^\wedge) = R\Gamma_Z(K) \quad and \quad (R\Gamma_Z(K))^\wedge = K^\wedge$$

in $D(A)$.

Proof. Choose $f_1, \ldots, f_r \in A$ generating I. Recall that

$$K^\wedge = R\,\mathrm{Hom}_A\left((A \to \prod A_{f_{i_0}} \to \prod A_{f_{i_0 i_1}} \to \ldots \to A_{f_1 \ldots f_r}), K\right)$$

by More on Algebra, Lemma 91.10. Hence the cone $C = \mathrm{Cone}(K \to K^\wedge)$ is given by

$$R\,\mathrm{Hom}_A\left((\prod A_{f_{i_0}} \to \prod A_{f_{i_0 i_1}} \to \ldots \to A_{f_1 \ldots f_r}), K\right)$$

which can be represented by a complex endowed with a finite filtration whose successive quotients are isomorphic to

$$R\,\mathrm{Hom}_A(A_{f_{i_0} \ldots f_{i_p}}, K), \quad p > 0$$

These complexes vanish on applying $R\Gamma_Z$, see Lemma 9.4. Applying $R\Gamma_Z$ to the distinguished triangle $K \to K^\wedge \to C \to K[1]$ we see that the first formula of the lemma is correct.

Recall that

$$R\Gamma_Z(K) = K \otimes^{\mathbf{L}} (A \to \prod A_{f_{i_0}} \to \prod A_{f_{i_0 i_1}} \to \ldots \to A_{f_1 \ldots f_r})$$

by Lemma 9.1. Hence the cone $C = \operatorname{Cone}(R\Gamma_Z(K) \to K)$ can be represented by a complex endowed with a finite filtration whose successive quotients are isomorphic to

$$K \otimes_A A_{f_{i_0} \ldots f_{i_p}}, \quad p > 0$$

These complexes vanish on applying $^\wedge$, see More on Algebra, Lemma 91.12. Applying derived completion to the distinguished triangle $R\Gamma_Z(K) \to K \to C \to R\Gamma_Z(K)[1]$ we see that the second formula of the lemma is correct. $\square$

The following result is a special case of a very general phenomenon concerning admissible subcategories of a triangulated category.

0A6X **Proposition 12.2.** *Let A be a ring and let $I \subset A$ be a finitely generated ideal. The functors $R\Gamma_Z$ and $^\wedge$ define quasi-inverse equivalences of categories*

$$D_{I^\infty\text{-}torsion}(A) \leftrightarrow D_{comp}(A, I)$$

This is a special case of [PSY14, Theorem 1.1].

Proof. Follows immediately from Lemma 12.1. $\square$

The following addendum of the proposition above makes the correspondence on morphisms more precise.

0A6Y **Lemma 12.3.** *With notation as in Lemma 12.1. For objects K, L in $D(A)$ there is a canonical isomorphism*

$$R\operatorname{Hom}_A(K^\wedge, L^\wedge) \longrightarrow R\operatorname{Hom}_A(R\Gamma_Z(K), R\Gamma_Z(L))$$

in $D(A)$.

Proof. Say $I = (f_1, \ldots, f_r)$. Denote $C = (A \to \prod A_{f_i} \to \ldots \to A_{f_1 \ldots f_r})$ the alternating Čech complex. Then derived completion is given by $R\operatorname{Hom}_A(C, -)$ (More on Algebra, Lemma 91.10) and local cohomology by $C \otimes^{\mathbf{L}} -$ (Lemma 9.1). Combining the isomorphism

$$R\operatorname{Hom}_A(K \otimes^{\mathbf{L}} C, L \otimes^{\mathbf{L}} C) = R\operatorname{Hom}_A(K, R\operatorname{Hom}_A(C, L \otimes^{\mathbf{L}} C))$$

(More on Algebra, Lemma 73.1) and the map

$$L \to R\operatorname{Hom}_A(C, L \otimes^{\mathbf{L}} C)$$

(More on Algebra, Lemma 73.6) we obtain a map

$$\gamma : R\operatorname{Hom}_A(K, L) \longrightarrow R\operatorname{Hom}_A(K \otimes^{\mathbf{L}} C, L \otimes^{\mathbf{L}} C)$$

On the other hand, the right hand side is derived complete as it is equal to

$$R\operatorname{Hom}_A(C, R\operatorname{Hom}_A(K, L \otimes^{\mathbf{L}} C)).$$

Thus γ factors through the derived completion of $R\operatorname{Hom}_A(K, L)$ by the universal property of derived completion. However, the derived completion goes inside the $R\operatorname{Hom}_A$ by More on Algebra, Lemma 91.13 and we obtain the desired map.

To show that the map of the lemma is an isomorphism we may assume that K and L are derived complete, i.e., $K = K^\wedge$ and $L = L^\wedge$. In this case we are looking at the map

$$\gamma : R\operatorname{Hom}_A(K, L) \longrightarrow R\operatorname{Hom}_A(R\Gamma_Z(K), R\Gamma_Z(L))$$

By Proposition 12.2 we know that the cohomology groups of the left and the right hand side coincide. In other words, we have to check that the map γ sends a morphism $\alpha : K \to L$ in $D(A)$ to the morphism $R\Gamma_Z(\alpha) : R\Gamma_Z(K) \to R\Gamma_Z(L)$. We omit the verification (hint: note that $R\Gamma_Z(\alpha)$ is just the map $\alpha \otimes \mathrm{id}_C : K \otimes^{\mathbf{L}} C \to L \otimes^{\mathbf{L}} C$ which is almost the same as the construction of the map in More on Algebra, Lemma 73.6). $\square$

0EEW **Lemma 12.4.** *Let I and J be ideals in a Noetherian ring A. Let M be a finite A-module. Set $Z = V(J)$. Consider the derived I-adic completion $R\Gamma_Z(M)^\wedge$ of local cohomology. Then*

(1) *we have $R\Gamma_Z(M)^\wedge = R\lim R\Gamma_Z(M/I^n M)$, and*
(2) *there are short exact sequences*

$$0 \to R^1\lim H_Z^{i-1}(M/I^n M) \to H^i(R\Gamma_Z(M)^\wedge) \to \lim H_Z^i(M/I^n M) \to 0$$

In particular $R\Gamma_Z(M)^\wedge$ has vanishing cohomology in negative degrees.

Proof. Suppose that $J = (g_1, \ldots, g_m)$. Then $R\Gamma_Z(M)$ is computed by the complex

$$M \to \prod M_{g_{j_0}} \to \prod M_{g_{j_0} g_{j_1}} \to \ldots \to M_{g_1 g_2 \ldots g_m}$$

by Lemma 9.1. By More on Algebra, Lemma 94.6 the derived I-adic completion of this complex is given by the complex

$$\lim M/I^n M \to \prod \lim (M/I^n M)_{g_{j_0}} \to \ldots \to \lim (M/I^n M)_{g_1 g_2 \ldots g_m}$$

of usual completions. Since $R\Gamma_Z(M/I^n M)$ is computed by the complex $M/I^n M \to \prod(M/I^n M)_{g_{j_0}} \to \ldots \to (M/I^n M)_{g_1 g_2 \ldots g_m}$ and since the transition maps between these complexes are surjective, we conclude that (1) holds by More on Algebra, Lemma 87.1. Part (2) then follows from More on Algebra, Lemma 87.4. $\square$

0EEX **Lemma 12.5.** *With notation and hypotheses as in Lemma 12.4 assume A is I-adically complete. Then*

$$H^0(R\Gamma_Z(M)^\wedge) = \mathrm{colim}\, H^0_{V(J')}(M)$$

where the filtered colimit is over $J' \subset J$ such that $V(J') \cap V(I) = V(J) \cap V(I)$.

Proof. Since M is a finite A-module, we have that M is I-adically complete. The proof of Lemma 12.4 shows that

$$H^0(R\Gamma_Z(M)^\wedge) = \mathrm{Ker}(M^\wedge \to \prod M_{g_j}^\wedge) = \mathrm{Ker}(M \to \prod M_{g_j}^\wedge)$$

where on the right hand side we have usual I-adic completion. The kernel K_j of $M_{g_j} \to M_{g_j}^\wedge$ is $\bigcap I^n M_{g_j}$. By Algebra, Lemma 51.5 for every $\mathfrak{p} \in V(IA_{g_j})$ we find an $f \in A_{g_j}$, $f \notin \mathfrak{p}$ such that $(K_j)_f = 0$.

Let $s \in H^0(R\Gamma_Z(M)^\wedge)$. By the above we may think of s as an element of M. The support Z' of s intersected with $D(g_j)$ is disjoint from $D(g_j) \cap V(I)$ by the arguments above. Thus Z' is a closed subset of $\mathrm{Spec}(A)$ with $Z' \cap V(I) \subset V(J)$. Then $Z' \cup V(J) = V(J')$ for some ideal $J' \subset J$ with $V(J') \cap V(I) \subset V(J)$ and we have $s \in H^0_{V(J')}(M)$. Conversely, any $s \in H^0_{V(J')}(M)$ with $J' \subset J$ and $V(J') \cap V(I) \subset V(J)$ maps to zero in $M_{g_j}^\wedge$ for all j. This proves the lemma. $\square$

13. Trivial duality for a ring map

0A6Z   Let $A \to B$ be a ring homomorphism. Consider the functor

$$\mathrm{Hom}_A(B, -) : \mathrm{Mod}_A \longrightarrow \mathrm{Mod}_B, \quad M \longmapsto \mathrm{Hom}_A(B, M)$$

This functor is left exact and has a derived extension $R\,\mathrm{Hom}(B, -) : D(A) \to D(B)$.

0A70   **Lemma 13.1.** *Let $A \to B$ be a ring homomorphism. The functor $R\,\mathrm{Hom}(B, -)$ constructed above is right adjoint to the restriction functor $D(B) \to D(A)$.*

Proof. This is a consequence of the fact that restriction and $\mathrm{Hom}_A(B, -)$ are adjoint functors by Algebra, Lemma 14.4. See Derived Categories, Lemma 30.3. $\square$

0C0F   **Lemma 13.2.** *Let $A \to B \to C$ be ring maps. Then $R\,\mathrm{Hom}(C, -) \circ R\,\mathrm{Hom}(B, -) : D(A) \to D(C)$ is the functor $R\,\mathrm{Hom}(C, -) : D(A) \to D(C)$.*

Proof. Follows from uniqueness of right adjoints and Lemma 13.1. $\square$

0A71   **Lemma 13.3.** *Let $\varphi : A \to B$ be a ring homomorphism. For K in $D(A)$ we have*

$$\varphi_* R\,\mathrm{Hom}(B, K) = R\,\mathrm{Hom}_A(B, K)$$

where $\varphi_* : D(B) \to D(A)$ is restriction. In particular $R^q\,\mathrm{Hom}(B, K) = \mathrm{Ext}_A^q(B, K)$.

Proof. Choose a K-injective complex $I^\bullet$ representing K. Then $R\,\mathrm{Hom}(B, K)$ is represented by the complex $\mathrm{Hom}_A(B, I^\bullet)$ of B-modules. Since this complex, as a complex of A-modules, represents $R\,\mathrm{Hom}_A(B, K)$ we see that the lemma is true. $\square$

Let A be a Noetherian ring. We will denote

$$D_{Coh}(A) \subset D(A)$$

the full subcategory consisting of those objects K of $D(A)$ whose cohomology modules are all finite A-modules. This makes sense by Derived Categories, Section 17 because as A is Noetherian, the subcategory of finite A-modules is a Serre subcategory of Mod_A.

0A72   **Lemma 13.4.** *With notation as above, assume $A \to B$ is a finite ring map of Noetherian rings. Then $R\,\mathrm{Hom}(B, -)$ maps $D_{Coh}^+(A)$ into $D_{Coh}^+(B)$.*

Proof. We have to show: if $K \in D^+(A)$ has finite cohomology modules, then the complex $R\,\mathrm{Hom}(B, K)$ has finite cohomology modules too. This follows for example from Lemma 13.3 if we can show the ext modules $\mathrm{Ext}_A^i(B, K)$ are finite A-modules. Since K is bounded below there is a convergent spectral sequence

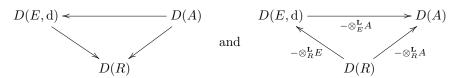$$\mathrm{Ext}_A^p(B, H^q(K)) \Rightarrow \mathrm{Ext}_A^{p+q}(B, K)$$

This finishes the proof as the modules $\mathrm{Ext}_A^p(B, H^q(K))$ are finite by Algebra, Lemma 71.9. $\square$

0A73   **Remark 13.5.** Let A be a ring and let $I \subset A$ be an ideal. Set $B = A/I$. In this case the functor $\mathrm{Hom}_A(B, -)$ is equal to the functor

$$\mathrm{Mod}_A \longrightarrow \mathrm{Mod}_B, \quad M \longmapsto M[I]$$

which sends M to the submodule of I-torsion.

0BZB **Situation 13.6.** Let $R \to A$ be a ring map. We will give an alternative construction of $R\operatorname{Hom}(A, -)$ which will stand us in good stead later in this chapter. Namely, suppose we have a differential graded algebra (E, d) over R and a quasi-isomorphism $E \to A$ where we view A as a differential graded algebra over R with zero differential. Then we have commutative diagrams

$$
\begin{array}{ccc}
D(E, \mathrm{d}) \longleftarrow D(A) & & D(E, \mathrm{d}) \xrightarrow{\ -\otimes_E^{\mathbf{L}} A\ } D(A) \\
\searrow \qquad \swarrow & \text{and} & {}_{-\otimes_R^{\mathbf{L}} E}\nwarrow \qquad \nearrow_{-\otimes_R^{\mathbf{L}} A} \\
D(R) & & D(R)
\end{array}
$$

where the horizontal arrows are equivalences of categories (Differential Graded Algebra, Lemma 37.1). It is clear that the first diagram commutes. The second diagram commutes because the first one does and our functors are their left adjoints (Differential Graded Algebra, Example 33.6) or because we have $E \otimes_E^{\mathbf{L}} A = E \otimes_E A$ and we can use Differential Graded Algebra, Lemma 34.1.

0BZC **Lemma 13.7.** *In Situation 13.6 the functor $R\operatorname{Hom}(A, -)$ is equal to the composition of $R\operatorname{Hom}(E, -) : D(R) \to D(E, d)$ and the equivalence $- \otimes_E^{\mathbf{L}} A : D(E, d) \to D(A)$.*

Proof. This is true because $R\operatorname{Hom}(E, -)$ is the right adjoint to $- \otimes_R^{\mathbf{L}} E$, see Differential Graded Algebra, Lemma 33.5. Hence this functor plays the same role as the functor $R\operatorname{Hom}(A, -)$ for the map $R \to A$ (Lemma 13.1), whence these functors must correspond via the equivalence $- \otimes_E^{\mathbf{L}} A : D(E, d) \to D(A)$. $\square$

0BZD **Lemma 13.8.** *In Situation 13.6 assume that*

  (1) *E viewed as an object of $D(R)$ is compact, and*
  (2) *$N = \operatorname{Hom}_R^\bullet(E^\bullet, R)$ computes $R\operatorname{Hom}(E, R)$.*

Then $R\operatorname{Hom}(E, -) : D(R) \to D(E)$ is isomorphic to $K \mapsto K \otimes_R^{\mathbf{L}} N$.

Proof. Special case of Differential Graded Algebra, Lemma 33.9. $\square$

0BZE **Lemma 13.9.** *In Situation 13.6 assume A is a perfect R-module. Then*

$$R\operatorname{Hom}(A, -) : D(R) \to D(A)$$

is given by $K \mapsto K \otimes_R^{\mathbf{L}} M$ where $M = R\operatorname{Hom}(A, R) \in D(A)$.

Proof. We apply Divided Power Algebra, Lemma 6.10 to choose a Tate resolution (E, d) of A over R. Note that $E^i = 0$ for $i > 0$, $E^0 = R[x_1, \ldots, x_n]$ is a polynomial algebra, and E^i is a finite free E^0-module for $i < 0$. It follows that E viewed as a complex of R-modules is a bounded above complex of free R-modules. We check the assumptions of Lemma 13.8. The first holds because A is perfect (hence compact by More on Algebra, Proposition 78.3) and the second by More on Algebra, Lemma 73.2. From the lemma conclude that $K \mapsto R\operatorname{Hom}(E, K)$ is isomorphic to $K \mapsto K \otimes_R^{\mathbf{L}} N$ for some differential graded E-module N. Observe that

$$(R \otimes_R E) \otimes_E^{\mathbf{L}} A = R \otimes_E E \otimes_E A$$

in $D(A)$. Hence by Differential Graded Algebra, Lemma 34.2 we conclude that the composition of $- \otimes_R^{\mathbf{L}} N$ and $- \otimes_R^{\mathbf{L}} A$ is of the form $- \otimes_R M$ for some $M \in D(A)$. To finish the proof we apply Lemma 13.7. $\square$

0BZH **Lemma 13.10.** *Let $R \to A$ be a surjective ring map whose kernel I is an invertible R-module. The functor $R\operatorname{Hom}(A, -) : D(R) \to D(A)$ is isomorphic to $K \mapsto K \otimes_R^{\mathbf{L}} N[-1]$ where N is inverse of the invertible A-module $I \otimes_R A$.*

Proof. Since A has the finite projective resolution

$$0 \to I \to R \to A \to 0$$

we see that A is a perfect R-module. By Lemma 13.9 it suffices to prove that $R\operatorname{Hom}(A, R)$ is represented by $N[-1]$ in $D(A)$. This means $R\operatorname{Hom}(A, R)$ has a unique nonzero cohomology module, namely N in degree 1. As $\operatorname{Mod}_A \to \operatorname{Mod}_R$ is fully faithful it suffice to prove this after applying the restriction functor $i_* : D(A) \to D(R)$. By Lemma 13.3 we have

$$i_* R\operatorname{Hom}(A, R) = R\operatorname{Hom}_R(A, R)$$

Using the finite projective resolution above we find that the latter is represented by the complex $R \to I^{\otimes -1}$ with R in degree 0. The map $R \to I^{\otimes -1}$ is injective and the cokernel is N. $\qquad\square$

14. Base change for trivial duality

0E28 In this section we consider a cocartesian square of rings

$$\begin{array}{ccc} A & \xrightarrow{\ \alpha\ } & A' \\ {\scriptstyle\varphi}\uparrow & & \uparrow{\scriptstyle\varphi'} \\ R & \xrightarrow{\ \rho\ } & R' \end{array}$$

In other words, we have $A' = A \otimes_R R'$. If A and R' are **tor independent over** R then there is a canonical base change map

0E29 (14.0.1) $$R\operatorname{Hom}(A, K) \otimes_A^{\mathbf{L}} A' \longrightarrow R\operatorname{Hom}(A', K \otimes_R^{\mathbf{L}} R')$$

in $D(A')$ functorial for K in $D(R)$. Namely, by the adjointness of Lemma 13.1 such an arrow is the same thing as a map

$$\varphi'_* \left(R\operatorname{Hom}(A, K) \otimes_A^{\mathbf{L}} A' \right) \longrightarrow K \otimes_R^{\mathbf{L}} R'$$

in $D(R')$ where $\varphi'_* : D(A') \to D(R')$ is the restriction functor. We may apply More on Algebra, Lemma 61.2 to the left hand side to get that this is the same thing as a map

$$\varphi_*(R\operatorname{Hom}(A, K)) \otimes_R^{\mathbf{L}} R' \longrightarrow K \otimes_R^{\mathbf{L}} R'$$

in $D(R')$ where $\varphi_* : D(A) \to D(R)$ is the restriction functor. For this we can choose $can \otimes^{\mathbf{L}} \operatorname{id}_{R'}$ where $can : \varphi_*(R\operatorname{Hom}(A, K)) \to K$ is the counit of the adjunction between $R\operatorname{Hom}(A, -)$ and φ_*.

0E2A **Lemma 14.1.** *In the situation above, the map (14.0.1) is an isomorphism if and only if the map*

$$R\operatorname{Hom}_R(A, K) \otimes_R^{\mathbf{L}} R' \longrightarrow R\operatorname{Hom}_R(A, K \otimes_R^{\mathbf{L}} R')$$

of More on Algebra, Lemma 73.5 is an isomorphism.

Proof. To see that the map is an isomorphism, it suffices to prove it is an isomorphism after applying φ'_*. Applying the functor φ'_* to (14.0.1) and using that $A' = A \otimes_R^{\mathbf{L}} R'$ we obtain the base change map $R\operatorname{Hom}_R(A, K) \otimes_R^{\mathbf{L}} R' \to R\operatorname{Hom}_{R'}(A \otimes_R^{\mathbf{L}} R', K \otimes_R^{\mathbf{L}} R')$ for derived hom of More on Algebra, Equation (99.1.1). Unwinding

the left and right hand side exactly as in the proof of More on Algebra, Lemma 99.2 and in particular using More on Algebra, Lemma 99.1 gives the desired result. $\square$

0BZM **Lemma 14.2.** *Let $R \to A$ and $R \to R'$ be ring maps and $A' = A \otimes_R R'$. Assume*

(1) *A is pseudo-coherent as an R-module,*
(2) *R' has finite tor dimension as an R-module (for example $R \to R'$ is flat),*
(3) *A and R' are tor independent over R.*

Then (14.0.1) is an isomorphism for $K \in D^+(R)$.

Proof. Follows from Lemma 14.1 and More on Algebra, Lemma 98.3 part (4). $\square$

0BZP **Lemma 14.3.** *Let $R \to A$ and $R \to R'$ be ring maps and $A' = A \otimes_R R'$. Assume*

(1) *A is perfect as an R-module,*
(2) *A and R' are tor independent over R.*

Then (14.0.1) is an isomorphism for all $K \in D(R)$.

Proof. Follows from Lemma 14.1 and More on Algebra, Lemma 98.3 part (1). $\square$

15. Dualizing complexes

0A7A In this section we define dualizing complexes for Noetherian rings.

0A7B **Definition 15.1.** Let A be a Noetherian ring. A *dualizing complex* is a complex of A-modules $\omega_A^\bullet$ such that

(1) $\omega_A^\bullet$ has finite injective dimension,
(2) $H^i(\omega_A^\bullet)$ is a finite A-module for all i, and
(3) $A \to R\operatorname{Hom}_A(\omega_A^\bullet, \omega_A^\bullet)$ is a quasi-isomorphism.

This definition takes some time getting used to. It is perhaps a good idea to prove some of the following lemmas yourself without reading the proofs.

0G4H **Lemma 15.2.** *Let A be a Noetherian ring. Let $K, L \in D_{Coh}(A)$ and assume L has finite injective dimension. Then $R\operatorname{Hom}_A(K, L)$ is in $D_{Coh}(A)$.*

Proof. Pick an integer n and consider the distinguished triangle

$$\tau_{\leq n}K \to K \to \tau_{\geq n+1}K \to \tau_{\leq n}K[1]$$

see Derived Categories, Remark 12.4. Since L has finite injective dimension we see that $R\operatorname{Hom}_A(\tau_{\geq n+1}K, L)$ has vanishing cohomology in degrees $\geq c - n$ for some constant c. Hence, given i, we see that $\operatorname{Ext}_A^i(K, L) \to \operatorname{Ext}_A^i(\tau_{\leq n}K, L)$ is an isomorphism for some $n \gg -i$. By Derived Categories of Schemes, Lemma 11.5 applied to $\tau_{\leq n}K$ and L we see conclude that $\operatorname{Ext}_A^i(K, L)$ is a finite A-module for all i. Hence $R\operatorname{Hom}_A(K, L)$ is indeed an object of $D_{Coh}(A)$. $\square$

0A7C **Lemma 15.3.** *Let A be a Noetherian ring. If $\omega_A^\bullet$ is a dualizing complex, then the functor*

$$D : K \longmapsto R\operatorname{Hom}_A(K, \omega_A^\bullet)$$

is an anti-equivalence $D_{Coh}(A) \to D_{Coh}(A)$ which exchanges $D_{Coh}^+(A)$ and $D_{Coh}^-(A)$ and induces an anti-equivalence $D_{Coh}^b(A) \to D_{Coh}^b(A)$. Moreover $D \circ D$ is isomorphic to the identity functor.

Proof. Let K be an object of $D_{Coh}(A)$. From Lemma 15.2 we see $R\operatorname{Hom}_A(K, \omega_A^\bullet)$ is an object of $D_{Coh}(A)$. By More on Algebra, Lemma 98.2 and the assumptions on the dualizing complex we obtain a canonical isomorphism

$$K = R\operatorname{Hom}_A(\omega_A^\bullet, \omega_A^\bullet) \otimes_A^{\mathbf{L}} K \longrightarrow R\operatorname{Hom}_A(R\operatorname{Hom}_A(K, \omega_A^\bullet), \omega_A^\bullet)$$

Thus our functor has a quasi-inverse and the proof is complete. $\qquad\square$

Let R be a ring. Recall that an object L of $D(R)$ is *invertible* if it is an invertible object for the symmetric monoidal structure on $D(R)$ given by derived tensor product. In More on Algebra, Lemma 126.4 we have seen this means L is perfect, $L = \bigoplus H^n(L)[-n]$, this is a finite sum, each $H^n(L)$ is finite projective, and there is an open covering $\operatorname{Spec}(R) = \bigcup D(f_i)$ such that $L \otimes_R R_{f_i} \cong R_{f_i}[-n_i]$ for some integers n_i.

0A7E **Lemma 15.4.** *Let A be a Noetherian ring. Let $F : D_{Coh}^b(A) \to D_{Coh}^b(A)$ be an A-linear equivalence of categories. Then $F(A)$ is an invertible object of $D(A)$.*

Proof. Let $\mathfrak{m} \subset A$ be a maximal ideal with residue field κ. Consider the object $F(\kappa)$. Since $\kappa = \operatorname{Hom}_{D(A)}(\kappa, \kappa)$ we find that all cohomology groups of $F(\kappa)$ are annihilated by $\mathfrak{m}$. We also see that

$$\operatorname{Ext}_A^i(\kappa, \kappa) = \operatorname{Ext}_A^i(F(\kappa), F(\kappa)) = \operatorname{Hom}_{D(A)}(F(\kappa), F(\kappa)[i])$$

is zero for $i < 0$. Say $H^a(F(\kappa)) \neq 0$ and $H^b(F(\kappa)) \neq 0$ with a minimal and b maximal (so in particular $a \leq b$). Then there is a nonzero map

$$F(\kappa) \to H^b(F(\kappa))[-b] \to H^a(F(\kappa))[-b] \to F(\kappa)[a-b]$$

in $D(A)$ (nonzero because it induces a nonzero map on cohomology). This proves that $b = a$. We conclude that $F(\kappa) = \kappa[-a]$.

Let G be a quasi-inverse to our functor F. Arguing as above we find an integer b such that $G(\kappa) = \kappa[-b]$. On composing we find $a + b = 0$. Let E be a finite A-module wich is annihilated by a power of $\mathfrak{m}$. Arguing by induction on the length of E we find that $G(E) = E'[-b]$ for some finite A-module E' annihilated by a power of $\mathfrak{m}$. Then $E[-a] = F(E')$. Next, we consider the groups

$$\operatorname{Ext}_A^i(A, E') = \operatorname{Ext}_A^i(F(A), F(E')) = \operatorname{Hom}_{D(A)}(F(A), E[-a+i])$$

The left hand side is nonzero if and only if $i = 0$ and then we get E'. Applying this with $E = E' = \kappa$ and using Nakayama's lemma this implies that $H^j(F(A))_{\mathfrak{m}}$ is zero for $j > a$ and generated by 1 element for $j = a$. On the other hand, if $H^j(F(A))_{\mathfrak{m}}$ is not zero for some $j < a$, then there is a map $F(A) \to E[-a+i]$ for some $i < 0$ and some E (More on Algebra, Lemma 65.7) which is a contradiction. Thus we see that $F(A)_{\mathfrak{m}} = M[-a]$ for some $A_{\mathfrak{m}}$-module M generated by 1 element. However, since

$$A_{\mathfrak{m}} = \operatorname{Hom}_{D(A)}(A, A)_{\mathfrak{m}} = \operatorname{Hom}_{D(A)}(F(A), F(A))_{\mathfrak{m}} = \operatorname{Hom}_{A_{\mathfrak{m}}}(M, M)$$

we see that $M \cong A_{\mathfrak{m}}$. We conclude that there exists an element $f \in A$, $f \notin \mathfrak{m}$ such that $F(A)_f$ is isomorphic to $A_f[-a]$. This finishes the proof. $\qquad\square$

0A7F **Lemma 15.5.** *Let A be a Noetherian ring. If $\omega_A^\bullet$ and $(\omega_A')^\bullet$ are dualizing complexes, then $(\omega_A')^\bullet$ is quasi-isomorphic to $\omega_A^\bullet \otimes_A^{\mathbf{L}} L$ for some invertible object L of $D(A)$.*

Proof. By Lemmas 15.3 and 15.4 the functor $K \mapsto R\operatorname{Hom}_A(R\operatorname{Hom}_A(K, \omega_A^\bullet), (\omega_A')^\bullet)$ maps A to an invertible object L. In other words, there is an isomorphism

$$L \longrightarrow R\operatorname{Hom}_A(\omega_A^\bullet, (\omega_A')^\bullet)$$

Since L has finite tor dimension, this means that we can apply More on Algebra, Lemma 98.2 to see that

$$R\operatorname{Hom}_A(\omega_A^\bullet, (\omega_A')^\bullet) \otimes_A^{\mathbf{L}} K \longrightarrow R\operatorname{Hom}_A(R\operatorname{Hom}_A(K, \omega_A^\bullet), (\omega_A')^\bullet)$$

is an isomorphism for K in $D_{Coh}^b(A)$. In particular, setting $K = \omega_A^\bullet$ finishes the proof. $\square$

0A7G **Lemma 15.6.** *Let A be a Noetherian ring. Let $B = S^{-1}A$ be a localization. If $\omega_A^\bullet$ is a dualizing complex, then $\omega_A^\bullet \otimes_A B$ is a dualizing complex for B.*

Proof. Let $\omega_A^\bullet \to I^\bullet$ be a quasi-isomorphism with $I^\bullet$ a bounded complex of injectives. Then $S^{-1}I^\bullet$ is a bounded complex of injective $B = S^{-1}A$-modules (Lemma 3.8) representing $\omega_A^\bullet \otimes_A B$. Thus $\omega_A^\bullet \otimes_A B$ has finite injective dimension. Since $H^i(\omega_A^\bullet \otimes_A B) = H^i(\omega_A^\bullet) \otimes_A B$ by flatness of $A \to B$ we see that $\omega_A^\bullet \otimes_A B$ has finite cohomology modules. Finally, the map

$$B \longrightarrow R\operatorname{Hom}_A(\omega_A^\bullet \otimes_A B, \omega_A^\bullet \otimes_A B)$$

is a quasi-isomorphism as formation of internal hom commutes with flat base change in this case, see More on Algebra, Lemma 99.2. $\square$

0A7H **Lemma 15.7.** *Let A be a Noetherian ring. Let $f_1, \ldots, f_n \in A$ generate the unit ideal. If $\omega_A^\bullet$ is a complex of A-modules such that $(\omega_A^\bullet)_{f_i}$ is a dualizing complex for A_{f_i} for all i, then $\omega_A^\bullet$ is a dualizing complex for A.*

Proof. Consider the double complex

$$\prod\nolimits_{i_0} (\omega_A^\bullet)_{f_{i_0}} \to \prod\nolimits_{i_0 < i_1} (\omega_A^\bullet)_{f_{i_0} f_{i_1}} \to \ldots$$

The associated total complex is quasi-isomorphic to $\omega_A^\bullet$ for example by Descent, Remark 3.10 or by Derived Categories of Schemes, Lemma 9.4. By assumption the complexes $(\omega_A^\bullet)_{f_i}$ have finite injective dimension as complexes of A_{f_i}-modules. This implies that each of the complexes $(\omega_A^\bullet)_{f_{i_0} \ldots f_{i_p}}$, $p > 0$ has finite injective dimension over $A_{f_{i_0} \ldots f_{i_p}}$, see Lemma 3.8. This in turn implies that each of the complexes $(\omega_A^\bullet)_{f_{i_0} \ldots f_{i_p}}$, $p > 0$ has finite injective dimension over A, see Lemma 3.2. Hence $\omega_A^\bullet$ has finite injective dimension as a complex of A-modules (as it can be represented by a complex endowed with a finite filtration whose graded parts have finite injective dimension). Since $H^n(\omega_A^\bullet)_{f_i}$ is a finite A_{f_i} module for each i we see that $H^i(\omega_A^\bullet)$ is a finite A-module, see Algebra, Lemma 23.2. Finally, the (derived) base change of the map $A \to R\operatorname{Hom}_A(\omega_A^\bullet, \omega_A^\bullet)$ to A_{f_i} is the map $A_{f_i} \to R\operatorname{Hom}_A((\omega_A^\bullet)_{f_i}, (\omega_A^\bullet)_{f_i})$ by More on Algebra, Lemma 99.2. Hence we deduce that $A \to R\operatorname{Hom}_A(\omega_A^\bullet, \omega_A^\bullet)$ is an isomorphism and the proof is complete. $\square$

0AX0 **Lemma 15.8.** *Let $A \to B$ be a finite ring map of Noetherian rings. Let $\omega_A^\bullet$ be a dualizing complex. Then $R\operatorname{Hom}(B, \omega_A^\bullet)$ is a dualizing complex for B.*

Proof. Let $\omega_A^\bullet \to I^\bullet$ be a quasi-isomorphism with $I^\bullet$ a bounded complex of injectives. Then $\operatorname{Hom}_A(B, I^\bullet)$ is a bounded complex of injective B-modules (Lemma

3.4) representing $R\operatorname{Hom}(B, \omega_A^\bullet)$. Thus $R\operatorname{Hom}(B, \omega_A^\bullet)$ has finite injective dimension. By Lemma 13.4 it is an object of $D_{Coh}(B)$. Finally, we compute

$$\operatorname{Hom}_{D(B)}(R\operatorname{Hom}(B, \omega_A^\bullet), R\operatorname{Hom}(B, \omega_A^\bullet)) = \operatorname{Hom}_{D(A)}(R\operatorname{Hom}(B, \omega_A^\bullet), \omega_A^\bullet) = B$$

and for $n \neq 0$ we compute

$$\operatorname{Hom}_{D(B)}(R\operatorname{Hom}(B, \omega_A^\bullet), R\operatorname{Hom}(B, \omega_A^\bullet)[n]) = \operatorname{Hom}_{D(A)}(R\operatorname{Hom}(B, \omega_A^\bullet), \omega_A^\bullet[n]) = 0$$

which proves the last property of a dualizing complex. In the displayed equations, the first equality holds by Lemma 13.1 and the second equality holds by Lemma 15.3. $\square$

0A7I **Lemma 15.9.** *Let $A \to B$ be a surjective homomorphism of Noetherian rings. Let $\omega_A^\bullet$ be a dualizing complex. Then $R\operatorname{Hom}(B, \omega_A^\bullet)$ is a dualizing complex for B.*

Proof. Special case of Lemma 15.8. $\square$

0A7J **Lemma 15.10.** *Let A be a Noetherian ring. If $\omega_A^\bullet$ is a dualizing complex, then $\omega_A^\bullet \otimes_A A[x]$ is a dualizing complex for $A[x]$.*

Proof. Set $B = A[x]$ and $\omega_B^\bullet = \omega_A^\bullet \otimes_A B$. It follows from Lemma 3.10 and More on Algebra, Lemma 69.5 that $\omega_B^\bullet$ has finite injective dimension. Since $H^i(\omega_B^\bullet) = H^i(\omega_A^\bullet) \otimes_A B$ by flatness of $A \to B$ we see that $\omega_A^\bullet \otimes_A B$ has finite cohomology modules. Finally, the map

$$B \longrightarrow R\operatorname{Hom}_B(\omega_B^\bullet, \omega_B^\bullet)$$

is a quasi-isomorphism as formation of internal hom commutes with flat base change in this case, see More on Algebra, Lemma 99.2. $\square$

0A7K **Proposition 15.11.** *Let A be a Noetherian ring which has a dualizing complex. Then any A-algebra essentially of finite type over A has a dualizing complex.*

Proof. This follows from a combination of Lemmas 15.6, 15.9, and 15.10. $\square$

0A7L **Lemma 15.12.** *Let A be a Noetherian ring. Let $\omega_A^\bullet$ be a dualizing complex. Let $\mathfrak{m} \subset A$ be a maximal ideal and set $\kappa = A/\mathfrak{m}$. Then $R\operatorname{Hom}_A(\kappa, \omega_A^\bullet) \cong \kappa[n]$ for some $n \in \mathbf{Z}$.*

Proof. This is true because $R\operatorname{Hom}_A(\kappa, \omega_A^\bullet)$ is a dualizing complex over κ (Lemma 15.9), because dualizing complexes over κ are unique up to shifts (Lemma 15.5), and because κ is a dualizing complex over κ. $\square$

16. Dualizing complexes over local rings

0A7M In this section $(A, \mathfrak{m}, \kappa)$ will be a Noetherian local ring endowed with a dualizing complex $\omega_A^\bullet$ such that the integer n of Lemma 15.12 is zero. More precisely, we assume that $R\operatorname{Hom}_A(\kappa, \omega_A^\bullet) = \kappa[0]$. In this case we will say that the dualizing complex is *normalized*. Observe that a normalized dualizing complex is unique up to isomorphism and that any other dualizing complex for A is isomorphic to a shift of a normalized one (Lemma 15.5).

0AX1 **Lemma 16.1.** *Let $(A, \mathfrak{m}, \kappa) \to (B, \mathfrak{m}', \kappa')$ be a finite local map of Noetherian local rings. Let $\omega_A^\bullet$ be a normalized dualizing complex. Then $\omega_B^\bullet = R\operatorname{Hom}(B, \omega_A^\bullet)$ is a normalized dualizing complex for B.*

Proof. By Lemma 15.8 the complex $\omega_B^\bullet$ is dualizing for B. We have

$$R\operatorname{Hom}_B(\kappa', \omega_B^\bullet) = R\operatorname{Hom}_B(\kappa', R\operatorname{Hom}(B, \omega_A^\bullet)) = R\operatorname{Hom}_A(\kappa', \omega_A^\bullet)$$

by Lemma 13.1. Since κ' is isomorphic to a finite direct sum of copies of κ as an A-module and since $\omega_A^\bullet$ is normalized, we see that this complex only has cohomology placed in degree 0. Thus $\omega_B^\bullet$ is a normalized dualizing complex as well. $\qquad\square$

0A7N **Lemma 16.2.** *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring with normalized dualizing complex $\omega_A^\bullet$. Let $A \to B$ be surjective. Then $\omega_B^\bullet = R\operatorname{Hom}_A(B, \omega_A^\bullet)$ is a normalized dualizing complex for B.*

Proof. Special case of Lemma 16.1. $\qquad\square$

0A7P **Lemma 16.3.** *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Let F be an A-linear self-equivalence of the category of finite length A-modules. Then F is isomorphic to the identity functor.*

Proof. Since κ is the unique simple object of the category we have $F(\kappa) \cong \kappa$. Since our category is abelian, we find that F is exact. Hence $F(E)$ has the same length as E for all finite length modules E. Since $\operatorname{Hom}(E, \kappa) = \operatorname{Hom}(F(E), F(\kappa)) \cong \operatorname{Hom}(F(E), \kappa)$ we conclude from Nakayama's lemma that E and $F(E)$ have the same number of generators. Hence $F(A/\mathfrak{m}^n)$ is a cyclic A-module. Pick a generator $e \in F(A/\mathfrak{m}^n)$. Since F is A-linear we conclude that $\mathfrak{m}^n e = 0$. The map $A/\mathfrak{m}^n \to F(A/\mathfrak{m}^n)$ has to be an isomorphism as the lengths are equal. Pick an element

$$e \in \lim F(A/\mathfrak{m}^n)$$

which maps to a generator for all n (small argument omitted). Then we obtain a system of isomorphisms $A/\mathfrak{m}^n \to F(A/\mathfrak{m}^n)$ compatible with all A-module maps $A/\mathfrak{m}^n \to A/\mathfrak{m}^{n'}$ (by A-linearity of F again). Since any finite length module is a cokernel of a map between direct sums of cyclic modules, we obtain the isomorphism of the lemma. $\qquad\square$

0A7Q **Lemma 16.4.** *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring with normalized dualizing complex $\omega_A^\bullet$. Let E be an injective hull of κ. Then there exists a functorial isomorphism*

$$R\operatorname{Hom}_A(N, \omega_A^\bullet) = \operatorname{Hom}_A(N, E)[0]$$

for N running through the finite length A-modules.

Proof. By induction on the length of N we see that $R\operatorname{Hom}_A(N, \omega_A^\bullet)$ is a module of finite length sitting in degree 0. Thus $R\operatorname{Hom}_A(-, \omega_A^\bullet)$ induces an anti-equivalence on the category of finite length modules. Since the same is true for $\operatorname{Hom}_A(-, E)$ by Proposition 7.8 we see that

$$N \longmapsto \operatorname{Hom}_A(R\operatorname{Hom}_A(N, \omega_A^\bullet), E)$$

is an equivalence as in Lemma 16.3. Hence it is isomorphic to the identity functor. Since $\operatorname{Hom}_A(-, E)$ applied twice is the identity (Proposition 7.8) we obtain the statement of the lemma. $\qquad\square$

0A7U **Lemma 16.5.** *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring with normalized dualizing complex $\omega_A^\bullet$. Let M be a finite A-module and let $d = \dim(Supp(M))$. Then*

(1) *if $\operatorname{Ext}_A^i(M, \omega_A^\bullet)$ is nonzero, then $i \in \{-d, \ldots, 0\}$,*
(2) *the dimension of the support of $\operatorname{Ext}_A^i(M, \omega_A^\bullet)$ is at most $-i$,*

(3) $depth(M)$ is the smallest integer $\delta \geq 0$ such that $\operatorname{Ext}_A^{-\delta}(M, \omega_A^\bullet) \neq 0$.

Proof. We prove this by induction on d. If $d = 0$, this follows from Lemma 16.4 and Matlis duality (Proposition 7.8) which guarantees that $\operatorname{Hom}_A(M, E)$ is nonzero if M is nonzero.

Assume the result holds for modules with support of dimension $< d$ and that M has depth > 0. Choose an $f \in \mathfrak{m}$ which is a nonzerodivisor on M and consider the short exact sequence

$$0 \to M \to M \to M/fM \to 0$$

Since $\dim(\operatorname{Supp}(M/fM)) = d - 1$ (Algebra, Lemma 63.10) we may apply the induction hypothesis. Writing $E^i = \operatorname{Ext}_A^i(M, \omega_A^\bullet)$ and $F^i = \operatorname{Ext}_A^i(M/fM, \omega_A^\bullet)$ we obtain a long exact sequence

$$\ldots \to F^i \to E^i \xrightarrow{f} E^i \to F^{i+1} \to \ldots$$

By induction $E^i/fE^i = 0$ for $i+1 \notin \{-\dim(\operatorname{Supp}(M/fM)), \ldots, -\operatorname{depth}(M/fM)\}$. By Nakayama's lemma (Algebra, Lemma 20.1) and Algebra, Lemma 72.7 we conclude $E^i = 0$ for $i \notin \{-\dim(\operatorname{Supp}(M)), \ldots, -\operatorname{depth}(M)\}$. Moreover, in the boundary case $i = -\operatorname{depth}(M)$ we deduce that E^i is nonzero as F^{i+1} is nonzero by induction. Since $E^i/fE^i \subset F^{i+1}$ we get

$$\dim(\operatorname{Supp}(F^{i+1})) \geq \dim(\operatorname{Supp}(E^i/fE^i)) \geq \dim(\operatorname{Supp}(E^i)) - 1$$

(see lemma used above) we also obtain the dimension estimate (2).

If M has depth 0 and $d > 0$ we let $N = M[\mathfrak{m}^\infty]$ and set $M' = M/N$ (compare with Lemma 11.6). Then M' has depth > 0 and $\dim(\operatorname{Supp}(M')) = d$. Thus we know the result for M' and since $R\operatorname{Hom}_A(N, \omega_A^\bullet) = \operatorname{Hom}_A(N, E)$ (Lemma 16.4) the long exact cohomology sequence of Ext's implies the result for M. $\square$

0BUJ   **Remark 16.6.** Let $(A, \mathfrak{m})$ and $\omega_A^\bullet$ be as in Lemma 16.5. By More on Algebra, Lemma 69.2 we see that $\omega_A^\bullet$ has injective-amplitude in $[-d, 0]$ because part (3) of that lemma applies. In particular, for any A-module M (not necessarily finite) we have $\operatorname{Ext}_A^i(M, \omega_A^\bullet) = 0$ for $i \notin \{-d, \ldots, 0\}$.

0B5A   **Lemma 16.7.** *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring with normalized dualizing complex $\omega_A^\bullet$. Let M be a finite A-module. The following are equivalent*

     (1) *M is Cohen-Macaulay,*
     (2) *$\operatorname{Ext}_A^i(M, \omega_A^\bullet)$ is nonzero for a single i,*
     (3) *$\operatorname{Ext}_A^{-i}(M, \omega_A^\bullet)$ is zero for $i \neq \dim(Supp(M))$.*

Denote CM_d the category of finite Cohen-Macaulay A-modules of depth d. Then $M \mapsto \operatorname{Ext}_A^{-d}(M, \omega_A^\bullet)$ defines an anti-auto-equivalence of CM_d.

Proof. We will use the results of Lemma 16.5 without further mention. Fix a finite module M. If M is Cohen-Macaulay, then only $\operatorname{Ext}_A^{-d}(M, \omega_A^\bullet)$ can be nonzero, hence (1) $\Rightarrow$ (3). The implication (3) $\Rightarrow$ (2) is immediate. Assume (2) and let $N = \operatorname{Ext}_A^{-\delta}(M, \omega_A^\bullet)$ be the nonzero Ext where $\delta = \operatorname{depth}(M)$. Then, since

$$M[0] = R\operatorname{Hom}_A(R\operatorname{Hom}_A(M, \omega_A^\bullet), \omega_A^\bullet) = R\operatorname{Hom}_A(N[\delta], \omega_A^\bullet)$$

(Lemma 15.3) we conclude that $M = \operatorname{Ext}_A^{-\delta}(N, \omega_A^\bullet)$. Thus $\delta \geq \dim(\operatorname{Supp}(M))$. However, since we also know that $\delta \leq \dim(\operatorname{Supp}(M))$ (Algebra, Lemma 72.3) we conclude that M is Cohen-Macaulay.

To prove the final statement, it suffices to show that $N = \operatorname{Ext}_A^{-d}(M, \omega_A^\bullet)$ is in CM_d for M in CM_d. Above we have seen that $M[0] = R\operatorname{Hom}_A(N[d], \omega_A^\bullet)$ and this proves the desired result by the equivalence of (1) and (3). $\qquad\square$

0A7R **Lemma 16.8.** *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring with normalized dualizing complex $\omega_A^\bullet$. If $\dim(A) = 0$, then $\omega_A^\bullet \cong E[0]$ where E is an injective hull of the residue field.*

Proof. Immediate from Lemma 16.4. $\qquad\square$

0A7S **Lemma 16.9.** *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring with normalized dualizing complex. Let $I \subset \mathfrak{m}$ be an ideal of finite length. Set $B = A/I$. Then there is a distinguished triangle*

$$\omega_B^\bullet \to \omega_A^\bullet \to \operatorname{Hom}_A(I, E)[0] \to \omega_B^\bullet[1]$$

in $D(A)$ where E is an injective hull of κ and $\omega_B^\bullet$ is a normalized dualizing complex for B.

Proof. Use the short exact sequence $0 \to I \to A \to B \to 0$ and Lemmas 16.4 and 16.2. $\qquad\square$

0A7T **Lemma 16.10.** *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring with normalized dualizing complex $\omega_A^\bullet$. Let $f \in \mathfrak{m}$ be a nonzerodivisor. Set $B = A/(f)$. Then there is a distinguished triangle*

$$\omega_B^\bullet \to \omega_A^\bullet \to \omega_A^\bullet \to \omega_B^\bullet[1]$$

in $D(A)$ where $\omega_B^\bullet$ is a normalized dualizing complex for B.

Proof. Use the short exact sequence $0 \to A \to A \to B \to 0$ and Lemma 16.2. $\qquad\square$

0A7V **Lemma 16.11.** *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring with normalized dualizing complex $\omega_A^\bullet$. Let $\mathfrak{p}$ be a minimal prime of A with $\dim(A/\mathfrak{p}) = e$. Then $H^i(\omega_A^\bullet)_\mathfrak{p}$ is nonzero if and only if $i = -e$.*

Proof. Since $A_\mathfrak{p}$ has dimension zero, there exists an integer $n > 0$ such that $\mathfrak{p}^n A_\mathfrak{p}$ is zero. Set $B = A/\mathfrak{p}^n$ and $\omega_B^\bullet = R\operatorname{Hom}_A(B, \omega_A^\bullet)$. Since $B_\mathfrak{p} = A_\mathfrak{p}$ we see that

$$(\omega_B^\bullet)_\mathfrak{p} = R\operatorname{Hom}_A(B, \omega_A^\bullet) \otimes_A^{\mathbf{L}} A_\mathfrak{p} = R\operatorname{Hom}_{A_\mathfrak{p}}(B_\mathfrak{p}, (\omega_A^\bullet)_\mathfrak{p}) = (\omega_A^\bullet)_\mathfrak{p}$$

The second equality holds by More on Algebra, Lemma 99.2. By Lemma 16.2 we may replace A by B. After doing so, we see that $\dim(A) = e$. Then we see that $H^i(\omega_A^\bullet)_\mathfrak{p}$ can only be nonzero if $i = -e$ by Lemma 16.5 parts (1) and (2). On the other hand, since $(\omega_A^\bullet)_\mathfrak{p}$ is a dualizing complex for the nonzero ring $A_\mathfrak{p}$ (Lemma 15.6) we see that the remaining module has to be nonzero. $\qquad\square$

17. Dualizing complexes and dimension functions

0A7W Our results in the local setting have the following consequence: a Noetherian ring which has a dualizing complex is a universally catenary ring of finite dimension.

0A7X **Lemma 17.1.** *Let A be a Noetherian ring. Let $\mathfrak{p}$ be a minimal prime of A. Then $H^i(\omega_A^\bullet)_\mathfrak{p}$ is nonzero for exactly one i.*

Proof. The complex $\omega_A^\bullet \otimes_A A_\mathfrak{p}$ is a dualizing complex for $A_\mathfrak{p}$ (Lemma 15.6). The dimension of $A_\mathfrak{p}$ is zero as $\mathfrak{p}$ is minimal. Hence the result follows from Lemma 16.8. $\qquad\square$

Let A be a Noetherian ring and let $\omega_A^\bullet$ be a dualizing complex. Lemma 15.12 allows us to define a function

$$\delta = \delta_{\omega_A^\bullet} : \mathrm{Spec}(A) \longrightarrow \mathbf{Z}$$

by mapping $\mathfrak{p}$ to the integer of Lemma 15.12 for the dualizing complex $(\omega_A^\bullet)_\mathfrak{p}$ over $A_\mathfrak{p}$ (Lemma 15.6) and the residue field $\kappa(\mathfrak{p})$. To be precise, we define $\delta(\mathfrak{p})$ to be the unique integer such that

$$(\omega_A^\bullet)_\mathfrak{p}[-\delta(\mathfrak{p})]$$

is a normalized dualizing complex over the Noetherian local ring $A_\mathfrak{p}$.

0A7Y **Lemma 17.2.** *Let A be a Noetherian ring and let $\omega_A^\bullet$ be a dualizing complex. Let $A \to B$ be a surjective ring map and let $\omega_B^\bullet = R\mathrm{Hom}(B, \omega_A^\bullet)$ be the dualizing complex for B of Lemma 15.9. Then we have*

$$\delta_{\omega_B^\bullet} = \delta_{\omega_A^\bullet}|_{\mathrm{Spec}(B)}$$

Proof. This follows from the definition of the functions and Lemma 16.2. $\square$

0A7Z **Lemma 17.3.** *Let A be a Noetherian ring and let $\omega_A^\bullet$ be a dualizing complex. The function $\delta = \delta_{\omega_A^\bullet}$ defined above is a dimension function (Topology, Definition 20.1).*

Proof. Let $\mathfrak{p} \subset \mathfrak{q}$ be an immediate specialization. We have to show that $\delta(\mathfrak{p}) = \delta(\mathfrak{q}) + 1$. We may replace A by $A/\mathfrak{p}$, the complex $\omega_A^\bullet$ by $\omega_{A/\mathfrak{p}}^\bullet = R\mathrm{Hom}(A/\mathfrak{p}, \omega_A^\bullet)$, the prime $\mathfrak{p}$ by (0), and the prime $\mathfrak{q}$ by $\mathfrak{q}/\mathfrak{p}$, see Lemma 17.2. Thus we may assume that A is a domain, $\mathfrak{p} = (0)$, and $\mathfrak{q}$ is a prime ideal of height 1.

Then $H^i(\omega_A^\bullet)_{(0)}$ is nonzero for exactly one i, say i_0, by Lemma 17.1. In fact $i_0 = -\delta((0))$ because $(\omega_A^\bullet)_{(0)}[-\delta((0))]$ is a normalized dualizing complex over the field $A_{(0)}$.

On the other hand $(\omega_A^\bullet)_\mathfrak{q}[-\delta(\mathfrak{q})]$ is a normalized dualizing complex for $A_\mathfrak{q}$. By Lemma 16.11 we see that

$$H^e((\omega_A^\bullet)_\mathfrak{q}[-\delta(\mathfrak{q})])_{(0)} = H^{e-\delta(\mathfrak{q})}(\omega_A^\bullet)_{(0)}$$

is nonzero only for $e = -\dim(A_\mathfrak{q}) = -1$. We conclude

$$-\delta((0)) = -1 - \delta(\mathfrak{q})$$

as desired. $\square$

0A80 **Lemma 17.4.** *Let A be a Noetherian ring which has a dualizing complex. Then A is universally catenary of finite dimension.*

Proof. Because $\mathrm{Spec}(A)$ has a dimension function by Lemma 17.3 it is catenary, see Topology, Lemma 20.2. Hence A is catenary, see Algebra, Lemma 105.2. It follows from Proposition 15.11 that A is universally catenary.

Because any dualizing complex $\omega_A^\bullet$ is in $D_{Coh}^b(A)$ the values of the function $\delta_{\omega_A^\bullet}$ in minimal primes are bounded by Lemma 17.1. On the other hand, for a maximal ideal $\mathfrak{m}$ with residue field κ the integer $i = -\delta(\mathfrak{m})$ is the unique integer such that $\mathrm{Ext}_A^i(\kappa, \omega_A^\bullet)$ is nonzero (Lemma 15.12). Since $\omega_A^\bullet$ has finite injective dimension these values are bounded too. Since the dimension of A is the maximal value of $\delta(\mathfrak{p}) - \delta(\mathfrak{m})$ where $\mathfrak{p} \subset \mathfrak{m}$ are a pair consisting of a minimal prime and a maximal prime we find that the dimension of $\mathrm{Spec}(A)$ is bounded. $\square$

0AWE **Lemma 17.5.** *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring with normalized dualizing complex $\omega_A^\bullet$. Let $d = \dim(A)$ and $\omega_A = H^{-d}(\omega_A^\bullet)$. Then*

(1) *the support of ω_A is the union of the irreducible components of $\mathrm{Spec}(A)$ of dimension d,*
(2) *ω_A satisfies (S_2), see Algebra, Definition 157.1.*

Proof. We will use Lemma 16.5 without further mention. By Lemma 16.11 the support of ω_A contains the irreducible components of dimension d. Let $\mathfrak{p} \subset A$ be a prime. By Lemma 17.3 the complex $(\omega_A^\bullet)_\mathfrak{p}[-\dim(A/\mathfrak{p})]$ is a normalized dualizing complex for $A_\mathfrak{p}$. Hence if $\dim(A/\mathfrak{p}) + \dim(A_\mathfrak{p}) < d$, then $(\omega_A)_\mathfrak{p} = 0$. This proves the support of ω_A is the union of the irreducible components of dimension d, because the complement of this union is exactly the primes $\mathfrak{p}$ of A for which $\dim(A/\mathfrak{p}) + \dim(A_\mathfrak{p}) < d$ as A is catenary (Lemma 17.4). On the other hand, if $\dim(A/\mathfrak{p}) + \dim(A_\mathfrak{p}) = d$, then

$$(\omega_A)_\mathfrak{p} = H^{-\dim(A_\mathfrak{p})}\left((\omega_A^\bullet)_\mathfrak{p}[-\dim(A/\mathfrak{p})]\right)$$

Hence in order to prove ω_A has (S_2) it suffices to show that the depth of ω_A is at least $\min(\dim(A), 2)$. We prove this by induction on $\dim(A)$. The case $\dim(A) = 0$ is trivial.

Assume $\mathrm{depth}(A) > 0$. Choose a nonzerodivisor $f \in \mathfrak{m}$ and set $B = A/fA$. Then $\dim(B) = \dim(A) - 1$ and we may apply the induction hypothesis to B. By Lemma 16.10 we see that multiplication by f is injective on ω_A and we get $\omega_A/f\omega_A \subset \omega_B$. This proves the depth of ω_A is at least 1. If $\dim(A) > 1$, then $\dim(B) > 0$ and ω_B has depth > 0. Hence ω_A has depth > 1 and we conclude in this case.

Assume $\dim(A) > 0$ and $\mathrm{depth}(A) = 0$. Let $I = A[\mathfrak{m}^\infty]$ and set $B = A/I$. Then B has depth ≥ 1 and $\omega_A = \omega_B$ by Lemma 16.9. Since we proved the result for ω_B above the proof is done. $\square$

18. The local duality theorem

0A81 The main result in this section is due to Grothendieck.

0A82 **Lemma 18.1.** *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Let $\omega_A^\bullet$ be a normalized dualizing complex. Let $Z = V(\mathfrak{m}) \subset \mathrm{Spec}(A)$. Then $E = R^0\Gamma_Z(\omega_A^\bullet)$ is an injective hull of κ and $R\Gamma_Z(\omega_A^\bullet) = E[0]$.*

Proof. By Lemma 10.1 we have $R\Gamma_\mathfrak{m} = R\Gamma_Z$. Thus

$$R\Gamma_Z(\omega_A^\bullet) = R\Gamma_\mathfrak{m}(\omega_A^\bullet) = \mathrm{hocolim}\, R\mathrm{Hom}_A(A/\mathfrak{m}^n, \omega_A^\bullet)$$

by Lemma 8.2. Let E' be an injective hull of the residue field. By Lemma 16.4 we can find isomorphisms

$$R\mathrm{Hom}_A(A/\mathfrak{m}^n, \omega_A^\bullet) \cong \mathrm{Hom}_A(A/\mathfrak{m}^n, E')[0]$$

compatible with transition maps. Since $E' = \bigcup E'[\mathfrak{m}^n] = \mathrm{colim}\,\mathrm{Hom}_A(A/\mathfrak{m}^n, E')$ by Lemma 7.3 we conclude that $E \cong E'$ and that all other cohomology groups of the complex $R\Gamma_Z(\omega_A^\bullet)$ are zero. $\square$

0A83 **Remark 18.2.** Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring with a normalized dualizing complex $\omega_A^\bullet$. By Lemma 18.1 above we see that $R\Gamma_Z(\omega_A^\bullet)$ is an injective hull of the residue field placed in degree 0. In fact, this gives a "construction" or "realization" of the injective hull which is slightly more canonical than just picking any old injective hull. Namely, a normalized dualizing complex is unique up to isomorphism, with group of automorphisms the group of units of A, whereas an injective hull of κ is

unique up to isomorphism, with group of automorphisms the group of units of the completion $A^\wedge$ of A with respect to $\mathfrak{m}$.

Here is the main result of this section.

0A84 **Theorem 18.3.** *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Let $\omega_A^\bullet$ be a normalized dualizing complex. Let E be an injective hull of the residue field. Let $Z = V(\mathfrak{m}) \subset \operatorname{Spec}(A)$. Denote $^\wedge$ derived completion with respect to $\mathfrak{m}$. Then*

$$R\operatorname{Hom}_A(K, \omega_A^\bullet)^\wedge \cong R\operatorname{Hom}_A(R\Gamma_Z(K), E[0])$$

for K in $D(A)$.

Proof. Observe that $E[0] \cong R\Gamma_Z(\omega_A^\bullet)$ by Lemma 18.1. By More on Algebra, Lemma 91.13 completion on the left hand side goes inside. Thus we have to prove

$$R\operatorname{Hom}_A(K^\wedge, (\omega_A^\bullet)^\wedge) = R\operatorname{Hom}_A(R\Gamma_Z(K), R\Gamma_Z(\omega_A^\bullet))$$

This follows from the equivalence between $D_{comp}(A, \mathfrak{m})$ and $D_{\mathfrak{m}^\infty\text{-torsion}}(A)$ given in Proposition 12.2. More precisely, it is a special case of Lemma 12.3. $\square$

Here is a special case of the theorem above.

0AAK **Lemma 18.4.** *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Let $\omega_A^\bullet$ be a normalized dualizing complex. Let E be an injective hull of the residue field. Let $K \in D_{Coh}(A)$. Then*

$$\operatorname{Ext}_A^{-i}(K, \omega_A^\bullet)^\wedge = \operatorname{Hom}_A(H_\mathfrak{m}^i(K), E)$$

where $^\wedge$ denotes $\mathfrak{m}$-adic completion.

Proof. By Lemma 15.3 we see that $R\operatorname{Hom}_A(K, \omega_A^\bullet)$ is an object of $D_{Coh}(A)$. It follows that the cohomology modules of the derived completion of $R\operatorname{Hom}_A(K, \omega_A^\bullet)$ are equal to the usual completions $\operatorname{Ext}_A^i(K, \omega_A^\bullet)^\wedge$ by More on Algebra, Lemma 94.4. On the other hand, we have $R\Gamma_\mathfrak{m} = R\Gamma_Z$ for $Z = V(\mathfrak{m})$ by Lemma 10.1. Moreover, the functor $\operatorname{Hom}_A(-, E)$ is exact hence factors through cohomology. Hence the lemma is consequence of Theorem 18.3. $\square$

19. Dualizing modules

0DW3 If $(A, \mathfrak{m}, \kappa)$ is a Noetherian local ring and $\omega_A^\bullet$ is a normalized dualizing complex, then we say the module $\omega_A = H^{-\dim(A)}(\omega_A^\bullet)$, described in Lemma 17.5, is a *dualizing module* for A. This module is a canonical module of A. It seems generally agreed upon to define a *canonical module* for a Noetherian local ring $(A, \mathfrak{m}, \kappa)$ to be a finite A-module K such that

$$\operatorname{Hom}_A(K, E) \cong H_\mathfrak{m}^{\dim(A)}(A)$$

where E is an injective hull of the residue field. A dualizing module is canonical because

$$\operatorname{Hom}_A(H_\mathfrak{m}^{\dim(A)}(A), E) = (\omega_A)^\wedge$$

by Lemma 18.4 and hence applying $\operatorname{Hom}_A(-, E)$ we get

$$\begin{aligned} \operatorname{Hom}_A(\omega_A, E) &= \operatorname{Hom}_A((\omega_A)^\wedge, E) \\ &= \operatorname{Hom}_A(\operatorname{Hom}_A(H_\mathfrak{m}^{\dim(A)}(A), E), E) \\ &= H_\mathfrak{m}^{\dim(A)}(A) \end{aligned}$$

the first equality because E is $\mathfrak{m}$-power torsion, the second by the above, and the third by Matlis duality (Proposition 7.8). The utility of the definition of a canonical

module given above lies in the fact that it makes sense even if A does not have a dualizing complex.

20. Cohen-Macaulay rings

0DW4   Cohen-Macaulay modules and rings were studied in Algebra, Sections 103 and 104.

0AWR   **Lemma 20.1.** *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring with normalized dualizing complex $\omega_A^\bullet$. Then $\mathrm{depth}(A)$ is equal to the smallest integer $\delta \geq 0$ such that $H^{-\delta}(\omega_A^\bullet) \neq 0$.*

Proof. This follows immediately from Lemma 16.5. Here are two other ways to see that it is true.

First alternative. By Nakayama's lemma we see that δ is the smallest integer such that $\mathrm{Hom}_A(H^{-\delta}(\omega_A^\bullet), \kappa) \neq 0$. In other words, it is the smallest integer such that $\mathrm{Ext}_A^{-\delta}(\omega_A^\bullet, \kappa)$ is nonzero. Using Lemma 15.3 and the fact that $\omega_A^\bullet$ is normalized this is equal to the smallest integer such that $\mathrm{Ext}_A^\delta(\kappa, A)$ is nonzero. This is equal to the depth of A by Algebra, Lemma 72.5.

Second alternative. By the local duality theorem (in the form of Lemma 18.4) δ is the smallest integer such that $H_\mathfrak{m}^\delta(A)$ is nonzero. This is equal to the depth of A by Lemma 11.1. $\qquad\square$

0AWS   **Lemma 20.2.** *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring with normalized dualizing complex $\omega_A^\bullet$ and dualizing module $\omega_A = H^{-\dim(A)}(\omega_A^\bullet)$. The following are equivalent*

(1) *A is Cohen-Macaulay,*
(2) *$\omega_A^\bullet$ is concentrated in a single degree, and*
(3) *$\omega_A^\bullet = \omega_A[\dim(A)]$.*

In this case ω_A is a maximal Cohen-Macaulay module.

Proof. Follows immediately from Lemma 16.7. $\qquad\square$

0DW5   **Lemma 20.3.** *Let A be a Noetherian ring. If there exists a finite A-module ω_A such that $\omega_A[0]$ is a dualizing complex, then A is Cohen-Macaulay.*

Proof. We may replace A by the localization at a prime (Lemma 15.6 and Algebra, Definition 104.6). In this case the result follows immediately from Lemma 20.2. $\qquad\square$

0EHS   **Lemma 20.4.** *Let A be a Noetherian ring with dualizing complex $\omega_A^\bullet$. Let M be a finite A-module. Then*

$$U = \{\mathfrak{p} \in \mathrm{Spec}(A) \mid M_\mathfrak{p} \text{ is Cohen-Macaulay}\}$$

is an open subset of $\mathrm{Spec}(A)$ whose intersection with $\mathrm{Supp}(M)$ is dense.

Proof. If $\mathfrak{p}$ is a generic point of $\mathrm{Supp}(M)$, then $\mathrm{depth}(M_\mathfrak{p}) = \dim(M_\mathfrak{p}) = 0$ and hence $\mathfrak{p} \in U$. This proves denseness. If $\mathfrak{p} \in U$, then we see that

$$R\,\mathrm{Hom}_A(M, \omega_A^\bullet)_\mathfrak{p} = R\,\mathrm{Hom}_{A_\mathfrak{p}}(M_\mathfrak{p}, (\omega_A^\bullet)_\mathfrak{p})$$

has a unique nonzero cohomology module, say in degree i_0, by Lemma 16.7. Since $R\,\mathrm{Hom}_A(M, \omega_A^\bullet)$ has only a finite number of nonzero cohomology modules H^i and since each of these is a finite A-module, we can find an $f \in A$, $f \notin \mathfrak{p}$ such that $(H^i)_f = 0$ for $i \neq i_0$. Then $R\,\mathrm{Hom}_A(M, \omega_A^\bullet)_f$ has a unique nonzero cohomology module and reversing the arguments just given we find that $D(f) \subset U$. $\qquad\square$

0EHT **Lemma 20.5.** *Let A be a Noetherian ring. If A has a dualizing complex $\omega_A^\bullet$, then $\{\mathfrak{p} \in \mathrm{Spec}(A) \mid A_\mathfrak{p}$ is Cohen-Macaulay$\}$ is a dense open subset of $\mathrm{Spec}(A)$.*

Proof. Immediate consequence of Lemma 20.4 and the definitions. $\square$

21. Gorenstein rings

0DW6 So far, the only explicit dualizing complex we've seen is κ on κ for a field κ, see proof of Lemma 15.12. By Proposition 15.11 this means that any finite type algebra over a field has a dualizing complex. However, it turns out that there are Noetherian (local) rings which do not have a dualizing complex. Namely, we have seen that a ring which has a dualizing complex is universally catenary (Lemma 17.4) but there are examples of Noetherian local rings which are not catenary, see Examples, Section 18.

Nonetheless many rings in algebraic geometry have dualizing complexes simply because they are quotients of Gorenstein rings. This condition is in fact both necessary and sufficient. That is: a Noetherian ring has a dualizing complex if and only if it is a quotient of a finite dimensional Gorenstein ring. This is Sharp's conjecture ([Sha79]) which can be found as [Kaw02, Corollary 1.4] in the literature. Returning to our current topic, here is the definition of Gorenstein rings.

0DW7 **Definition 21.1.** Gorenstein rings.
(1) Let A be a Noetherian local ring. We say A is *Gorenstein* if $A[0]$ is a dualizing complex for A.
(2) Let A be a Noetherian ring. We say A is *Gorenstein* if $A_\mathfrak{p}$ is Gorenstein for every prime $\mathfrak{p}$ of A.

This definition makes sense, because if $A[0]$ is a dualizing complex for A, then $S^{-1}A[0]$ is a dualizing complex for $S^{-1}A$ by Lemma 15.6. We will see later that a finite dimensional Noetherian ring is Gorenstein if it has finite injective dimension as a module over itself.

0DW8 **Lemma 21.2.** *A Gorenstein ring is Cohen-Macaulay.*

Proof. Follows from Lemma 20.2. $\square$

An example of a Gorenstein ring is a regular ring.

0AWX **Lemma 21.3.** *A regular local ring is Gorenstein. A regular ring is Gorenstein.*

Proof. Let A be a regular ring of finite dimension d. Then A has finite global dimension d, see Algebra, Lemma 110.8. Hence $\mathrm{Ext}_A^{d+1}(M, A) = 0$ for all A-modules M, see Algebra, Lemma 109.8. Thus A has finite injective dimension as an A-module by More on Algebra, Lemma 69.2. It follows that $A[0]$ is a dualizing complex, hence A is Gorenstein by the remark following the definition. $\square$

0DW9 **Lemma 21.4.** *Let A be a Noetherian ring.*
(1) *If A has a dualizing complex $\omega_A^\bullet$, then*
   (a) *A is Gorenstein $\Leftrightarrow \omega_A^\bullet$ is an invertible object of $D(A)$,*
   (b) *$A_\mathfrak{p}$ is Gorenstein $\Leftrightarrow (\omega_A^\bullet)_\mathfrak{p}$ is an invertible object of $D(A_\mathfrak{p})$,*
   (c) *$\{\mathfrak{p} \in \mathrm{Spec}(A) \mid A_\mathfrak{p}$ is Gorenstein$\}$ is an open subset.*
(2) *If A is Gorenstein, then A has a dualizing complex if and only if $A[0]$ is a dualizing complex.*

Proof. For invertible objects of $D(A)$, see More on Algebra, Lemma 126.4 and the discussion in Section 15.

By Lemma 15.6 for every $\mathfrak{p}$ the complex $(\omega_A^\bullet)_\mathfrak{p}$ is a dualizing complex over $A_\mathfrak{p}$. By definition and uniqueness of dualizing complexes (Lemma 15.5) we see that (1)(b) holds.

To see (1)(c) assume that $A_\mathfrak{p}$ is Gorenstein. Let n_x be the unique integer such that $H^{n_x}((\omega_A^\bullet)_\mathfrak{p})$ is nonzero and isomorphic to $A_\mathfrak{p}$. Since $\omega_A^\bullet$ is in $D_{Coh}^b(A)$ there are finitely many nonzero finite A-modules $H^i(\omega_A^\bullet)$. Thus there exists some $f \in A$, $f \notin \mathfrak{p}$ such that only $H^{n_x}((\omega_A^\bullet)_f)$ is nonzero and generated by 1 element over A_f. Since dualizing complexes are faithful (by definition) we conclude that $A_f \cong H^{n_x}((\omega_A^\bullet)_f)$. In this way we see that $A_\mathfrak{q}$ is Gorenstein for every $\mathfrak{q} \in D(f)$. This proves that the set in (1)(c) is open.

Proof of (1)(a). The implication $\Leftarrow$ follows from (1)(b). The implication $\Rightarrow$ follows from the discussion in the previous paragraph, where we showed that if $A_\mathfrak{p}$ is Gorenstein, then for some $f \in A$, $f \notin \mathfrak{p}$ the complex $(\omega_A^\bullet)_f$ has only one nonzero cohomology module which is invertible.

If $A[0]$ is a dualizing complex then A is Gorenstein by part (1). Conversely, we see that part (1) shows that $\omega_A^\bullet$ is locally isomorphic to a shift of A. Since being a dualizing complex is local (Lemma 15.7) the result is clear. $\square$

0BJI **Lemma 21.5.** *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Then A is Gorenstein if and only if $\mathrm{Ext}_A^i(\kappa, A)$ is zero for $i \gg 0$.*

Proof. Observe that $A[0]$ is a dualizing complex for A if and only if A has finite injective dimension as an A-module (follows immediately from Definition 15.1). Thus the lemma follows from More on Algebra, Lemma 69.7. $\square$

0BJJ **Lemma 21.6.** *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Let $f \in \mathfrak{m}$ be a nonzero-divisor. Set $B = A/(f)$. Then A is Gorenstein if and only if B is Gorenstein.*

Proof. If A is Gorenstein, then B is Gorenstein by Lemma 16.10. Conversely, suppose that B is Gorenstein. Then $\mathrm{Ext}_B^i(\kappa, B)$ is zero for $i \gg 0$ (Lemma 21.5). Recall that $R\mathrm{Hom}(B, -) : D(A) \to D(B)$ is a right adjoint to restriction (Lemma 13.1). Hence

$$R\mathrm{Hom}_A(\kappa, A) = R\mathrm{Hom}_B(\kappa, R\mathrm{Hom}(B, A)) = R\mathrm{Hom}_B(\kappa, B[1])$$

The final equality by direct computation or by Lemma 13.10. Thus we see that $\mathrm{Ext}_A^i(\kappa, A)$ is zero for $i \gg 0$ and A is Gorenstein (Lemma 21.5). $\square$

0DWA **Lemma 21.7.** *If $A \to B$ is a local complete intersection homomorphism of rings and A is a Noetherian Gorenstein ring, then B is a Gorenstein ring.*

Proof. By More on Algebra, Definition 33.2 we can write $B = A[x_1, \ldots, x_n]/I$ where I is a Koszul-regular ideal. Observe that a polynomial ring over a Gorenstein ring A is Gorenstein: reduce to A local and then use Lemmas 15.10 and 21.4. A Koszul-regular ideal is by definition locally generated by a Koszul-regular sequence, see More on Algebra, Section 32. Looking at local rings of $A[x_1, \ldots, x_n]$ we see it suffices to show: if R is a Noetherian local Gorenstein ring and $f_1, \ldots, f_c \in \mathfrak{m}_R$ is a Koszul regular sequence, then $R/(f_1, \ldots, f_c)$ is Gorenstein. This follows from Lemma 21.6 and the fact that a Koszul regular sequence in R is just a regular sequence (More on Algebra, Lemma 30.7). $\square$

0BJL **Lemma 21.8.** *Let $A \to B$ be a flat local homomorphism of Noetherian local rings. The following are equivalent*

(1) *B is Gorenstein, and*
(2) *A and $B/\mathfrak{m}_A B$ are Gorenstein.*

Proof. Below we will use without further mention that a local Gorenstein ring has finite injective dimension as well as Lemma 21.5. By More on Algebra, Lemma 65.4 we have

$$\operatorname{Ext}_A^i(\kappa_A, A) \otimes_A B = \operatorname{Ext}_B^i(B/\mathfrak{m}_A B, B)$$

for all i.

Assume (2). Using that $R\operatorname{Hom}(B/\mathfrak{m}_A B, -) : D(B) \to D(B/\mathfrak{m}_A B)$ is a right adjoint to restriction (Lemma 13.1) we obtain

$$R\operatorname{Hom}_B(\kappa_B, B) = R\operatorname{Hom}_{B/\mathfrak{m}_A B}(\kappa_B, R\operatorname{Hom}(B/\mathfrak{m}_A B, B))$$

The cohomology modules of $R\operatorname{Hom}(B/\mathfrak{m}_A B, B)$ are the modules $\operatorname{Ext}_B^i(B/\mathfrak{m}_A B, B) = \operatorname{Ext}_A^i(\kappa_A, A) \otimes_A B$. Since A is Gorenstein, we conclude only a finite number of these are nonzero and each is isomorphic to a direct sum of copies of $B/\mathfrak{m}_A B$. Hence since $B/\mathfrak{m}_A B$ is Gorenstein we conclude that $R\operatorname{Hom}_B(B/\mathfrak{m}_B, B)$ has only a finite number of nonzero cohomology modules. Hence B is Gorenstein.

Assume (1). Since B has finite injective dimension, $\operatorname{Ext}_B^i(B/\mathfrak{m}_A B, B)$ is 0 for $i \gg 0$. Since $A \to B$ is faithfully flat we conclude that $\operatorname{Ext}_A^i(\kappa_A, A)$ is 0 for $i \gg 0$. We conclude that A is Gorenstein. This implies that $\operatorname{Ext}_A^i(\kappa_A, A)$ is nonzero for exactly one i, namely for $i = \dim(A)$, and $\operatorname{Ext}_A^{\dim(A)}(\kappa_A, A) \cong \kappa_A$ (see Lemmas 16.1, 20.2, and 21.2). Thus we see that $\operatorname{Ext}_B^i(B/\mathfrak{m}_A B, B)$ is zero except for one i, namely $i = \dim(A)$ and $\operatorname{Ext}_B^{\dim(A)}(B/\mathfrak{m}_A B, B) \cong B/\mathfrak{m}_A B$. Thus $B/\mathfrak{m}_A B$ is Gorenstein by Lemma 16.1. $\square$

0EBT **Lemma 21.9.** *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local Gorenstein ring of dimension d. Let E be the injective hull of κ. Then $\operatorname{Tor}_i^A(E, \kappa)$ is zero for $i \neq d$ and $\operatorname{Tor}_d^A(E, \kappa) = \kappa$.*

Proof. Since A is Gorenstein $\omega_A^\bullet = A[d]$ is a normalized dualizing complex for A. Also E is the only nonzero cohomology module of $R\Gamma_{\mathfrak{m}}(\omega_A^\bullet)$ sitting in degree 0, see Lemma 18.1. By Lemma 9.5 we have

$$E \otimes_A^{\mathbf{L}} \kappa = R\Gamma_{\mathfrak{m}}(\omega_A^\bullet) \otimes_A^{\mathbf{L}} \kappa = R\Gamma_{\mathfrak{m}}(\omega_A^\bullet \otimes_A^{\mathbf{L}} \kappa) = R\Gamma_{\mathfrak{m}}(\kappa[d]) = \kappa[d]$$

and the lemma follows. $\square$

22. The ubiquity of dualizing complexes

0DWB Many Noetherian rings have dualizing complexes.

0AWD **Lemma 22.1.** *Let $A \to B$ be a local homomorphism of Noetherian local rings. Let $\omega_A^\bullet$ be a normalized dualizing complex. If $A \to B$ is flat and $\mathfrak{m}_A B = \mathfrak{m}_B$, then $\omega_A^\bullet \otimes_A B$ is a normalized dualizing complex for B.*

Proof. It is clear that $\omega_A^\bullet \otimes_A B$ is in $D_{Coh}^b(B)$. Let κ_A and κ_B be the residue fields of A and B. By More on Algebra, Lemma 99.2 we see that

$$R\operatorname{Hom}_B(\kappa_B, \omega_A^\bullet \otimes_A B) = R\operatorname{Hom}_A(\kappa_A, \omega_A^\bullet) \otimes_A B = \kappa_A[0] \otimes_A B = \kappa_B[0]$$

Thus $\omega_A^\bullet \otimes_A B$ has finite injective dimension by More on Algebra, Lemma 69.7. Finally, we can use the same arguments to see that

$$R\operatorname{Hom}_B(\omega_A^\bullet \otimes_A B, \omega_A^\bullet \otimes_A B) = R\operatorname{Hom}_A(\omega_A^\bullet, \omega_A^\bullet) \otimes_A B = A \otimes_A B = B$$

as desired. $\square$

0DWC **Lemma 22.2.** *Let $A \to B$ be a flat map of Noetherian rings. Let $I \subset A$ be an ideal such that $A/I = B/IB$ and such that IB is contained in the Jacobson radical of B. Let $\omega_A^\bullet$ be a dualizing complex. Then $\omega_A^\bullet \otimes_A B$ is a dualizing complex for B.*

Proof. It is clear that $\omega_A^\bullet \otimes_A B$ is in $D_{Coh}^b(B)$. By More on Algebra, Lemma 99.2 we see that

$$R\operatorname{Hom}_B(K \otimes_A B, \omega_A^\bullet \otimes_A B) = R\operatorname{Hom}_A(K, \omega_A^\bullet) \otimes_A B$$

for any $K \in D_{Coh}^b(A)$. For any ideal $IB \subset J \subset B$ there is a unique ideal $I \subset J' \subset A$ such that $A/J' \otimes_A B = B/J$. Thus $\omega_A^\bullet \otimes_A B$ has finite injective dimension by More on Algebra, Lemma 69.6. Finally, we also have

$$R\operatorname{Hom}_B(\omega_A^\bullet \otimes_A B, \omega_A^\bullet \otimes_A B) = R\operatorname{Hom}_A(\omega_A^\bullet, \omega_A^\bullet) \otimes_A B = A \otimes_A B = B$$

as desired. $\square$

0DWD **Lemma 22.3.** *Let A be a Noetherian ring and let $I \subset A$ be an ideal. Let $\omega_A^\bullet$ be a dualizing complex.*
   (1) *$\omega_A^\bullet \otimes_A A^h$ is a dualizing complex on the henselization (A^h, I^h) of the pair (A, I),*
   (2) *$\omega_A^\bullet \otimes_A A^\wedge$ is a dualizing complex on the I-adic completion $A^\wedge$, and*
   (3) *if A is local, then $\omega_A^\bullet \otimes_A A^h$, resp. $\omega_A^\bullet \otimes_A A^{sh}$ is a dualzing complex on the henselization, resp. strict henselization of A.*

Proof. Immediate from Lemmas 22.1 and 22.2. See More on Algebra, Sections 11, 43, and 45 and Algebra, Sections 96 and 97 for information on completions and henselizations. $\square$

0BFR **Lemma 22.4.** *The following types of rings have a dualizing complex:*
   (1) *fields,*
   (2) *Noetherian complete local rings,*
   (3) *$\mathbf{Z}$,*
   (4) *Dedekind domains,*
   (5) *any ring which is obtained from one of the rings above by taking an algebra essentially of finite type, or by taking an ideal-adic completion, or by taking a henselization, or by taking a strict henselization.*

Proof. Part (5) follows from Proposition 15.11 and Lemma 22.3. By Lemma 21.3 a regular local ring has a dualizing complex. A complete Noetherian local ring is the quotient of a regular local ring by the Cohen structure theorem (Algebra, Theorem 160.8). Let A be a Dedekind domain. Then every ideal I is a finite projective A-module (follows from Algebra, Lemma 78.2 and the fact that the local rings of A are discrete valuation ring and hence PIDs). Thus every A-module has finite injective dimension at most 1 by More on Algebra, Lemma 69.2. It follows easily that $A[0]$ is a dualizing complex. $\square$

23. Formal fibres

0BJM This section is a continuation of More on Algebra, Section 51. There we saw there is a (fairly) good theory of Noetherian rings A whose local rings have Cohen-Macaulay formal fibres. Namely, we proved (1) it suffices to check the formal fibres of localizations at maximal ideals are Cohen-Macaulay, (2) the property is inherited by rings of finite type over A, (3) the fibres of $A \to A^\wedge$ are Cohen-Macaulay for any completion $A^\wedge$ of A, and (4) the property is inherited by henselizations of A. See More on Algebra, Lemma 51.4, Proposition 51.5, Lemma 51.6, and Lemma 51.7. Similarly, for Noetherian rings whose local rings have formal fibres which are geometrically reduced, geometrically normal, (S_n), and geometrically (R_n). In this section we will see that the same is true for Noetherian rings whose local rings have formal fibres which are Gorenstein or local complete intersections. This is relevant to this chapter because a Noetherian ring which has a dualizing complex is an example.

0BJN **Lemma 23.1.** *Properties (A), (B), (C), (D), and (E) of More on Algebra, Section 51 hold for $P(k \to R) = $ "R is a Gorenstein ring".*

Proof. Since we already know the result holds for Cohen-Macaulay instead of Gorenstein, we may in each step assume the ring we have is Cohen-Macaulay. This is not particularly helpful for the proof, but psychologically may be useful.

Part (A). Let K/k be a finitely generated field extension. Let R be a Gorenstein k-algebra. We can find a global complete intersection $A = k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ over k such that K is isomorphic to the fraction field of A, see Algebra, Lemma 158.11. Then $R \to R \otimes_k A$ is a relative global complete intersection. Hence $R \otimes_k A$ is Gorenstein by Lemma 21.7. Thus $R \otimes_k K$ is too as a localization.

Proof of (B). This is clear because a ring is Gorenstein if and only if all of its local rings are Gorenstein.

Part (C). Let $A \to B \to C$ be flat maps of Noetherian rings. Assume the fibres of $A \to B$ are Gorenstein and $B \to C$ is regular. We have to show the fibres of $A \to C$ are Gorenstein. Clearly, we may assume $A = k$ is a field. Then we may assume that $B \to C$ is a regular local homomorphism of Noetherian local rings. Then B is Gorenstein and $C/\mathfrak{m}_B C$ is regular, in particular Gorenstein (Lemma 21.3). Then C is Gorenstein by Lemma 21.8.

Part (D). This follows from Lemma 21.8. Part (E) is immediate as the condition does not refer to the ground field. $\square$

0AWY **Lemma 23.2.** *Let A be a Noetherian local ring. If A has a dualizing complex, then the formal fibres of A are Gorenstein.*

Proof. Let $\mathfrak{p}$ be a prime of A. The formal fibre of A at $\mathfrak{p}$ is isomorphic to the formal fibre of $A/\mathfrak{p}$ at (0). The quotient $A/\mathfrak{p}$ has a dualizing complex (Lemma 15.9). Thus it suffices to check the statement when A is a local domain and $\mathfrak{p} = (0)$. Let $\omega_A^\bullet$ be a dualizing complex for A. Then $\omega_A^\bullet \otimes_A A^\wedge$ is a dualizing complex for the completion $A^\wedge$ (Lemma 22.1). Then $\omega_A^\bullet \otimes_A K$ is a dualizing complex for the fraction field K of A (Lemma 15.6). Hence $\omega_A^\bullet \otimes_A K$ is isomorphic ot $K[n]$ for some $n \in \mathbf{Z}$. Similarly, we conclude a dualizing complex for the formal fibre $A^\wedge \otimes_A K$ is

$$\omega_A^\bullet \otimes_A A^\wedge \otimes_{A^\wedge} (A^\wedge \otimes_A K) = (\omega_A^\bullet \otimes_A K) \otimes_K (A^\wedge \otimes_A K) \cong (A^\wedge \otimes_A K)[n]$$

as desired. $\square$

Here is the verification promised in Divided Power Algebra, Remark 9.3.

0BJP **Lemma 23.3.** *Properties (A), (B), (C), (D), and (E) of More on Algebra, Section 51 hold for $P(k \to R) =$ "R is a local complete intersection". See Divided Power Algebra, Definition 8.5.*

Proof. Part (A). Let K/k be a finitely generated field extension. Let R be a k-algebra which is a local complete intersection. We can find a global complete intersection $A = k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ over k such that K is isomorphic to the fraction field of A, see Algebra, Lemma 158.11. Then $R \to R \otimes_k A$ is a relative global complete intersection. It follows that $R \otimes_k A$ is a local complete intersection by Divided Power Algebra, Lemma 8.9.

Proof of (B). This is clear because a ring is a local complete intersection if and only if all of its local rings are complete intersections.

Part (C). Let $A \to B \to C$ be flat maps of Noetherian rings. Assume the fibres of $A \to B$ are local complete intersections and $B \to C$ is regular. We have to show the fibres of $A \to C$ are local complete intersections. Clearly, we may assume $A = k$ is a field. Then we may assume that $B \to C$ is a regular local homomorphism of Noetherian local rings. Then B is a complete intersection and $C/\mathfrak{m}_B C$ is regular, in particular a complete intersection (by definition). Then C is a complete intersection by Divided Power Algebra, Lemma 8.9.

Part (D). This follows by the same arguments as in (C) from the other implication in Divided Power Algebra, Lemma 8.9. Part (E) is immediate as the condition does not refer to the ground field. $\square$

24. Upper shriek algebraically

0BZI For a finite type homomorphism $R \to A$ of Noetherian rings we will construct a functor $\varphi^! : D(R) \to D(A)$ well defined up to nonunique isomorphism which as we will see in Duality for Schemes, Remark 17.5 agrees up to isomorphism with the upper shriek functors one encounters in the duality theory for schemes. To motivate the construction we mention two additional properties:

(1) $\varphi^!$ sends a dualizing complex for R (if it exists) to a dualizing complex for A, and
(2) $\omega^\bullet_{A/R} = \varphi^!(R)$ is a kind of relative dualizing complex: it lies in $D^b_{Coh}(A)$ and restricts to a dualizing complex on the fibres provided $R \to A$ is flat.

These statemens are Lemmas 24.3 and 25.2.

Let $\varphi : R \to A$ be a finite type homomorphism of Noetherian rings. We will define a functor $\varphi^! : D(R) \to D(A)$ in the following way

(1) If $\varphi : R \to A$ is surjective we set $\varphi^!(K) = R\operatorname{Hom}(A, K)$. Here we use the functor $R\operatorname{Hom}(A, -) : D(R) \to D(A)$ of Section 13, and
(2) in general we choose a surjection $\psi : P \to A$ with $P = R[x_1, \ldots, x_n]$ and we set $\varphi^!(K) = \psi^!(K \otimes^{\mathbf{L}}_R P)[n]$. Here we use the functor $- \otimes^{\mathbf{L}}_R P : D(R) \to D(P)$ of More on Algebra, Section 60.

Note the shift $[n]$ by the number of variables in the polynomial ring. This construction is **not** canonical and the functor $\varphi^!$ will only be well defined up to a (nonunique) isomorphism of functors[1].

0BZJ **Lemma 24.1.** *Let $\varphi : R \to A$ be a finite type homomorphism of Noetherian rings. The functor $\varphi^!$ is well defined up to isomorphism.*

Proof. Suppose that $\psi_1 : P_1 = R[x_1, \ldots, x_n] \to A$ and $\psi_2 : P_2 = R[y_1, \ldots, y_m] \to A$ are two surjections from polynomial rings onto A. Then we get a commutative diagram

$$
\begin{array}{ccc}
R[x_1, \ldots, x_n, y_1, \ldots, y_m] & \xrightarrow{\ y_j \mapsto f_j\ } & R[x_1, \ldots, x_n] \\
\Big\downarrow{\scriptstyle x_i \mapsto g_i} & & \Big\downarrow \\
R[y_1, \ldots, y_m] & \xrightarrow{\hspace{4cm}} & A
\end{array}
$$

where f_j and g_i are chosen such that $\psi_1(f_j) = \psi_2(y_j)$ and $\psi_2(g_i) = \psi_1(x_i)$. By symmetry it suffices to prove the functors defined using $P \to A$ and $P[y_1, \ldots, y_m] \to A$ are isomorphic. By induction we may assume $m = 1$. This reduces us to the case discussed in the next paragraph.

Here $\psi : P \to A$ is given and $\chi : P[y] \to A$ induces ψ on P. Write $Q = P[y]$. Choose $g \in P$ with $\psi(g) = \chi(y)$. Denote $\pi : Q \to P$ the P-algebra map with $\pi(y) = g$. Then $\chi = \psi \circ \pi$ and hence $\chi^! = \psi^! \circ \pi^!$ as both are adjoint to the restriction functor $D(A) \to D(Q)$ by the material in Section 13. Thus

$$
\chi^! \left(K \otimes_R^{\mathbf{L}} Q \right)[n+1] = \psi^! \left(\pi^! \left(K \otimes_R^{\mathbf{L}} Q \right)[1] \right)[n]
$$

Hence it suffices to show that $\pi^!(K \otimes_R^{\mathbf{L}} Q[1]) = K \otimes_R^{\mathbf{L}} P$ Thus it suffices to show that the functor $\pi^!(-) : D(Q) \to D(P)$ is isomorphic to $K \mapsto K \otimes_Q^{\mathbf{L}} P[-1]$. This follows from Lemma 13.10. $\square$

0BZK **Lemma 24.2.** *Let $\varphi : R \to A$ be a finite type homomorphism of Noetherian rings.*
   (1) *$\varphi^!$ maps $D^+(R)$ into $D^+(A)$ and $D_{Coh}^+(R)$ into $D_{Coh}^+(A)$.*
   (2) *if φ is perfect, then $\varphi^!$ maps $D^-(R)$ into $D^-(A)$, $D_{Coh}^-(R)$ into $D_{Coh}^-(A)$, and $D_{Coh}^b(R)$ into $D_{Coh}^b(A)$.*

Proof. Choose a factorization $R \to P \to A$ as in the definition of $\varphi^!$. The functor $-\otimes_R^{\mathbf{L}} : D(R) \to D(P)$ preserves the subcategories $D^+, D_{Coh}^+, D^-, D_{Coh}^-, D_{Coh}^b$. The functor $R\operatorname{Hom}(A, -) : D(P) \to D(A)$ preserves D^+ and D_{Coh}^+ by Lemma 13.4. If $R \to A$ is perfect, then A is perfect as a P-module, see More on Algebra, Lemma 82.2. Recall that the restriction of $R\operatorname{Hom}(A, K)$ to $D(P)$ is $R\operatorname{Hom}_P(A, K)$. By More on Algebra, Lemma 74.15 we have $R\operatorname{Hom}_P(A, K) = E \otimes_P^{\mathbf{L}} K$ for some perfect $E \in D(P)$. Since we can represent E by a finite complex of finite projective P-modules it is clear that $R\operatorname{Hom}_P(A, K)$ is in $D^-(P), D_{Coh}^-(P), D_{Coh}^b(P)$ as soon as K is. Since the restriction functor $D(A) \to D(P)$ reflects these subcategories, the proof is complete. $\square$

0BZL **Lemma 24.3.** *Let φ be a finite type homomorphism of Noetherian rings. If $\omega_R^\bullet$ is a dualizing complex for R, then $\varphi^!(\omega_R^\bullet)$ is a dualizing complex for A.*

---

[1]It is possible to make the construction canonical: use $\Omega_{P/R}^n[n]$ instead of $P[n]$ in the construction and use this in Lemma 24.1. The material in this section becomes a lot more involved if one wants to do this.

Proof. Follows from Lemmas 15.10 and 15.9, □

0BZN **Lemma 24.4.** *Let $R \to R'$ be a flat homomorphism of Noetherian rings. Let $\varphi : R \to A$ be a finite type ring map. Let $\varphi' : R' \to A' = A \otimes_R R'$ be the map induced by φ. Then we have a functorial maps*

$$\varphi^!(K) \otimes_A^{\mathbf{L}} A' \longrightarrow (\varphi')^!(K \otimes_R^{\mathbf{L}} R')$$

for K in $D(R)$ which are isomorphisms for $K \in D^+(R)$.

Proof. Choose a factorization $R \to P \to A$ where P is a polynomial ring over R. This gives a corresponding factorization $R' \to P' \to A'$ by base change. Since we have $(K \otimes_R^{\mathbf{L}} P) \otimes_P^{\mathbf{L}} P' = (K \otimes_R^{\mathbf{L}} R') \otimes_{R'}^{\mathbf{L}} P'$ by More on Algebra, Lemma 60.5 it suffices to construct maps

$$R \operatorname{Hom}(A, K \otimes_R^{\mathbf{L}} P[n]) \otimes_A^{\mathbf{L}} A' \longrightarrow R \operatorname{Hom}(A', (K \otimes_R^{\mathbf{L}} P[n]) \otimes_P^{\mathbf{L}} P')$$

functorial in K. For this we use the map (14.0.1) constructed in Section 14 for P, A, P', A'. The map is an isomorphism for $K \in D^+(R)$ by Lemma 14.2. □

0BZR **Lemma 24.5.** *Let $R \to R'$ be a homomorphism of Noetherian rings. Let $\varphi : R \to A$ be a perfect ring map (More on Algebra, Definition 82.1) such that R' and A are tor independent over R. Let $\varphi' : R' \to A' = A \otimes_R R'$ be the map induced by φ. Then we have a functorial isomorphism*

$$\varphi^!(K) \otimes_A^{\mathbf{L}} A' = (\varphi')^!(K \otimes_R^{\mathbf{L}} R')$$

for K in $D(R)$.

Proof. We may choose a factorization $R \to P \to A$ where P is a polynomial ring over R such that A is a perfect P-module, see More on Algebra, Lemma 82.2. This gives a corresponding factorization $R' \to P' \to A'$ by base change. Since we have $(K \otimes_R^{\mathbf{L}} P) \otimes_P^{\mathbf{L}} P' = (K \otimes_R^{\mathbf{L}} R') \otimes_{R'}^{\mathbf{L}} P'$ by More on Algebra, Lemma 60.5 it suffices to construct maps

$$R \operatorname{Hom}(A, K \otimes_R^{\mathbf{L}} P[n]) \otimes_A^{\mathbf{L}} A' \longrightarrow R \operatorname{Hom}(A', (K \otimes_R^{\mathbf{L}} P[n]) \otimes_P^{\mathbf{L}} P')$$

functorial in K. We have

$$A \otimes_P^{\mathbf{L}} P' = A \otimes_R^{\mathbf{L}} R' = A'$$

The first equality by More on Algebra, Lemma 61.2 applied to R, R', P, P'. The second equality because A and R' are tor independent over R. Hence A and P' are tor independent over P and we can use the map (14.0.1) constructed in Section 14 for P, A, P', A' get the desired arrow. By Lemma 14.3 to finish the proof it suffices to prove that A is a perfect P-module which we saw above. □

0BZS **Lemma 24.6.** *Let $R \to R'$ be a homomorphism of Noetherian rings. Let $\varphi : R \to A$ be flat of finite type. Let $\varphi' : R' \to A' = A \otimes_R R'$ be the map induced by φ. Then we have a functorial isomorphism*

$$\varphi^!(K) \otimes_A^{\mathbf{L}} A' = (\varphi')^!(K \otimes_R^{\mathbf{L}} R')$$

for K in $D(R)$.

Proof. Special case of Lemma 24.5 by More on Algebra, Lemma 82.4. □

0BZT **Lemma 24.7.** *Let $A \xrightarrow{a} B \xrightarrow{b} C$ be finite type homomorphisms of Noetherian rings. Then there is a transformation of functors $b^! \circ a^! \to (b \circ a)^!$ which is an isomorphism on $D^+(A)$.*

Proof. Choose a polynomial ring $P = A[x_1, \ldots, x_n]$ over A and a surjection $P \to B$. Choose elements $c_1, \ldots, c_m \in C$ generating C over B. Set $Q = P[y_1, \ldots, y_m]$ and denote $Q' = Q \otimes_P B = B[y_1, \ldots, y_m]$. Let $\chi : Q' \to C$ be the surjection sending y_j to c_j. Picture

$$
\begin{array}{ccccc}
Q & \xrightarrow{\ \psi'\ } & Q' & \xrightarrow{\ \chi\ } & C \\
\uparrow & & \uparrow & & \\
\uparrow & & \uparrow & & \\
A \longrightarrow & P & \xrightarrow{\ \psi\ } & B &
\end{array}
$$

By Lemma 14.2 for $M \in D(P)$ we have an arrow $\psi^!(M) \otimes_B^{\mathbf{L}} Q' \to (\psi')^!(M \otimes_P^{\mathbf{L}} Q)$ which is an isomorphism whenever M is bounded below. Also we have $\chi^! \circ (\psi')^! = (\chi \circ \psi')^!$ as both functors are adjoint to the restriction functor $D(C) \to D(Q)$ by Section 13. Then we see

$$
\begin{aligned}
b^!(a^!(K)) &= \chi^!(\psi^!(K \otimes_A^{\mathbf{L}} P)[n] \otimes_B^{\mathbf{L}} Q)[m] \\
&\to \chi^!((\psi')^!(K \otimes_A^{\mathbf{L}} P \otimes_P^{\mathbf{L}} Q))[n + m] \\
&= (\chi \circ \psi')^!(K \otimes_A^{\mathbf{L}} Q)[n + m] \\
&= (b \circ a)^!(K)
\end{aligned}
$$

where we have used in addition to the above More on Algebra, Lemma 60.5. $\square$

0C0G **Lemma 24.8.** *Let $\varphi : R \to A$ be a finite map of Noetherian rings. Then $\varphi^!$ is isomorphic to the functor $R\operatorname{Hom}(A, -) : D(R) \to D(A)$ from Section 13.*

Proof. Suppose that A is generated by $n > 1$ elements over R. Then can factor $R \to A$ as a composition of two finite ring maps where in both steps the number of generators is $< n$. Since we have Lemma 24.7 and Lemma 13.2 we conclude that it suffices to prove the lemma when A is generated by one element over R. Since A is finite over R, it follows that A is a quotient of $B = R[x]/(f)$ where f is a monic polynomial in x (Algebra, Lemma 36.3). Again using the lemmas on composition and the fact that we have agreement for surjections by definition, we conclude that it suffices to prove the lemma for $R \to B = R[x]/(f)$. In this case, the functor $\varphi^!$ is isomorphic to $K \mapsto K \otimes_R^{\mathbf{L}} B$; you prove this by using Lemma 13.10 for the map $R[x] \to B$ (note that the shift in the definition of $\varphi^!$ and in the lemma add up to zero). For the functor $R\operatorname{Hom}(B, -) : D(R) \to D(B)$ we can use Lemma 13.9 to see that it suffices to show $\operatorname{Hom}_R(B, R) \cong B$ as B-modules. Suppose that f has degree d. Then an R-basis for B is given by $1, x, \ldots, x^{d-1}$. Let $\delta_i : B \to R$, $i = 0, \ldots, d-1$ be the R-linear map which picks off the coefficient of x^i with respect to the given basis. Then $\delta_0, \ldots, \delta_{d-1}$ is a basis for $\operatorname{Hom}_R(B, R)$. Finally, for $0 \le i \le d - 1$ a computation shows that

$$
x^i \delta_{d-1} = \delta_{d-1-i} + b_1 \delta_{d-i} + \ldots + b_i \delta_{d-1}
$$

for some $c_1, \ldots, c_d \in R^2$. Hence $\operatorname{Hom}_R(B, R)$ is a principal B-module with generator δ_{d-1}. By looking at ranks we conclude that it is a rank 1 free B-module. $\square$

0C0H **Lemma 24.9.** *Let R be a Noetherian ring and let $f \in R$. If φ denotes the map $R \to R_f$, then $\varphi^!$ is isomorphic to $- \otimes_R^{\mathbf{L}} R_f$. More generally, if $\varphi : R \to R'$ is a map such that $\operatorname{Spec}(R') \to \operatorname{Spec}(R)$ is an open immersion, then $\varphi^!$ is isomorphic to $- \otimes_R^{\mathbf{L}} R'$.*

---

[2]If $f = x^d + a_1 x^{d-1} + \ldots + a_d$, then $c_1 = -a_1$, $c_2 = a_1^2 - a_2$, $c_3 = -a_1^3 + 2a_1 a_2 - a_3$, etc.

Proof. Choose the presentation $R \to R[x] \to R[x]/(fx - 1) = R_f$ and observe that $fx - 1$ is a nonzerodivisor in $R[x]$. Thus we can apply using Lemma 13.10 to compute the functor $\varphi^!$. Details omitted; note that the shift in the definition of $\varphi^!$ and in the lemma add up to zero.

In the general case note that $R' \otimes_R R' = R'$. Hence the result follows from the base change results above. Either Lemma 24.4 or Lemma 24.5 will do. $\square$

0BZU   **Lemma 24.10.** *Let $\varphi : R \to A$ be a perfect homomorphism of Noetherian rings (for example φ is flat of finite type). Then $\varphi^!(K) = K \otimes_R^{\mathbf{L}} \varphi^!(R)$ for $K \in D(R)$.*

Proof. (The parenthetical statement follows from More on Algebra, Lemma 82.4.) We can choose a factorization $R \to P \to A$ where P is a polynomial ring in n variables over R and then A is a perfect P-module, see More on Algebra, Lemma 82.2. Recall that $\varphi^!(K) = R\operatorname{Hom}(A, K \otimes_R^{\mathbf{L}} P[n])$. Thus the result follows from Lemma 13.9 and More on Algebra, Lemma 60.5. $\square$

0E9L   **Lemma 24.11.** *Let $\varphi : A \to B$ be a finite type homomorphism of Noetherian rings. Let $\omega_A^\bullet$ be a dualizing complex for A. Set $\omega_B^\bullet = \varphi^!(\omega_A^\bullet)$. Denote $D_A(K) = R\operatorname{Hom}_A(K, \omega_A^\bullet)$ for $K \in D_{Coh}(A)$ and $D_B(L) = R\operatorname{Hom}_B(L, \omega_B^\bullet)$ for $L \in D_{Coh}(B)$. Then there is a functorial isomorphism*

$$\varphi^!(K) = D_B(D_A(K) \otimes_A^{\mathbf{L}} B)$$

for $K \in D_{Coh}(A)$.

Proof. Observe that $\omega_B^\bullet$ is a dualizing complex for B by Lemma 24.3. Let $A \to B \to C$ be finite type homomorphisms of Noetherian rings. If the lemma holds for $A \to B$ and $B \to C$, then the lemma holds for $A \to C$. This follows from Lemma 24.7 and the fact that $D_B \circ D_B \cong \operatorname{id}$ by Lemma 15.3. Thus it suffices to prove the lemma in case $A \to B$ is a surjection and in the case where B is a polynomial ring over A.

Assume $B = A[x_1, \ldots, x_n]$. Since $D_A \circ D_A \cong \operatorname{id}$, it suffices to prove $D_B(K \otimes_A B) \cong D_A(K) \otimes_A B[n]$ for K in $D_{Coh}(A)$. Choose a bounded complex $I^\bullet$ of injectives representing $\omega_A^\bullet$. Choose a quasi-isomorphism $I^\bullet \otimes_A B \to J^\bullet$ where $J^\bullet$ is a bounded complex of B-modules. Given a complex $K^\bullet$ of A-modules, consider the obvious map of complexes

$$\operatorname{Hom}^\bullet(K^\bullet, I^\bullet) \otimes_A B[n] \longrightarrow \operatorname{Hom}^\bullet(K^\bullet \otimes_A B, J^\bullet[n])$$

The left hand side represents $D_A(K) \otimes_A B[n]$ and the right hand side represents $D_B(K \otimes_A B)$. Thus it suffices to prove this map is a quasi-isomorphism if the cohomology modules of $K^\bullet$ are finite A-modules. Observe that the cohomology of the complex in degree r (on either side) only depends on finitely many of the K^i. Thus we may replace $K^\bullet$ by a truncation, i.e., we may assume $K^\bullet$ represents an object of $D_{Coh}^-(A)$. Then $K^\bullet$ is quasi-isomorphic to a bounded above complex of finite free A-modules. Therefore we may assume $K^\bullet$ is a bounded above complex of finite free A-modules. In this case it is easy to that the displayed map is an isomorphism of complexes which finishes the proof in this case.

Assume that $A \to B$ is surjective. Denote $i_* : D(B) \to D(A)$ the restriction functor and recall that $\varphi^!(-) = R\operatorname{Hom}(A, -)$ is a right adjoint to i_* (Lemma 13.1). For

$F \in D(B)$ we have

$$\begin{aligned}
\operatorname{Hom}_B(F, D_B(D_A(K) \otimes_A^{\mathbf{L}} B)) &= \operatorname{Hom}_B((D_A(K) \otimes_A^{\mathbf{L}} B) \otimes_B^{\mathbf{L}} F, \omega_B^\bullet) \\
&= \operatorname{Hom}_A(D_A(K) \otimes_A^{\mathbf{L}} i_* F, \omega_A^\bullet) \\
&= \operatorname{Hom}_A(i_* F, D_A(D_A(K))) \\
&= \operatorname{Hom}_A(i_* F, K) \\
&= \operatorname{Hom}_B(F, \varphi^!(K))
\end{aligned}$$

The first equality follows from More on Algebra, Lemma 73.1 and the definition of D_B. The second equality by the adjointness mentioned above and the equality $i_*((D_A(K) \otimes_A^{\mathbf{L}} B) \otimes_B^{\mathbf{L}} F) = D_A(K) \otimes_A^{\mathbf{L}} i_* F$ (More on Algebra, Lemma 60.1). The third equality follows from More on Algebra, Lemma 73.1. The fourth because $D_A \circ D_A = \mathrm{id}$. The final equality by adjointness again. Thus the result holds by the Yoneda lemma. $\square$

25. Relative dualizing complexes in the Noetherian case

0E9M   Let $\varphi : R \to A$ be a finite type homomorphism of Noetherian rings. Then we define the *relative dualizing complex of A over R* as the object

$$\omega_{A/R}^\bullet = \varphi^!(R)$$

of $D(A)$. Here $\varphi^!$ is as in Section 24. From the material in that section we see that $\omega_{A/R}^\bullet$ is well defined up to (non-unique) isomorphism.

0BZV   **Lemma 25.1.** *Let $R \to R'$ be a homomorphism of Noetherian rings. Let $R \to A$ be of finite type. Set $A' = A \otimes_R R'$. If*

(1) *$R \to R'$ is flat, or*
(2) *$R \to A$ is flat, or*
(3) *$R \to A$ is perfect and R' and A are tor independent over R,*

then there is an isomorphism $\omega_{A/R}^\bullet \otimes_A^{\mathbf{L}} A' \to \omega_{A'/R'}^\bullet$ in $D(A')$.

Proof. Follows from Lemmas 24.4, 24.6, and 24.5 and the definitions. $\square$

0BZW   **Lemma 25.2.** *Let $\varphi : R \to A$ be a flat finite type map of Noetherian rings. Then*

(1) *$\omega_{A/R}^\bullet$ is in $D_{Coh}^b(A)$ and R-perfect (More on Algebra, Definition 83.1),*
(2) *$A \to R\operatorname{Hom}_A(\omega_{A/R}^\bullet, \omega_{A/R}^\bullet)$ is an isomorphism, and*
(3) *for every map $R \to k$ to a field the base change $\omega_{A/R}^\bullet \otimes_A^{\mathbf{L}} (A \otimes_R k)$ is a dualizing complex for $A \otimes_R k$.*

Proof. Choose $R \to P \to A$ as in the definition of $\varphi^!$. Recall that $R \to A$ is a perfect ring map (More on Algebra, Lemma 82.4) and hence A is perfect as a P-mdoule (More on Algebra, Lemma 82.2). This shows that $\omega_{A/R}^\bullet$ is in $D_{Coh}^b(A)$ by Lemma 24.2. To show $\omega_{A/R}^\bullet$ is R-perfect it suffices to show it has finite tor dimension as a complex of R-modules. This is true because $\omega_{A/R}^\bullet = \varphi^!(R) = R\operatorname{Hom}(A, P)[n]$ maps to $R\operatorname{Hom}_P(A, P)[n]$ in $D(P)$, which is perfect in $D(P)$ (More on Algebra, Lemma 74.15), hence has finite tor dimension in $D(R)$ as $R \to P$ is flat. This proves (1).

Proof of (2). The object $R \operatorname{Hom}_A(\omega_{A/R}^\bullet, \omega_{A/R}^\bullet)$ of $D(A)$ maps in $D(P)$ to

$$R \operatorname{Hom}_P(\omega_{A/R}^\bullet, R \operatorname{Hom}(A, P)[n]) = R \operatorname{Hom}_P(R \operatorname{Hom}_P(A, P)[n], P)[n]$$
$$= R \operatorname{Hom}_P(R \operatorname{Hom}_P(A, P), P)$$

This is equal to A by the already used More on Algebra, Lemma 74.15.

Proof of (3). By Lemma 25.1 there is an isomorphism

$$\omega_{A/R}^\bullet \otimes_A^{\mathbf{L}} (A \otimes_R k) \cong \omega_{A \otimes_R k/k}^\bullet$$

and the right hand side is a dualizing complex by Lemma 24.3. $\square$

0E0P **Lemma 25.3.** *Let K/k be an extension of fields. Let A be a finite type k-algebra. Let $A_K = A \otimes_k K$. If $\omega_A^\bullet$ is a dualizing complex for A, then $\omega_A^\bullet \otimes_A A_K$ is a dualizing complex for A_K.*

Proof. By the uniqueness of dualizing complexes, it doesn't matter which dualizing complex we pick for A; we omit the detailed proof. Denote $\varphi : k \to A$ the algebra structure. We may take $\omega_A^\bullet = \varphi^!(k[0])$ by Lemma 24.3. We conclude by Lemma 25.2. $\square$

0E4B **Lemma 25.4.** *Let $\varphi : R \to A$ be a local complete intersection homomorphism of Noetherian rings. Then $\omega_{A/R}^\bullet$ is an invertible object of $D(A)$ and $\varphi^!(K) = K \otimes_R^{\mathbf{L}} \omega_{A/R}^\bullet$ for all $K \in D(R)$.*

Proof. Recall that a local complete intersection homomorphism is a perfect ring map by More on Algebra, Lemma 82.6. Hence the final statement holds by Lemma 24.10. By More on Algebra, Definition 33.2 we can write $A = R[x_1, \ldots, x_n]/I$ where I is a Koszul-regular ideal. The construction of $\varphi^!$ in Section 24 shows that it suffices to show the lemma in case $A = R/I$ where $I \subset R$ is a Koszul-regular ideal. Checking $\omega_{A/R}^\bullet$ is invertible in $D(A)$ is local on $\operatorname{Spec}(A)$ by More on Algebra, Lemma 126.4. Moreover, formation of $\omega_{A/R}^\bullet$ commutes with localization on R by Lemma 24.4. Combining More on Algebra, Definition 32.1 and Lemma 30.7 and Algebra, Lemma 68.6 we can find $g_1, \ldots, g_r \in R$ generating the unit ideal in A such that $I_{g_j} \subset R_{g_j}$ is generated by a regular sequence. Thus we may assume $A = R/(f_1, \ldots, f_c)$ where $f_1, \ldots, f_c$ is a regular sequence in R. Then we consider the ring maps

$$R \to R/(f_1) \to R/(f_1, f_2) \to \ldots \to R/(f_1, \ldots, f_c) = A$$

and we use Lemma 24.7 (and the final statement already proven) to see that it suffices to prove the lemma for each step. Finally, in case $A = R/(f)$ for some nonzerodivisor f we see that the lemma is true since $\varphi^!(R) = R \operatorname{Hom}(A, R)$ is invertible by Lemma 13.10. $\square$

0E4C **Lemma 25.5.** *Let $\varphi : R \to A$ be a flat finite type homomorphism of Noetherian rings. The following are equivalent*

(1) *the fibres $A \otimes_R \kappa(\mathfrak{p})$ are Gorenstein for all primes $\mathfrak{p} \subset R$, and*
(2) *$\omega_{A/R}^\bullet$ is an invertible object of $D(A)$, see More on Algebra, Lemma 126.4.*

Proof. If (2) holds, then the fibre rings $A \otimes_R \kappa(\mathfrak{p})$ have invertible dualizing complexes, and hence are Gorenstein. See Lemmas 25.2 and 21.4.

For the converse, assume (1). Observe that $\omega_{A/R}^\bullet$ is in $D_{Coh}^b(A)$ by Lemma 24.2 (since flat finite type homomorphisms of Noetherian rings are perfect, see More on Algebra, Lemma 82.4). Take a prime $\mathfrak{q} \subset A$ lying over $\mathfrak{p} \subset R$. Then

$$\omega_{A/R}^\bullet \otimes_A^{\mathbf{L}} \kappa(\mathfrak{q}) = \omega_{A/R}^\bullet \otimes_A^{\mathbf{L}} (A \otimes_R \kappa(\mathfrak{p})) \otimes_{(A \otimes_R \kappa(\mathfrak{p}))}^{\mathbf{L}} \kappa(\mathfrak{q})$$

Applying Lemmas 25.2 and 21.4 and assumption (1) we find that this complex has 1 nonzero cohomology group which is a 1-dimensional $\kappa(\mathfrak{q})$-vector space. By More on Algebra, Lemma 77.1 we conclude that $(\omega_{A/R}^\bullet)_f$ is an invertible object of $D(A_f)$ for some $f \in A$, $f \notin \mathfrak{q}$. This proves (2) holds. $\square$

The following lemma is useful to see how dimension functions change when passing to a finite type algebra over a Noetherian ring.

0E9N **Lemma 25.6.** *Let $\varphi : R \to A$ be a finite type homomorphism of Noetherian rings. Assume R local and let $\mathfrak{m} \subset A$ be a maximal ideal lying over the maximal ideal of R. If $\omega_R^\bullet$ is a normalized dualizing complex for R, then $\varphi^!(\omega_R^\bullet)_\mathfrak{m}$ is a normalized dualizing complex for $A_\mathfrak{m}$.*

Proof. We already know that $\varphi^!(\omega_R^\bullet)$ is a dualizing complex for A, see Lemma 24.3. Choose a factorization $R \to P \to A$ with $P = R[x_1, \ldots, x_n]$ as in the construction of $\varphi^!$. If we can prove the lemma for $R \to P$ and the maximal ideal $\mathfrak{m}'$ of P corresponding to $\mathfrak{m}$, then we obtain the result for $R \to A$ by applying Lemma 16.2 to $P_{\mathfrak{m}'} \to A_\mathfrak{m}$ or by applying Lemma 17.2 to $P \to A$. In the case $A = R[x_1, \ldots, x_n]$ we see that $\dim(A_\mathfrak{m}) = \dim(R) + n$ for example by Algebra, Lemma 112.7 (combined with Algebra, Lemma 114.1 to compute the dimension of the fibre). The fact that $\omega_R^\bullet$ is normalized means that $i = -\dim(R)$ is the smallest index such that $H^i(\omega_R^\bullet)$ is nonzero (follows from Lemmas 16.5 and 16.11). Then $\varphi^!(\omega_R^\bullet)_\mathfrak{m} = \omega_R^\bullet \otimes_R A_\mathfrak{m}[n]$ has its first nonzero cohomology module in degree $-\dim(R) - n$ and therefore is the normalized dualizing complex for $A_\mathfrak{m}$. $\square$

0E9P **Lemma 25.7.** *Let $R \to A$ be a finite type homomorphism of Noetherian rings. Let $\mathfrak{q} \subset A$ be a prime ideal lying over $\mathfrak{p} \subset R$. Then*

$$H^i(\omega_{A/R}^\bullet)_\mathfrak{q} \neq 0 \Rightarrow -d \leq i$$

where d is the dimension of the fibre of $\mathrm{Spec}(A) \to \mathrm{Spec}(R)$ over $\mathfrak{p}$ at the point $\mathfrak{q}$.

Proof. Choose a factorization $R \to P \to A$ with $P = R[x_1, \ldots, x_n]$ as in Section 24 so that $\omega_{A/R}^\bullet = R\mathrm{Hom}(A, P)[n]$. We have to show that $R\mathrm{Hom}(A, P)_\mathfrak{q}$ has vanishing cohomology in degrees $< n - d$. By Lemma 13.3 this means we have to show that $\mathrm{Ext}_P^i(P/I, P)_\mathfrak{r} = 0$ for $i < n - d$ where $\mathfrak{r} \subset P$ is the prime corresponding to $\mathfrak{q}$ and I is the kernel of $P \to A$. We may rewrite this as $\mathrm{Ext}_{P_\mathfrak{r}}^i(P_\mathfrak{r}/IP_\mathfrak{r}, P_\mathfrak{r})$ by More on Algebra, Lemma 65.4. Thus we have to show

$$\mathrm{depth}_{IP_\mathfrak{r}}(P_\mathfrak{r}) \geq n - d$$

by Lemma 11.1. By Lemma 11.5 we have

$$\mathrm{depth}_{IP_\mathfrak{r}}(P_\mathfrak{r}) \geq \dim((P \otimes_R \kappa(\mathfrak{p}))_\mathfrak{r}) - \dim((P/I \otimes_R \kappa(\mathfrak{p}))_\mathfrak{r})$$

The two expressions on the right hand side agree by Algebra, Lemma 116.4. $\square$

0E9Q **Lemma 25.8.** *Let $R \to A$ be a flat finite type homomorphism of Noetherian rings. Let $\mathfrak{q} \subset A$ be a prime ideal lying over $\mathfrak{p} \subset R$. Then*

$$H^i(\omega_{A/R}^\bullet)_\mathfrak{q} \neq 0 \Rightarrow -d \leq i \leq 0$$

where d is the dimension of the fibre of $\mathrm{Spec}(A) \to \mathrm{Spec}(R)$ over $\mathfrak{p}$ at the point $\mathfrak{q}$. If all fibres of $\mathrm{Spec}(A) \to \mathrm{Spec}(R)$ have dimension $\leq d$, then $\omega^\bullet_{A/R}$ has tor amplitude in $[-d, 0]$ as a complex of R-modules.

Proof. The lower bound has been shown in Lemma 25.7. Choose a factorization $R \to P \to A$ with $P = R[x_1, \ldots, x_n]$ as in Section 24 so that $\omega^\bullet_{A/R} = R\,\mathrm{Hom}(A, P)[n]$. The upper bound means that $\mathrm{Ext}^i_P(A, P)$ is zero for $i > n$. This follows from More on Algebra, Lemma 77.5 which shows that A is a perfect P-module with tor amplitude in $[-n, 0]$.

Proof of the final statement. Let $R \to R'$ be a ring homomorphism of Noetherian rings. Set $A' = A \otimes_R R'$. Then

$$\omega^\bullet_{A'/R'} = \omega^\bullet_{A/R} \otimes^{\mathbf{L}}_A A' = \omega^\bullet_{A/R} \otimes^{\mathbf{L}}_R R'$$

The first isomorphism by Lemma 25.1 and the second, which takes place in $D(R')$, by More on Algebra, Lemma 61.2. By the first part of the proof (note that the fibres of $\mathrm{Spec}(A') \to \mathrm{Spec}(R')$ have dimension $\leq d$) we conclude that $\omega^\bullet_{A/R} \otimes^{\mathbf{L}}_R R'$ has cohomology only in degrees $[-d, 0]$. Taking $R' = R \oplus M$ to be the square zero thickening of R by a finite R-module M, we see that $R\,\mathrm{Hom}(A, P) \otimes^{\mathbf{L}}_R M$ has cohomology only in the interval $[-d, 0]$ for any finite R-module M. Since any R-module is a filtered colimit of finite R-modules and since tensor products commute with colimits we conclude. $\square$

0E9R   **Lemma 25.9.** *Let $R \to A$ be a finite type homomorphism of Noetherian rings. Let $\mathfrak{p} \subset R$ be a prime ideal. Assume*

  (1) *$R_\mathfrak{p}$ is Cohen-Macaulay, and*
  (2) *for any minimal prime $\mathfrak{q} \subset A$ we have $\mathrm{trdeg}_{\kappa(R \cap \mathfrak{q})} \kappa(\mathfrak{q}) \leq r$.*

Then

$$H^i(\omega^\bullet_{A/R})_\mathfrak{p} \neq 0 \Rightarrow -r \leq i$$

and $H^{-r}(\omega^\bullet_{A/R})_\mathfrak{p}$ is (S_2) as an $A_\mathfrak{p}$-module.

Proof. We may replace R by $R_\mathfrak{p}$ by Lemma 25.1. Thus we may assume R is a Cohen-Macaulay local ring and we have to show the assertions of the lemma for the A-modules $H^i(\omega^\bullet_{A/R})$.

Let $R^\wedge$ be the completion of R. The map $R \to R^\wedge$ is flat and $R^\wedge$ is Cohen-Macaulay (More on Algebra, Lemma 43.3). Observe that the minimal primes of $A \otimes_R R^\wedge$ lie over minimal primes of A by the flatness of $A \to A \otimes_R R^\wedge$ (and going down for flatness, see Algebra, Lemma 39.19). Thus condition (2) holds for the finite type ring map $R^\wedge \to A \otimes_R R^\wedge$ by Morphisms, Lemma 28.3. Appealing to Lemma 25.1 once again it suffices to prove the lemma for $R^\wedge \to A \otimes_R R^\wedge$. In this way, using Lemma 22.4, we may assume R is a Noetherian local Cohen-Macaulay ring which has a dualizing complex $\omega^\bullet_R$.

Let $\mathfrak{m} \subset A$ be a maximal ideal. It suffices to show that the assertions of the lemma hold for $H^i(\omega^\bullet_{A/R})_\mathfrak{m}$. If $\mathfrak{m}$ does not lie over the maximal ideal of R, then we replace R by a localization to reduce to this case (small detail omitted).

We may assume $\omega^\bullet_R$ is normalized. Setting $d = \dim(R)$ we see that $\omega^\bullet_R = \omega_R[d]$ for some R-module ω_R, see Lemma 20.2. Set $\omega^\bullet_A = \varphi^!(\omega^\bullet_R)$. By Lemma 24.11 we have

$$\omega^\bullet_{A/R} = R\,\mathrm{Hom}_A(\omega_R[d] \otimes^{\mathbf{L}}_R A, \omega^\bullet_A)$$

By the dimension formula we have $\dim(A_{\mathfrak{m}}) \leq d + r$, see Morphisms, Lemma 52.2 and use that $\kappa(\mathfrak{m})$ is finite over the residue field of R by the Hilbert Nullstellensatz. By Lemma 25.6 we see that $(\omega_A^\bullet)_{\mathfrak{m}}$ is a normalized dualizing complex for $A_{\mathfrak{m}}$. Hence $H^i((\omega_A^\bullet)_{\mathfrak{m}})$ is nonzero only for $-d - r \leq i \leq 0$, see Lemma 16.5. Since $\omega_R[d] \otimes_R^{\mathbf{L}} A$ lives in degrees $\leq -d$ we conclude the vanishing holds. Finally, we also see that

$$H^{-r}(\omega_{A/R}^\bullet)_{\mathfrak{m}} = \operatorname{Hom}_A(\omega_R \otimes_R A, H^{-d-r}(\omega_A^\bullet))_{\mathfrak{m}}$$

Since $H^{-d-r}(\omega_A^\bullet)_{\mathfrak{m}}$ is (S_2) by Lemma 17.5 we find that the final statement is true by More on Algebra, Lemma 23.11. $\square$

26. More on dualizing complexes

0E49   Some lemmas which don't fit anywhere else very well.

0E4A   **Lemma 26.1.** *Let $A \to B$ be a faithfully flat map of Noetherian rings. If $K \in D(A)$ and $K \otimes_A^{\mathbf{L}} B$ is a dualizing complex for B, then K is a dualizing complex for A.*

Proof. Since $A \to B$ is flat we have $H^i(K) \otimes_A B = H^i(K \otimes_A^{\mathbf{L}} B)$. Since $K \otimes_A^{\mathbf{L}} B$ is in $D_{Coh}^b(B)$ we first find that K is in $D^b(A)$ and then we see that $H^i(K)$ is a finite A-module by Algebra, Lemma 83.2. Let M be a finite A-module. Then

$$R\operatorname{Hom}_A(M, K) \otimes_A B = R\operatorname{Hom}_B(M \otimes_A B, K \otimes_A^{\mathbf{L}} B)$$

by More on Algebra, Lemma 99.2. Since $K \otimes_A^{\mathbf{L}} B$ has finite injective dimension, say injective-amplitude in $[a, b]$, we see that the right hand side has vanishing cohomology in degrees $> b$. Since $A \to B$ is faithfully flat, we find that $R\operatorname{Hom}_A(M, K)$ has vanishing cohomology in degrees $> b$. Thus K has finite injective dimension by More on Algebra, Lemma 69.2. To finish the proof we have to show that the map $A \to R\operatorname{Hom}_A(K, K)$ is an isomorphism. For this we again use More on Algebra, Lemma 99.2 and the fact that $B \to R\operatorname{Hom}_B(K \otimes_A^{\mathbf{L}} B, K \otimes_A^{\mathbf{L}} B)$ is an isomorphism. $\square$

0E4D   **Lemma 26.2.** *Let $\varphi : A \to B$ be a homomorphism of Noetherian rings. Assume*

(1) *$A \to B$ is syntomic and induces a surjective map on spectra, or*
(2) *$A \to B$ is a faithfully flat local complete intersection, or*
(3) *$A \to B$ is faithfully flat of finite type with Gorenstein fibres.*

Then $K \in D(A)$ is a dualizing complex for A if and only if $K \otimes_A^{\mathbf{L}} B$ is a dualizing complex for B.

Proof. Observe that $A \to B$ satisfies (1) if and only if $A \to B$ satisfies (2) by More on Algebra, Lemma 33.5. Observe that in both (2) and (3) the relative dualzing complex $\varphi^!(A) = \omega_{B/A}^\bullet$ is an invertible object of $D(B)$, see Lemmas 25.4 and 25.5. Moreover we have $\varphi^!(K) = K \otimes_A^{\mathbf{L}} \omega_{B/A}^\bullet$ in both cases, see Lemma 24.10 for case (3). Thus $\varphi^!(K)$ is the same as $K \otimes_A^{\mathbf{L}} B$ up to tensoring with an invertible object of $D(B)$. Hence $\varphi^!(K)$ is a dualizing complex for B if and only if $K \otimes_A^{\mathbf{L}} B$ is (as being a dualizing complex is local and invariant under shifts). Thus we see that if K is dualizing for A, then $K \otimes_A^{\mathbf{L}} B$ is dualizing for B by Lemma 24.3. To descend the property, see Lemma 26.1. $\square$

0E4E   **Lemma 26.3.** *Let $(A, \mathfrak{m}, \kappa) \to (B, \mathfrak{n}, l)$ be a flat local homorphism of Noetherian rings such that $\mathfrak{n} = \mathfrak{m}B$. If E is the injective hull of κ, then $E \otimes_A B$ is the injective hull of l.*

Proof. Write $E = \bigcup E_n$ as in Lemma 7.3. It suffices to show that $E_n \otimes_{A/\mathfrak{m}^n} B/\mathfrak{n}^n$ is the injective hull of l over $B/\mathfrak{n}$. This reduces us to the case where A and B are Artinian local. Observe that $\text{length}_A(A) = \text{length}_B(B)$ and $\text{length}_A(E) = \text{length}_B(E \otimes_A B)$ by Algebra, Lemma 52.13. By Lemma 6.1 we have $\text{length}_A(E) = \text{length}_A(A)$ and $\text{length}_B(E') = \text{length}_B(B)$ where E' is the injective hull of l over B. We conclude $\text{length}_B(E') = \text{length}_B(E \otimes_A B)$. Observe that

$$\dim_l((E \otimes_A B)[\mathfrak{n}]) = \dim_l(E[\mathfrak{m}] \otimes_A B) = \dim_\kappa(E[\mathfrak{m}]) = 1$$

where we have used flatness of $A \to B$ and $\mathfrak{n} = \mathfrak{m}B$. Thus there is an injective B-module map $E \otimes_A B \to E'$ by Lemma 7.2. By equality of lengths shown above this is an isomorphism. $\square$

0E4F **Lemma 26.4.** *Let $\varphi : A \to B$ be a flat homorphism of Noetherian rings such that for all primes $\mathfrak{q} \subset B$ we have $\mathfrak{p}B_\mathfrak{q} = \mathfrak{q}B_\mathfrak{q}$ where $\mathfrak{p} = \varphi^{-1}(\mathfrak{q})$, for example if φ is étale. If I is an injective A-module, then $I \otimes_A B$ is an injective B-module.*

Proof. Étale maps satisfy the assumption by Algebra, Lemma 143.5. By Lemma 3.7 and Proposition 5.9 we may assume I is the injective hull of $\kappa(\mathfrak{p})$ for some prime $\mathfrak{p} \subset A$. Then I is a module over $A_\mathfrak{p}$. It suffices to prove $I \otimes_A B = I \otimes_{A_\mathfrak{p}} B_\mathfrak{p}$ is injective as a $B_\mathfrak{p}$-module, see Lemma 3.2. Thus we may assume $(A, \mathfrak{m}, \kappa)$ is local Noetherian and $I = E$ is the injective hull of the residue field κ. Our assumption implies that the Noetherian ring $B/\mathfrak{m}B$ is a product of fields (details omitted). Thus there are finitely many prime ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ in B lying over $\mathfrak{m}$ and they are all maximal ideals. Write $E = \bigcup E_n$ as in Lemma 7.3. Then $E \otimes_A B = \bigcup E_n \otimes_A B$ and $E_n \otimes_A B$ is a finite B-module with support $\{\mathfrak{m}_1, \ldots, \mathfrak{m}_n\}$ hence decomposes as a product over the localizations at $\mathfrak{m}_i$. Thus $E \otimes_A B = \prod (E \otimes_A B)_{\mathfrak{m}_i}$. Since $(E \otimes_A B)_{\mathfrak{m}_i} = E \otimes_A B_{\mathfrak{m}_i}$ is the injective hull of the residue field of $\mathfrak{m}_i$ by Lemma 26.3 we conclude. $\square$

27. Relative dualizing complexes

0E2B For a finite type ring map $\varphi : R \to A$ of Noetherian rings we have the relative dualizing complex $\omega^\bullet_{A/R} = \varphi^!(R)$ considered in Section 25. If R is not Noetherian, a similarly constructed complex will in general not have good properties. In this section, we give a definition of a relative dualizing complex for a flat and finitely presented ring maps $R \to A$ of non-Noetherian rings. The definition is chosen to globalize to flat and finitely presented morphisms of schemes, see Duality for Schemes, Section 28. We will show that relative dualizing complexes exist (when the definition applies), are unique up to (noncanonical) isomorphism, and that in the Noetherian case we recover the complex of Section 25.

The Noetherian reader may safely skip this section!

0E2C **Definition 27.1.** Let $R \to A$ be a flat ring map of finite presentation. A *relative dualizing complex* is an object $K \in D(A)$ such that
(1) K is R-perfect (More on Algebra, Definition 83.1), and
(2) $R\text{Hom}_{A \otimes_R A}(A, K \otimes^\mathbf{L}_A (A \otimes_R A))$ is isomorphic to A.

To understand this definition you may have to read and understand some of the following lemmas. Lemmas 27.3 and 27.2 show this definition does not clash with the definition in Section 25.

0E2D **Lemma 27.2.** *Let $R \to A$ be a flat ring map of finite presentation. Any two relative dualizing complexes for $R \to A$ are isomorphic.*

Proof. Let K and L be two relative dualizing complexes for $R \to A$. Denote $K_1 = K \otimes_A^{\mathbf{L}} (A \otimes_R A)$ and $L_2 = (A \otimes_R A) \otimes_A^{\mathbf{L}} L$ the derived base changes via the first and second coprojections $A \to A \otimes_R A$. By symmetry the assumption on L_2 implies that $R \operatorname{Hom}_{A \otimes_R A}(A, L_2)$ is isomorphic to A. By More on Algebra, Lemma 98.3 part (3) applied twice we have
$$A \otimes_{A \otimes_R A}^{\mathbf{L}} L_2 \cong R \operatorname{Hom}_{A \otimes_R A}(A, K_1 \otimes_{A \otimes_R A}^{\mathbf{L}} L_2) \cong A \otimes_{A \otimes_R A}^{\mathbf{L}} K_1$$
Applying the restriction functor $D(A \otimes_R A) \to D(A)$ for either coprojection we obtain the desired result. $\qquad\square$

0E2E **Lemma 27.3.** *Let $\varphi : R \to A$ be a flat finite type ring map of Noetherian rings. Then the relative dualizing complex $\omega_{A/R}^{\bullet} = \varphi^!(R)$ of Section 25 is a relative dualizing complex in the sense of Definition 27.1.*

Proof. From Lemma 25.2 we see that $\varphi^!(R)$ is R-perfect. Denote $\delta : A \otimes_R A \to A$ the multiplication map and $p_1, p_2 : A \to A \otimes_R A$ the coprojections. Then
$$\varphi^!(R) \otimes_A^{\mathbf{L}} (A \otimes_R A) = \varphi^!(R) \otimes_{A, p_1}^{\mathbf{L}} (A \otimes_R A) = p_2^!(A)$$
by Lemma 24.4. Recall that $R \operatorname{Hom}_{A \otimes_R A}(A, \varphi^!(R) \otimes_A^{\mathbf{L}} (A \otimes_R A))$ is the image of $\delta^!(\varphi^!(R) \otimes_A^{\mathbf{L}} (A \otimes_R A))$ under the restriction map $\delta_* : D(A) \to D(A \otimes_R A)$. Use the definition of $\delta^!$ from Section 24 and Lemma 13.3. Since $\delta^!(p_2^!(A)) \cong A$ by Lemma 24.7 we conclude. $\qquad\square$

0E2F **Lemma 27.4.** *Let $R \to A$ be a flat ring map of finite presentation. Then*
1. *there exists a relative dualizing complex K in $D(A)$, and*
2. *for any ring map $R \to R'$ setting $A' = A \otimes_R R'$ and $K' = K \otimes_A^{\mathbf{L}} A'$, then K' is a relative dualizing complex for $R' \to A'$.*

Moreover, if
$$\xi : A \longrightarrow K \otimes_A^{\mathbf{L}} (A \otimes_R A)$$
is a generator for the cyclic module $\operatorname{Hom}_{D(A \otimes_R A)}(A, K \otimes_A^{\mathbf{L}} (A \otimes_R A))$ then in (2) the derived base change of ξ by $A \otimes_R A \to A' \otimes_{R'} A'$ is a generator for the cyclic module $\operatorname{Hom}_{D(A' \otimes_{R'} A')}(A', K' \otimes_{A'}^{\mathbf{L}} (A' \otimes_{R'} A'))$

Proof. We first reduce to the Noetherian case. By Algebra, Lemma 168.1 there exists a finite type $\mathbf{Z}$ subalgebra $R_0 \subset R$ and a flat finite type ring map $R_0 \to A_0$ such that $A = A_0 \otimes_{R_0} R$. By Lemma 27.3 there exists a relative dualizing complex $K_0 \in D(A_0)$. Thus if we show (2) for K_0, then we find that $K_0 \otimes_{A_0}^{\mathbf{L}} A$ is a dualizing complex for $R \to A$ and that it also satisfies (2) by transitivity of derived base change. The uniqueness of relative dualizing complexes (Lemma 27.2) then shows that this holds for any relative dualizing complex.

Assume R Noetherian and let K be a relative dualizing complex for $R \to A$. Given a ring map $R \to R'$ set $A' = A \otimes_R R'$ and $K' = K \otimes_A^{\mathbf{L}} A'$. To finish the proof we have to show that K' is a relative dualizing complex for $R' \to A'$. By More on Algebra, Lemma 83.5 we see that K' is R'-perfect in all cases. By Lemmas 25.1 and 27.3 if R' is Noetherian, then K' is a relative dualizing complex for $R' \to A'$ (in either sense). Transitivity of derived tensor product shows that $K \otimes_A^{\mathbf{L}} (A \otimes_R A) \otimes_{A \otimes_R A}^{\mathbf{L}} (A' \otimes_{R'} A') = K' \otimes_{A'}^{\mathbf{L}} (A' \otimes_{R'} A')$. Flatness of $R \to$

A guarantees that $A \otimes^{\mathbf{L}}_{A \otimes_R A} (A' \otimes_{R'} A') = A'$; namely $A \otimes_R A$ and R' are tor independent over R so we can apply More on Algebra, Lemma 61.2. Finally, A is pseudo-coherent as an $A \otimes_R A$-module by More on Algebra, Lemma 82.8. Thus we have checked all the assumptions of More on Algebra, Lemma 83.6. We find there exists a bounded below complex $E^\bullet$ of R-flat finitely presented $A \otimes_R A$-modules such that $E^\bullet \otimes_R R'$ represents $R\operatorname{Hom}_{A' \otimes_{R'} A'}(A', K' \otimes^{\mathbf{L}}_{A'} (A' \otimes_{R'} A'))$ and these identifications are compatible with derived base change. Let $n \in \mathbf{Z}$, $n \neq 0$. Define Q^n by the sequence

$$E^{n-1} \to E^n \to Q^n \to 0$$

Since $\kappa(\mathfrak{p})$ is a Noetherian ring, we know that $H^n(E^\bullet \otimes_R \kappa(\mathfrak{p})) = 0$, see remarks above. Chasing diagrams this means that

$$Q^n \otimes_R \kappa(\mathfrak{p}) \to E^{n+1} \otimes_R \kappa(\mathfrak{p})$$

is injective. Hence for a prime $\mathfrak{q}$ of $A \otimes_R A$ lying over $\mathfrak{p}$ we have $Q^n_{\mathfrak{q}}$ is $R_{\mathfrak{p}}$-flat and $Q^n_{\mathfrak{p}} \to E^{n+1}_{\mathfrak{q}}$ is $R_{\mathfrak{p}}$-universally injective, see Algebra, Lemma 99.1. Since this holds for all primes, we conclude that Q^n is R-flat and $Q^n \to E^{n+1}$ is R-universally injective. In particular $H^n(E^\bullet \otimes_R R') = 0$ for any ring map $R \to R'$. Let $Z^0 = \operatorname{Ker}(E^0 \to E^1)$. Since there is an exact sequence $0 \to Z^0 \to E^0 \to E^1 \to Q^1 \to 0$ we see that Z^0 is R-flat and that $Z^0 \otimes_R R' = \operatorname{Ker}(E^0 \otimes_R R' \to E^1 \otimes_R R')$ for all $R \to R'$. Then the short exact sequence $0 \to Q^{-1} \to Z^0 \to H^0(E^\bullet) \to 0$ shows that

$$H^0(E^\bullet \otimes_R R') = H^0(E^\bullet) \otimes_R R' = A \otimes_R R' = A'$$

as desired. This equality furthermore gives the final assertion of the lemma. $\square$

0E2G **Lemma 27.5.** *Let $R \to A$ be a flat ring map of finite presentation. Let K be a relative dualizing complex. Then $A \to R\operatorname{Hom}_A(K, K)$ is an isomorphism.*

Proof. By Algebra, Lemma 168.1 there exists a finite type $\mathbf{Z}$ subalgebra $R_0 \subset R$ and a flat finite type ring map $R_0 \to A_0$ such that $A = A_0 \otimes_{R_0} R$. By Lemmas 27.2, 27.3, and 27.4 there exists a relative dualizing complex $K_0 \in D(A_0)$ and its derived base change is K. This reduces us to the situation discussed in the next paragraph.

Assume R Noetherian and let K be a relative dualizing complex for $R \to A$. Given a ring map $R \to R'$ set $A' = A \otimes_R R'$ and $K' = K \otimes^{\mathbf{L}}_A A'$. To finish the proof we show $R\operatorname{Hom}_{A'}(K', K') = A'$. By Lemma 25.2 we know this is true whenever R' is Noetherian. Since a general R' is a filtered colimit of Noetherian R-algebras, we find the result holds by More on Algebra, Lemma 83.7. $\square$

0E2H **Lemma 27.6.** *Let $R \to A \to B$ be a ring maps which are flat and of finite presentation. Let $K_{A/R}$ and $K_{B/A}$ be relative dualizing complexes for $R \to A$ and $A \to B$. Then $K = K_{A/R} \otimes^{\mathbf{L}}_A K_{B/A}$ is a relative dualizing complex for $R \to B$.*

Proof. We will use reduction to the Noetherian case. Namely, by Algebra, Lemma 168.1 there exists a finite type $\mathbf{Z}$ subalgebra $R_0 \subset R$ and a flat finite type ring map $R_0 \to A_0$ such that $A = A_0 \otimes_{R_0} R$. After increasing R_0 and correspondingly replacing A_0 we may assume there is a flat finite type ring map $A_0 \to B_0$ such that $B = B_0 \otimes_{R_0} R$ (use the same lemma). If we prove the lemma for $R_0 \to A_0 \to B_0$, then the lemma follows by Lemmas 27.2, 27.3, and 27.4. This reduces us to the situation discussed in the next paragraph.

Assume R is Noetherian and denote $\varphi : R \to A$ and $\psi : A \to B$ the given ring maps. Then $K_{A/R} \cong \varphi^!(R)$ and $K_{B/A} \cong \psi^!(A)$, see references given above. Then

$$K = K_{A/R} \otimes_A^{\mathbf{L}} K_{B/A} \cong \varphi^!(R) \otimes_A^{\mathbf{L}} \psi^!(A) \cong \psi^!(\varphi^!(R)) \cong (\psi \circ \varphi)^!(R)$$

by Lemmas 24.10 and 24.7. Thus K is a relative dualizing complex for $R \to B$. $\square$

28. Other chapters

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