DEFORMATION THEORY

08KW

Contents

1. Introduction

08KX   The goal of this chapter is to give a (relatively) gentle introduction to deformation theory of modules, morphisms, etc. In this chapter we deal with those results that can be proven using the naive cotangent complex. In the chapter on the cotangent complex we will extend these results a little bit. The advanced reader may wish to consult the treatise by Illusie on this subject, see [Ill72].

2. Deformations of rings and the naive cotangent complex

08S3   In this section we use the naive cotangent complex to do a little bit of deformation theory. We start with a surjective ring map $A' \to A$ whose kernel is an ideal I of square zero. Moreover we assume given a ring map $A \to B$, a B-module N, and an A-module map $c : I \to N$. In this section we ask ourselves whether we can find

the question mark fitting into the following diagram

08S4 (2.0.1)

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & N & \longrightarrow & ? & \longrightarrow & B & \longrightarrow & 0 \\
& & \big\uparrow{\scriptstyle c} & & \big\uparrow & & \big\uparrow & & \\
0 & \longrightarrow & I & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & 0
\end{array}
$$

and moreover how unique the solution is (if it exists). More precisely, we look for a surjection of A'-algebras $B' \to B$ whose kernel is an ideal of square zero and is identified with N such that $A' \to B'$ induces the given map c. We will say B' is a *solution* to (2.0.1).

08S5 **Lemma 2.1.** *Given a commutative diagram*

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & N_2 & \longrightarrow & B'_2 & \longrightarrow & B_2 & \longrightarrow & 0 \\
& & \uparrow\uparrow{\scriptstyle c_2} & & \uparrow & & \uparrow\uparrow & & \\
0 & \longrightarrow & I_2 & \longrightarrow & A'_2 & \longrightarrow & A_2 & \longrightarrow & 0 \\
& & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & N_1 & \longrightarrow & B'_1 & \longrightarrow & B_1 & \longrightarrow & 0 \\
& & \uparrow{\scriptstyle c_1} & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & I_1 & \longrightarrow & A'_1 & \longrightarrow & A_1 & \longrightarrow & 0
\end{array}
$$

with front and back solutions to (2.0.1) we have

(1) *There exist a canonical element in $\mathrm{Ext}^1_{B_1}(NL_{B_1/A_1}, N_2)$ whose vanishing is a necessary and sufficient condition for the existence of a ring map $B'_1 \to B'_2$ fitting into the diagram.*
(2) *If there exists a map $B'_1 \to B'_2$ fitting into the diagram the set of all such maps is a principal homogeneous space under $\mathrm{Hom}_{B_1}(\Omega_{B_1/A_1}, N_2)$.*

Proof. Let $E = B_1$ viewed as a set. Consider the surjection $A_1[E] \to B_1$ with kernel J used to define the naive cotangent complex by the formula

$$
NL_{B_1/A_1} = (J/J^2 \to \Omega_{A_1[E]/A_1} \otimes_{A_1[E]} B_1)
$$

in Algebra, Section 134. Since $\Omega_{A_1[E]/A_1} \otimes B_1$ is a free B_1-module we have

$$
\mathrm{Ext}^1_{B_1}(NL_{B_1/A_1}, N_2) = \frac{\mathrm{Hom}_{B_1}(J/J^2, N_2)}{\mathrm{Hom}_{B_1}(\Omega_{A_1[E]/A_1} \otimes B_1, N_2)}
$$

We will construct an obstruction in the module on the right. Let $J' = \mathrm{Ker}(A'_1[E] \to B_1)$. Note that there is a surjection $J' \to J$ whose kernel is $I_1 A_1[E]$. For every $e \in E$ denote $x_e \in A_1[E]$ the corresponding variable. Choose a lift $y_e \in B'_1$ of the image of x_e in B_1 and a lift $z_e \in B'_2$ of the image of x_e in B_2. These choices determine A'_1-algebra maps

$$
A'_1[E] \to B'_1 \quad \text{and} \quad A'_1[E] \to B'_2
$$

The first of these gives a map $J' \to N_1$, $f' \mapsto f'(y_e)$ and the second gives a map $J' \to N_2$, $f' \mapsto f'(z_e)$. A calculation shows that these maps annihilate $(J')^2$. Because the left square of the diagram (involving c_1 and c_2) commutes we see that these maps agree on $I_1 A_1[E]$ as maps into N_2. Observe that B'_1 is the pushout

of $J' \to A_1'[B_1]$ and $J' \to N_1$. Thus, if the maps $J' \to N_1 \to N_2$ and $J' \to N_2$ agree, then we obtain a map $B_1' \to B_2'$ fitting into the diagram. Thus we let the obstruction be the class of the map

$$J/J^2 \to N_2, \quad f \mapsto f'(z_e) - \nu(f'(y_e))$$

where $\nu : N_1 \to N_2$ is the given map and where $f' \in J'$ is a lift of f. This is well defined by our remarks above. Note that we have the freedom to modify our choices of z_e into $z_e + \delta_{2,e}$ and y_e into $y_e + \delta_{1,e}$ for some $\delta_{i,e} \in N_i$. This will modify the map above into

$$f \mapsto f'(z_e + \delta_{2,e}) - \nu(f'(y_e + \delta_{1,e})) = f'(z_e) - \nu(f'(z_e)) + \sum(\delta_{2,e} - \nu(\delta_{1,e}))\frac{\partial f}{\partial x_e}$$

This means exactly that we are modifying the map $J/J^2 \to N_2$ by the composition $J/J^2 \to \Omega_{A_1[E]/A_1} \otimes B_1 \to N_2$ where the second map sends dx_e to $\delta_{2,e} - \nu(\delta_{1,e})$. Thus our obstruction is well defined and is zero if and only if a lift exists.

Part (2) comes from the observation that given two maps $\varphi, \psi : B_1' \to B_2'$ fitting into the diagram, then $\varphi - \psi$ factors through a map $D : B_1 \to N_2$ which is an A_1-derivation:

$$\begin{aligned} D(fg) &= \varphi(f'g') - \psi(f'g') \\ &= \varphi(f')\varphi(g') - \psi(f')\psi(g') \\ &= (\varphi(f') - \psi(f'))\varphi(g') + \psi(f')(\varphi(g') - \psi(g')) \\ &= gD(f) + fD(g) \end{aligned}$$

Thus D corresponds to a unique B_1-linear map $\Omega_{B_1/A_1} \to N_2$. Conversely, given such a linear map we get a derivation D and given a ring map $\psi : B_1' \to B_2'$ fitting into the diagram the map $\psi + D$ is another ring map fitting into the diagram. $\square$

08S7 **Lemma 2.2.** *If there exists a solution to (2.0.1), then the set of isomorphism classes of solutions is principal homogeneous under $\mathrm{Ext}_B^1(NL_{B/A}, N)$.*

Proof. We observe right away that given two solutions B_1' and B_2' to (2.0.1) we obtain by Lemma 2.1 an obstruction element $o(B_1', B_2') \in \mathrm{Ext}_B^1(NL_{B/A}, N)$ to the existence of a map $B_1' \to B_2'$. Clearly, this element is the obstruction to the existence of an isomorphism, hence separates the isomorphism classes. To finish the proof it therefore suffices to show that given a solution B' and an element $\xi \in \mathrm{Ext}_B^1(NL_{B/A}, N)$ we can find a second solution B_ξ' such that $o(B', B_\xi') = \xi$.

Let $E = B$ viewed as a set. Consider the surjection $A[E] \to B$ with kernel J used to define the naive cotangent complex by the formula

$$NL_{B/A} = (J/J^2 \to \Omega_{A[E]/A} \otimes_{A[E]} B)$$

in Algebra, Section 134. Since $\Omega_{A[E]/A} \otimes B$ is a free B-module we have

$$\mathrm{Ext}_B^1(NL_{B/A}, N) = \frac{\mathrm{Hom}_B(J/J^2, N)}{\mathrm{Hom}_B(\Omega_{A[E]/A} \otimes B, N)}$$

Thus we may represent ξ as the class of a morphism $\delta : J/J^2 \to N$.

For every $e \in E$ denote $x_e \in A[E]$ the corresponding variable. Choose a lift $y_e \in B'$ of the image of x_e in B. These choices determine an A'-algebra map $\varphi : A'[E] \to B'$.

Let $J' = \mathrm{Ker}(A'[E] \to B)$. Observe that φ induces a map $\varphi|_{J'} : J' \to N$ and that B' is the pushout, as in the following diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & N & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & 0 \\
& & \uparrow{\scriptstyle \varphi|_{J'}} & & \uparrow & & \uparrow{\scriptstyle =} & & \\
0 & \longrightarrow & J' & \longrightarrow & A'[E] & \longrightarrow & B & \longrightarrow & 0
\end{array}
$$

Let $\psi : J' \to N$ be the sum of the map $\varphi|_{J'}$ and the composition

$$J' \to J'/(J')^2 \to J/J^2 \xrightarrow{\delta} N.$$

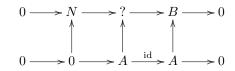Then the pushout along ψ is an other ring extension B'_ξ fitting into a diagram as above. A calculation shows that $o(B', B'_\xi) = \xi$ as desired. $\square$

0GPT **Lemma 2.3.** *Let A be a ring. Let B be an A-algebra. Let N be a B-module. The set of isomorphism classes of extensions of A-algebras*

$$0 \to N \to B' \to B \to 0$$

where N is an ideal of square zero is canonically bijective to $\mathrm{Ext}^1_B(NL_{B/A}, N)$.

Proof. To prove this we apply the previous results to the case where (2.0.1) is given by the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & N & \longrightarrow & ? & \longrightarrow & B & \longrightarrow & 0 \\
& & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & A & \xrightarrow{\ \mathrm{id}\ } & A & \longrightarrow & 0
\end{array}
$$

Thus our lemma follows from Lemma 2.2 and the fact that there exists a solution, namely $N \oplus B$. (See remark below for a direct construction of the bijection.) $\square$

0GPU **Remark 2.4.** Let $A \to B$ and N be as in Lemma 2.3. Let $\alpha : P \to B$ be a presentation of B over A, see Algebra, Section 134. With $J = \mathrm{Ker}(\alpha)$ the naive cotangent complex $NL(\alpha)$ associated to α is the complex $J/J^2 \to \Omega_{P/A} \otimes_P B$. We have

$$\mathrm{Ext}^1_B(NL(\alpha), N) = \mathrm{Coker}\left(\mathrm{Hom}_B(\Omega_{P/A} \otimes_P B, N) \to \mathrm{Hom}_B(J/J^2, N)\right)$$

because $\Omega_{P/A}$ is a free module. Consider a extension $0 \to N \to B' \to B \to 0$ as in the lemma. Since P is a polynomial algebra over A we can lift α to an A-algebra map $\alpha' : P' \to B'$. Then $\alpha'|_J : J \to N$ factors as $J \to J/J^2 \to N$ as N has square zero in B'. The lemma sends our extension to the class of this map $J/J^2 \to N$ in the displayed cokernel.

0GPV **Lemma 2.5.** *Given ring maps $A \to B \to C$, a B-module M, a C-module N, a B-linear map $c : M \to N$, and extensions of A-algebras with square zero kernels*

    (a) $0 \to M \to B' \to B \to 0$ *corresponding to $\xi \in \mathrm{Ext}^1_B(NL_{B/A}, M)$, and*
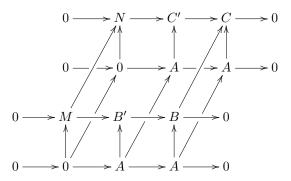   (b) $0 \to N \to C' \to C \to 0$ *corresponding to $\zeta \in \mathrm{Ext}^1_C(NL_{C/A}, N)$.*

See Lemma 2.3. Then there is an A-algebra map $B' \to C'$ compatible with $B \to C$ and c if and only if ξ and ζ map to the same element of $\mathrm{Ext}^1_B(NL_{B/A}, N)$.

Proof. The stament makes sense as we have the maps

$$\mathrm{Ext}^1_B(NL_{B/A}, M) \to \mathrm{Ext}^1_B(NL_{B/A}, N)$$

using the map $M \to N$ and

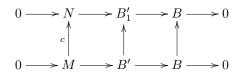$$\mathrm{Ext}^1_C(NL_{C/A}, N) \to \mathrm{Ext}^1_B(NL_{C/A}, N) \to \mathrm{Ext}^1_B(NL_{B/A}, N)$$

where the first arrows uses the restriction map $D(C) \to D(B)$ and the second arrow uses the canonical map of complexes $NL_{B/A} \to NL_{C/A}$. The statement of the lemma can be deduced from Lemma 2.1 applied to the diagram
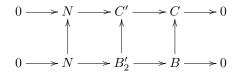

and a compatibility between the constructions in the proofs of Lemmas 2.3 and 2.1 whose statement and proof we omit. (See remark below for a direct argument.) $\square$

0GPW **Remark 2.6.** Let $A \to B \to C$, M, N, $c : M \to N$, $0 \to M \to B' \to B \to 0$, $\xi \in \mathrm{Ext}^1_B(NL_{B/A}, M)$, $0 \to N \to C' \to C \to 0$, and $\zeta \in \mathrm{Ext}^1_C(NL_{C/A}, N)$ be as in Lemma 2.5. Using pushout along $c : M \to N$ we can construct an extension

by setting $B'_1 = (N \times B')/M$ where M is antidiagonally embedded. Using pullback along $B \to C$ we can construct an extension

by setting $B'_2 = C' \times_C B$ (fibre product of rings). A simple diagram chase tells us that there exists an A-algebra map $B' \to C'$ compatible with $B \to C$ and c if and only if B'_1 is isomorphic to B'_2 as A-algebra extensions of B by N. Thus to see Lemma 2.5 is true, it suffices to show that B'_1 corresponds via the bijection of Lemma 2.3 to the image of ξ by the map $\mathrm{Ext}^1_B(NL_{B/A}, M) \to \mathrm{Ext}^1_B(NL_{B/A}, N)$ and that B'_2 correspond to the image of ζ by the map $\mathrm{Ext}^1_C(NL_{C/A}, N) \to \mathrm{Ext}^1_B(NL_{B/A}, N)$. The first of these two statements is immediate from the construction of the class in

Remark 2.4. For the second, choose a commutative diagram

$$
\begin{array}{ccc}
Q & \xrightarrow{\ \beta\ } & C \\
{\scriptstyle\varphi}\big\uparrow & & \big\uparrow \\
P & \xrightarrow{\ \alpha\ } & B
\end{array}
$$

of A-algebras, such that α is a presentation of B over A and β is a presentation of C over A. See Remark 2.4 and references therein. Set $J = \mathrm{Ker}(\alpha)$ and $K = \mathrm{Ker}(\beta)$. The map φ induces a map of complexes $NL(\alpha) \to NL(\beta)$ and in particular $\bar{\varphi} : J/J^2 \to K/K^2$. Choose A-algebra homomorphism $\beta' : Q \to C'$ which is a lift of β. Then $\alpha' = (\beta' \circ \varphi, \alpha) : P \to B_2' = C' \times_C B$ is a lift of α. With these choices the composition of the map $K/K^2 \to N$ induced by β' and the map $\bar{\varphi} : J/J^2 \to K/K^2$ is the restriction of α' to J/J^2. Unwinding the constructions of our classes in Remark 2.4 this indeed shows that B_2' correspond to the image of ζ by the map $\mathrm{Ext}^1_C(NL_{C/A}, N) \to \mathrm{Ext}^1_B(NL_{B/A}, N)$.

0GPX **Lemma 2.7.** *Let $0 \to I \to A' \to A \to 0$, $A \to B$, and $c : I \to N$ be as in (2.0.1). Denote $\xi \in \mathrm{Ext}^1_A(NL_{A/A'}, I)$ the element corresponding to the extension A' of A by I via Lemma 2.3. The set of isomorphism classes of solutions is canonically bijective to the fibre of*

$$
\mathrm{Ext}^1_B(NL_{B/A'}, N) \to \mathrm{Ext}^1_A(NL_{A'/A}, N)
$$

over the image of ξ.

Proof. By Lemma 2.3 applied to $A' \to B$ and the B-module N we see that elements ζ of $\mathrm{Ext}^1_B(NL_{B/A'}, N)$ parametrize extensions $0 \to N \to B' \to B \to 0$ of A'-algebras. By Lemma 2.5 applied to $A' \to A \to B$ and $c : I \to N$ we see that there is an A'-algebra map $A' \to B'$ compatible with c and $A \to B$ if and only if ζ maps to ξ. Of course this is the same thing as saying B' is a solution of (2.0.1). $\square$

0GPY **Remark 2.8.** Observe that in the situation of Lemma 2.7 we have

$$
\mathrm{Ext}^1_A(NL_{A'/A}, N) = \mathrm{Ext}^1_B(NL_{A'/A} \otimes^{\mathbf{L}}_A B, N) = \mathrm{Ext}^1_B(NL_{A'/A} \otimes_A B, N)
$$

The first equality by More on Algebra, Lemma 60.3 and the second by More on Algebra, Lemma 85.1. We have maps of complexes

$$
NL_{A'/A} \otimes_A B \to NL_{B/A'} \to NL_{B/A}
$$

which is close to being a distinguished triangle, see Algebra, Lemma 134.4. If it were a distinguished triangle we would conclude that the image of ξ in $\mathrm{Ext}^2_B(NL_{B/A}, N)$ would be the obstruction to the existence of a solution to (2.0.1).

If our ring map $A \to B$ is a local complete intersection, then there is a solutuion. This is a kind of lifting result; observe that for syntomic ring maps we have proved a rather strong lifting result in Smoothing Ring Maps, Proposition 3.2.

08S6 **Lemma 2.9.** *If $A \to B$ is a local complete intersection ring map, then there exists a solution to (2.0.1).*

First proof. Write $B = A[x_1, \ldots, x_n]/J$. By More on Algebra, Definition 33.2 the ideal J is Koszul-regular. This implies J is H_1-regular and quasi-regular, see More on Algebra, Section 32. Let $J' \subset A'[x_1, \ldots, x_n]$ be the inverse image of J. Denote $I[x_1, \ldots, x_n]$ the kernel of $A'[x_1, \ldots, x_n] \to A[x_1, \ldots, x_n]$. By More on Algebra,

Lemma 32.5 we have $I[x_1, \ldots, x_n] \cap (J')^2 = J'I[x_1, \ldots, x_n] = JI[x_1, \ldots, x_n]$. Hence we obtain a short exact sequence

$$0 \to I \otimes_A B \to J'/(J')^2 \to J/J^2 \to 0$$

Since J/J^2 is projective (More on Algebra, Lemma 32.3) we can choose a splitting of this sequence

$$J'/(J')^2 = I \otimes_A B \oplus J/J^2$$

Let $(J')^2 \subset J'' \subset J'$ be the elements which map to the second summand in the decomposition above. Then

$$0 \to I \otimes_A B \to A'[x_1, \ldots, x_n]/J'' \to B \to 0$$

is a solution to (2.0.1) with $N = I \otimes_A B$. The general case is obtained by doing a pushout along the given map $I \otimes_A B \to N$. $\square$

Second proof. Please read Remark 2.8 before reading this proof. By More on Algebra, Lemma 33.6 the maps $NL_{A'/A} \otimes_A B \to NL_{B/A'} \to NL_{B/A}$ do form a distinguished triangle in $D(B)$. Hence it suffices to show that $\operatorname{Ext}^2_{B/A}(NL_{B/A}, N)$ vanishes. By More on Algebra, Lemma 85.4 the complex $NL_{B/A}$ is perfect of tor-amplitude in $[-1, 0]$. This implies our Ext^2 vanishes for example by More on Algebra, Lemma 76.1 part (1). $\square$

3. Thickenings of ringed spaces

08KY In the following few sections we will use the following notions:
  (1) A sheaf of ideals $\mathcal{I} \subset \mathcal{O}_{X'}$ on a ringed space $(X', \mathcal{O}_{X'})$ is *locally nilpotent* if any local section of $\mathcal{I}$ is locally nilpotent. Compare with Algebra, Item 29.
  (2) A *thickening* of ringed spaces is a morphism $i : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ of ringed spaces such that
    (a) i induces a homeomorphism $X \to X'$,
    (b) the map $i^\sharp : \mathcal{O}_{X'} \to i_* \mathcal{O}_X$ is surjective, and
    (c) the kernel of $i^\sharp$ is a locally nilpotent sheaf of ideals.
  (3) A *first order thickening* of ringed spaces is a thickening $i : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ of ringed spaces such that $\operatorname{Ker}(i^\sharp)$ has square zero.
  (4) It is clear how to define *morphisms of thickenings*, *morphisms of thickenings over a base ringed space*, etc.

If $i : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ is a thickening of ringed spaces then we identify the underlying topological spaces and think of $\mathcal{O}_X$, $\mathcal{O}_{X'}$, and $\mathcal{I} = \operatorname{Ker}(i^\sharp)$ as sheaves on $X = X'$. We obtain a short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0$$

of $\mathcal{O}_{X'}$-modules. By Modules, Lemma 13.4 the category of $\mathcal{O}_X$-modules is equivalent to the category of $\mathcal{O}_{X'}$-modules annihilated by $\mathcal{I}$. In particular, if i is a first order thickening, then $\mathcal{I}$ is a $\mathcal{O}_X$-module.

08KZ **Situation 3.1.** A morphism of thickenings (f, f') is given by a commutative diagram

08L0 (3.1.1)
$$\begin{array}{ccc} (X, \mathcal{O}_X) & \xrightarrow{\ i\ } & (X', \mathcal{O}_{X'}) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ (S, \mathcal{O}_S) & \xrightarrow{\ t\ } & (S', \mathcal{O}_{S'}) \end{array}$$

of ringed spaces whose horizontal arrows are thickenings. In this situation we set $\mathcal{I} = \operatorname{Ker}(i^\sharp) \subset \mathcal{O}_{X'}$ and $\mathcal{J} = \operatorname{Ker}(t^\sharp) \subset \mathcal{O}_{S'}$. As $f = f'$ on underlying topological spaces we will identify the (topological) pullback functors f^{-1} and $(f')^{-1}$. Observe that $(f')^\sharp : f^{-1}\mathcal{O}_{S'} \to \mathcal{O}_{X'}$ induces in particular a map $f^{-1}\mathcal{J} \to \mathcal{I}$ and therefore a map of $\mathcal{O}_{X'}$-modules

$$(f')^*\mathcal{J} \longrightarrow \mathcal{I}$$

If i and t are first order thickenings, then $(f')^*\mathcal{J} = f^*\mathcal{J}$ and the map above becomes a map $f^*\mathcal{J} \to \mathcal{I}$.

08L1   **Definition 3.2.** In Situation 3.1 we say that (f, f') is a *strict morphism of thickenings* if the map $(f')^*\mathcal{J} \longrightarrow \mathcal{I}$ is surjective.

The following lemma in particular shows that a morphism $(f, f') : (X \subset X') \to (S \subset S')$ of thickenings of schemes is strict if and only if $X = S \times_{S'} X'$.

08L2   **Lemma 3.3.** *In Situation 3.1 the morphism (f, f') is a strict morphism of thickenings if and only if (3.1.1) is cartesian in the category of ringed spaces.*

Proof. Omitted. $\square$

4. Modules on first order thickenings of ringed spaces

08L3   In this section we discuss some preliminaries to the deformation theory of modules. Let $i : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ be a first order thickening of ringed spaces. We will freely use the notation introduced in Section 3, in particular we will identify the underlying topological spaces. In this section we consider short exact sequences

08L4   (4.0.1) $0 \to \mathcal{K} \to \mathcal{F}' \to \mathcal{F} \to 0$

of $\mathcal{O}_{X'}$-modules, where $\mathcal{F}$, $\mathcal{K}$ are $\mathcal{O}_X$-modules and $\mathcal{F}'$ is an $\mathcal{O}_{X'}$-module. In this situation we have a canonical $\mathcal{O}_X$-module map

$$c_{\mathcal{F}'} : \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F} \longrightarrow \mathcal{K}$$

where $\mathcal{I} = \operatorname{Ker}(i^\sharp)$. Namely, given local sections f of $\mathcal{I}$ and s of $\mathcal{F}$ we set $c_{\mathcal{F}'}(f \otimes s) = fs'$ where s' is a local section of $\mathcal{F}'$ lifting s.

08L5   **Lemma 4.1.** *Let $i : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ be a first order thickening of ringed spaces. Assume given extensions*

$$0 \to \mathcal{K} \to \mathcal{F}' \to \mathcal{F} \to 0 \quad and \quad 0 \to \mathcal{L} \to \mathcal{G}' \to \mathcal{G} \to 0$$

as in (4.0.1) and maps $\varphi : \mathcal{F} \to \mathcal{G}$ and $\psi : \mathcal{K} \to \mathcal{L}$.

   (1) *If there exists an $\mathcal{O}_{X'}$-module map $\varphi' : \mathcal{F}' \to \mathcal{G}'$ compatible with φ and ψ, then the diagram*

$$\begin{array}{ccc} \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F} & \xrightarrow{\ c_{\mathcal{F}'}\ } & \mathcal{K} \\ {\scriptstyle 1 \otimes \varphi}\downarrow & & \downarrow{\scriptstyle \psi} \\ \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{G} & \xrightarrow{\ c_{\mathcal{G}'}\ } & \mathcal{L} \end{array}$$

   is commutative.
   (2) *The set of $\mathcal{O}_{X'}$-module maps $\varphi' : \mathcal{F}' \to \mathcal{G}'$ compatible with φ and ψ is, if nonempty, a principal homogeneous space under $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{L})$.*

Proof. Part (1) is immediate from the description of the maps. For (2), if φ' and φ'' are two maps $\mathcal{F}' \to \mathcal{G}'$ compatible with φ and ψ, then $\varphi' - \varphi''$ factors as
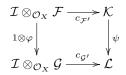
$$\mathcal{F}' \to \mathcal{F} \to \mathcal{L} \to \mathcal{G}'$$

The map in the middle comes from a unique element of $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{L})$ by Modules, Lemma 13.4. Conversely, given an element α of this group we can add the composition (as displayed above with α in the middle) to φ'. Some details omitted. $\square$

08L6 **Lemma 4.2.** *Let $i : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ be a first order thickening of ringed spaces. Assume given extensions*

$$0 \to \mathcal{K} \to \mathcal{F}' \to \mathcal{F} \to 0 \quad and \quad 0 \to \mathcal{L} \to \mathcal{G}' \to \mathcal{G} \to 0$$

as in (4.0.1) and maps $\varphi : \mathcal{F} \to \mathcal{G}$ and $\psi : \mathcal{K} \to \mathcal{L}$. Assume the diagram

$$
\begin{array}{ccc}
\mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F} & \xrightarrow{\ c_{\mathcal{F}'}\ } & \mathcal{K} \\
{\scriptstyle 1 \otimes \varphi} \downarrow & & \downarrow {\scriptstyle \psi} \\
\mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{G} & \xrightarrow{\ c_{\mathcal{G}'}\ } & \mathcal{L}
\end{array}
$$

is commutative. Then there exists an element

$$o(\varphi, \psi) \in \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{L})$$

whose vanishing is a necessary and sufficient condition for the existence of a map $\varphi' : \mathcal{F}' \to \mathcal{G}'$ compatible with φ and ψ.

Proof. We can construct explicitly an extension

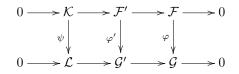$$0 \to \mathcal{L} \to \mathcal{H} \to \mathcal{F} \to 0$$

by taking $\mathcal{H}$ to be the cohomology of the complex

$$\mathcal{K} \xrightarrow{\ 1, -\psi\ } \mathcal{F}' \oplus \mathcal{G}' \xrightarrow{\ \varphi, 1\ } \mathcal{G}$$

in the middle (with obvious notation). A calculation with local sections using the assumption that the diagram of the lemma commutes shows that $\mathcal{H}$ is annihilated by $\mathcal{I}$. Hence $\mathcal{H}$ defines a class in

$$\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{L}) \subset \mathrm{Ext}^1_{\mathcal{O}_{X'}}(\mathcal{F}, \mathcal{L})$$

Finally, the class of $\mathcal{H}$ is the difference of the pushout of the extension $\mathcal{F}'$ via ψ and the pullback of the extension $\mathcal{G}'$ via φ (calculations omitted). Thus the vanishing of the class of $\mathcal{H}$ is equivalent to the existence of a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\
& & {\scriptstyle \psi} \downarrow & & {\scriptstyle \varphi'} \downarrow & & {\scriptstyle \varphi} \downarrow & & \\
0 & \longrightarrow & \mathcal{L} & \longrightarrow & \mathcal{G}' & \longrightarrow & \mathcal{G} & \longrightarrow & 0
\end{array}
$$

as desired. $\square$

08L7 **Lemma 4.3.** *Let $i : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ be a first order thickening of ringed spaces. Assume given $\mathcal{O}_X$-modules $\mathcal{F}, \mathcal{K}$ and an $\mathcal{O}_X$-linear map $c : \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{K}$. If there exists a sequence (4.0.1) with $c_{\mathcal{F}'} = c$ then the set of isomorphism classes of these extensions is principal homogeneous under $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{K})$.*

Proof. Assume given extensions

$$0 \to \mathcal{K} \to \mathcal{F}'_1 \to \mathcal{F} \to 0 \quad \text{and} \quad 0 \to \mathcal{K} \to \mathcal{F}'_2 \to \mathcal{F} \to 0$$

with $c_{\mathcal{F}'_1} = c_{\mathcal{F}'_2} = c$. Then the difference (in the extension group, see Homology, Section 6) is an extension

$$0 \to \mathcal{K} \to \mathcal{E} \to \mathcal{F} \to 0$$

where $\mathcal{E}$ is annihilated by $\mathcal{I}$ (local computation omitted). Hence the sequence is an extension of $\mathcal{O}_X$-modules, see Modules, Lemma 13.4. Conversely, given such an extension $\mathcal{E}$ we can add the extension $\mathcal{E}$ to the $\mathcal{O}_{X'}$-extension $\mathcal{F}'$ without affecting the map $c_{\mathcal{F}'}$. Some details omitted. $\qquad\square$

08L8 **Lemma 4.4.** *Let $i : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ be a first order thickening of ringed spaces. Assume given $\mathcal{O}_X$-modules $\mathcal{F}$, $\mathcal{K}$ and an $\mathcal{O}_X$-linear map $c : \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{K}$. Then there exists an element*

$$o(\mathcal{F}, \mathcal{K}, c) \in \mathrm{Ext}^2_{\mathcal{O}_X}(\mathcal{F}, \mathcal{K})$$

whose vanishing is a necessary and sufficient condition for the existence of a sequence (4.0.1) with $c_{\mathcal{F}'} = c$.
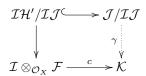
Proof. We first show that if $\mathcal{K}$ is an injective $\mathcal{O}_X$-module, then there does exist a sequence (4.0.1) with $c_{\mathcal{F}'} = c$. To do this, choose a flat $\mathcal{O}_{X'}$-module $\mathcal{H}'$ and a surjection $\mathcal{H}' \to \mathcal{F}$ (Modules, Lemma 17.6). Let $\mathcal{J} \subset \mathcal{H}'$ be the kernel. Since $\mathcal{H}'$ is flat we have

$$\mathcal{I} \otimes_{\mathcal{O}_{X'}} \mathcal{H}' = \mathcal{I}\mathcal{H}' \subset \mathcal{J} \subset \mathcal{H}'$$
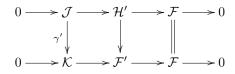
Observe that the map

$$\mathcal{I}\mathcal{H}' = \mathcal{I} \otimes_{\mathcal{O}_{X'}} \mathcal{H}' \longrightarrow \mathcal{I} \otimes_{\mathcal{O}_{X'}} \mathcal{F} = \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F}$$

annihilates $\mathcal{I}\mathcal{J}$. Namely, if f is a local section of $\mathcal{I}$ and s is a local section of $\mathcal{H}$, then fs is mapped to $f \otimes \bar{s}$ where $\bar{s}$ is the image of s in $\mathcal{F}$. Thus we obtain

$$
\begin{array}{ccc}
\mathcal{I}\mathcal{H}'/\mathcal{I}\mathcal{J} & \lhook\joinrel\longrightarrow & \mathcal{J}/\mathcal{I}\mathcal{J} \\
\downarrow & & \downarrow{\scriptstyle\gamma} \\
\mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F} & \xrightarrow{\;c\;} & \mathcal{K}
\end{array}
$$

a diagram of $\mathcal{O}_X$-modules. If $\mathcal{K}$ is injective as an $\mathcal{O}_X$-module, then we obtain the dotted arrow. Denote $\gamma' : \mathcal{J} \to \mathcal{K}$ the composition of γ with $\mathcal{J} \to \mathcal{J}/\mathcal{I}\mathcal{J}$. A local calculation shows the pushout

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{J} & \longrightarrow & \mathcal{H}' & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle\gamma'} & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{F} & \longrightarrow & 0
\end{array}
$$

is a solution to the problem posed by the lemma.

General case. Choose an embedding $\mathcal{K} \subset \mathcal{K}'$ with $\mathcal{K}'$ an injective $\mathcal{O}_X$-module. Let $\mathcal{Q}$ be the quotient, so that we have an exact sequence

$$0 \to \mathcal{K} \to \mathcal{K}' \to \mathcal{Q} \to 0$$

Denote $c' : \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{K}'$ be the composition. By the paragraph above there exists a sequence

$$0 \to \mathcal{K}' \to \mathcal{E}' \to \mathcal{F} \to 0$$

as in (4.0.1) with $c_{\mathcal{E}'} = c'$. Note that c' composed with the map $\mathcal{K}' \to \mathcal{Q}$ is zero, hence the pushout of $\mathcal{E}'$ by $\mathcal{K}' \to \mathcal{Q}$ is an extension

$$0 \to \mathcal{Q} \to \mathcal{D}' \to \mathcal{F} \to 0$$

as in (4.0.1) with $c_{\mathcal{D}'} = 0$. This means exactly that $\mathcal{D}'$ is annihilated by $\mathcal{I}$, in other words, the $\mathcal{D}'$ is an extension of $\mathcal{O}_X$-modules, i.e., defines an element

$$o(\mathcal{F}, \mathcal{K}, c) \in \operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{Q}) = \operatorname{Ext}^2_{\mathcal{O}_X}(\mathcal{F}, \mathcal{K})$$

(the equality holds by the long exact cohomology sequence associated to the exact sequence above and the vanishing of higher ext groups into the injective module $\mathcal{K}'$). If $o(\mathcal{F}, \mathcal{K}, c) = 0$, then we can choose a splitting $s : \mathcal{F} \to \mathcal{D}'$ and we can set

$$\mathcal{F}' = \operatorname{Ker}(\mathcal{E}' \to \mathcal{D}'/s(\mathcal{F}))$$

so that we obtain the following diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{K}' & \longrightarrow & \mathcal{E}' & \longrightarrow & \mathcal{F} & \longrightarrow & 0
\end{array}
$$

with exact rows which shows that $c_{\mathcal{F}'} = c$. Conversely, if $\mathcal{F}'$ exists, then the pushout of $\mathcal{F}'$ by the map $\mathcal{K} \to \mathcal{K}'$ is isomorphic to $\mathcal{E}'$ by Lemma 4.3 and the vanishing of higher ext groups into the injective module $\mathcal{K}'$. This gives a diagram as above, which implies that $\mathcal{D}'$ is split as an extension, i.e., the class $o(\mathcal{F}, \mathcal{K}, c)$ is zero. $\quad\square$

08L9 **Remark 4.5.** Let $(X, \mathcal{O}_X)$ be a ringed space. A first order thickening $i : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ is said to be *trivial* if there exists a morphism of ringed spaces $\pi : (X', \mathcal{O}_{X'}) \to (X, \mathcal{O}_X)$ which is a left inverse to i. The choice of such a morphism π is called a *trivialization* of the first order thickening. Given π we obtain a splitting

08LA (4.5.1) $$\mathcal{O}_{X'} = \mathcal{O}_X \oplus \mathcal{I}$$

as sheaves of algebras on X by using $\pi^\sharp$ to split the surjection $\mathcal{O}_{X'} \to \mathcal{O}_X$. Conversely, such a splitting determines a morphism π. The category of trivialized first order thickenings of $(X, \mathcal{O}_X)$ is equivalent to the category of $\mathcal{O}_X$-modules.

08LB **Remark 4.6.** Let $i : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ be a trivial first order thickening of ringed spaces and let $\pi : (X', \mathcal{O}_{X'}) \to (X, \mathcal{O}_X)$ be a trivialization. Then given any triple $(\mathcal{F}, \mathcal{K}, c)$ consisting of a pair of $\mathcal{O}_X$-modules and a map $c : \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{K}$ we may set
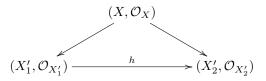
$$\mathcal{F}'_{c,triv} = \mathcal{F} \oplus \mathcal{K}$$

and use the splitting (4.5.1) associated to π and the map c to define the $\mathcal{O}_{X'}$-module structure and obtain an extension (4.0.1). We will call $\mathcal{F}'_{c,triv}$ the *trivial extension* of $\mathcal{F}$ by $\mathcal{K}$ corresponding to c and the trivialization π. Given any extension $\mathcal{F}'$ as in (4.0.1) we can use $\pi^\sharp : \mathcal{O}_X \to \mathcal{O}_{X'}$ to think of $\mathcal{F}'$ as an $\mathcal{O}_X$-module extension, hence a class $\xi_{\mathcal{F}'}$ in $\operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{K})$. Lemma 4.3 assures that $\mathcal{F}' \mapsto \xi_{\mathcal{F}'}$ induces a bijection

$$\left\{ \begin{array}{c} \text{isomorphism classes of extensions} \\ \mathcal{F}' \text{ as in (4.0.1) with } c = c_{\mathcal{F}'} \end{array} \right\} \longrightarrow \operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{K})$$

Moreover, the trivial extension $\mathcal{F}'_{c,triv}$ maps to the zero class.

08LC **Remark 4.7.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $(X, \mathcal{O}_X) \to (X_i', \mathcal{O}_{X_i'})$, $i = 1, 2$ be first order thickenings with ideal sheaves $\mathcal{I}_i$. Let $h : (X_1', \mathcal{O}_{X_1'}) \to (X_2', \mathcal{O}_{X_2'})$ be a morphism of first order thickenings of $(X, \mathcal{O}_X)$. Picture

$$
\begin{array}{ccc}
& (X, \mathcal{O}_X) & \\
\swarrow & & \searrow \\
(X_1', \mathcal{O}_{X_1'}) & \xrightarrow{\quad h \quad} & (X_2', \mathcal{O}_{X_2'})
\end{array}
$$

Observe that $h^\sharp : \mathcal{O}_{X_2'} \to \mathcal{O}_{X_1'}$ in particular induces an $\mathcal{O}_X$-module map $\mathcal{I}_2 \to \mathcal{I}_1$. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Let $(\mathcal{K}_i, c_i)$, $i = 1, 2$ be a pair consisting of an $\mathcal{O}_X$-module $\mathcal{K}_i$ and a map $c_i : \mathcal{I}_i \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{K}_i$. Assume furthermore given a map of $\mathcal{O}_X$-modules $\mathcal{K}_2 \to \mathcal{K}_1$ such that

$$
\begin{array}{ccc}
\mathcal{I}_2 \otimes_{\mathcal{O}_X} \mathcal{F} & \xrightarrow{\ c_2\ } & \mathcal{K}_2 \\
\downarrow & & \downarrow \\
\mathcal{I}_1 \otimes_{\mathcal{O}_X} \mathcal{F} & \xrightarrow{\ c_1\ } & \mathcal{K}_1
\end{array}
$$

is commutative. Then there is a canonical functoriality

$$
\left\{ \begin{array}{l} \mathcal{F}_2' \text{ as in } (4.0.1) \text{ with} \\ c_2 = c_{\mathcal{F}_2'} \text{ and } \mathcal{K} = \mathcal{K}_2 \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} \mathcal{F}_1' \text{ as in } (4.0.1) \text{ with} \\ c_1 = c_{\mathcal{F}_1'} \text{ and } \mathcal{K} = \mathcal{K}_1 \end{array} \right\}
$$

Namely, thinking of all sheaves $\mathcal{O}_X$, $\mathcal{O}_{X_i'}$, $\mathcal{F}$, $\mathcal{K}_i$, etc as sheaves on X, we set given $\mathcal{F}_2'$ the sheaf $\mathcal{F}_1'$ equal to the pushout, i.e., fitting into the following diagram of extensions

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{K}_2 & \longrightarrow & \mathcal{F}_2' & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{K}_1 & \longrightarrow & \mathcal{F}_1' & \longrightarrow & \mathcal{F} & \longrightarrow & 0
\end{array}
$$

We omit the construction of the $\mathcal{O}_{X_1'}$-module structure on the pushout (this uses the commutativity of the diagram involving c_1 and c_2).

08LD **Remark 4.8.** Let $(X, \mathcal{O}_X)$, $(X, \mathcal{O}_X) \to (X_i', \mathcal{O}_{X_i'})$, $\mathcal{I}_i$, and $h : (X_1', \mathcal{O}_{X_1'}) \to (X_2', \mathcal{O}_{X_2'})$ be as in Remark 4.7. Assume that we are given trivializations $\pi_i : X_i' \to X$ such that $\pi_1 = h \circ \pi_2$. In other words, assume h is a morphism of trivialized first order thickening of $(X, \mathcal{O}_X)$. Let $(\mathcal{K}_i, c_i)$, $i = 1, 2$ be a pair consisting of an $\mathcal{O}_X$-module $\mathcal{K}_i$ and a map $c_i : \mathcal{I}_i \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{K}_i$. Assume furthermore given a map of $\mathcal{O}_X$-modules $\mathcal{K}_2 \to \mathcal{K}_1$ such that

$$
\begin{array}{ccc}
\mathcal{I}_2 \otimes_{\mathcal{O}_X} \mathcal{F} & \xrightarrow{\ c_2\ } & \mathcal{K}_2 \\
\downarrow & & \downarrow \\
\mathcal{I}_1 \otimes_{\mathcal{O}_X} \mathcal{F} & \xrightarrow{\ c_1\ } & \mathcal{K}_1
\end{array}
$$

is commutative. In this situation the construction of Remark 4.6 induces a commutative diagram

$$\{\mathcal{F}_2' \text{ as in } (4.0.1) \text{ with } c_2 = c_{\mathcal{F}_2'} \text{ and } \mathcal{K} = \mathcal{K}_2\} \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{K}_2)$$

$$\{\mathcal{F}_1' \text{ as in } (4.0.1) \text{ with } c_1 = c_{\mathcal{F}_1'} \text{ and } \mathcal{K} = \mathcal{K}_1\} \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{K}_1)$$

where the vertical map on the right is given by functoriality of Ext and the map $\mathcal{K}_2 \to \mathcal{K}_1$ and the vertical map on the left is the one from Remark 4.7.

08LE **Remark 4.9.** Let $(X, \mathcal{O}_X)$ be a ringed space. We define a sequence of morphisms of first order thickenings

$$(X_1', \mathcal{O}_{X_1'}) \to (X_2', \mathcal{O}_{X_2'}) \to (X_3', \mathcal{O}_{X_3'})$$

of $(X, \mathcal{O}_X)$ to be a *complex* if the corresponding maps between the ideal sheaves $\mathcal{I}_i$ give a complex of $\mathcal{O}_X$-modules $\mathcal{I}_3 \to \mathcal{I}_2 \to \mathcal{I}_1$ (i.e., the composition is zero). In this case the composition $(X_1', \mathcal{O}_{X_1'}) \to (X_3', \mathcal{O}_{X_3'})$ factors through $(X, \mathcal{O}_X) \to (X_3', \mathcal{O}_{X_3'})$, i.e., the first order thickening $(X_1', \mathcal{O}_{X_1'})$ of $(X, \mathcal{O}_X)$ is trivial and comes with a canonical trivialization $\pi : (X_1', \mathcal{O}_{X_1'}) \to (X, \mathcal{O}_X)$.

We say a sequence of morphisms of first order thickenings

$$(X_1', \mathcal{O}_{X_1'}) \to (X_2', \mathcal{O}_{X_2'}) \to (X_3', \mathcal{O}_{X_3'})$$

of $(X, \mathcal{O}_X)$ is *a short exact sequence* if the corresponding maps between ideal sheaves is a short exact sequence

$$0 \to \mathcal{I}_3 \to \mathcal{I}_2 \to \mathcal{I}_1 \to 0$$

of $\mathcal{O}_X$-modules.

08LF **Remark 4.10.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Let

$$(X_1', \mathcal{O}_{X_1'}) \to (X_2', \mathcal{O}_{X_2'}) \to (X_3', \mathcal{O}_{X_3'})$$

be a complex first order thickenings of $(X, \mathcal{O}_X)$, see Remark 4.9. Let $(\mathcal{K}_i, c_i)$, $i = 1, 2, 3$ be pairs consisting of an $\mathcal{O}_X$-module $\mathcal{K}_i$ and a map $c_i : \mathcal{I}_i \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{K}_i$. Assume given a short exact sequence of $\mathcal{O}_X$-modules

$$0 \to \mathcal{K}_3 \to \mathcal{K}_2 \to \mathcal{K}_1 \to 0$$

such that

$$\begin{array}{ccc} \mathcal{I}_2 \otimes_{\mathcal{O}_X} \mathcal{F} \xrightarrow{c_2} \mathcal{K}_2 \\ \downarrow \qquad \qquad \downarrow \\ \mathcal{I}_1 \otimes_{\mathcal{O}_X} \mathcal{F} \xrightarrow{c_1} \mathcal{K}_1 \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{I}_3 \otimes_{\mathcal{O}_X} \mathcal{F} \xrightarrow{c_3} \mathcal{K}_3 \\ \downarrow \qquad \qquad \downarrow \\ \mathcal{I}_2 \otimes_{\mathcal{O}_X} \mathcal{F} \xrightarrow{c_2} \mathcal{K}_2 \end{array}$$

are commutative. Finally, assume given an extension

$$0 \to \mathcal{K}_2 \to \mathcal{F}_2' \to \mathcal{F} \to 0$$

as in (4.0.1) with $\mathcal{K} = \mathcal{K}_2$ of $\mathcal{O}_{X_2'}$-modules with $c_{\mathcal{F}_2'} = c_2$. In this situation we can apply the functoriality of Remark 4.7 to obtain an extension $\mathcal{F}_1'$ on X_1' (we'll describe $\mathcal{F}_1'$ in this special case below). By Remark 4.6 using the canonical splitting $\pi : (X_1', \mathcal{O}_{X_1'}) \to (X, \mathcal{O}_X)$ of Remark 4.9 we obtain $\xi_{\mathcal{F}_1'} \in \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{K}_1)$. Finally, we have the obstruction

$$o(\mathcal{F}, \mathcal{K}_3, c_3) \in \mathrm{Ext}^2_{\mathcal{O}_X}(\mathcal{F}, \mathcal{K}_3)$$

see Lemma 4.4. In this situation we **claim** that the canonical map

$$\partial : \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{K}_1) \longrightarrow \mathrm{Ext}^2_{\mathcal{O}_X}(\mathcal{F}, \mathcal{K}_3)$$

coming from the short exact sequence $0 \to \mathcal{K}_3 \to \mathcal{K}_2 \to \mathcal{K}_1 \to 0$ sends $\xi_{\mathcal{F}'_1}$ to the obstruction class $o(\mathcal{F}, \mathcal{K}_3, c_3)$.

To prove this claim choose an embedding $j : \mathcal{K}_3 \to \mathcal{K}$ where $\mathcal{K}$ is an injective $\mathcal{O}_X$-module. We can lift j to a map $j' : \mathcal{K}_2 \to \mathcal{K}$. Set $\mathcal{E}'_2 = j'_* \mathcal{F}'_2$ equal to the pushout of $\mathcal{F}'_2$ by j' so that $c_{\mathcal{E}'_2} = j' \circ c_2$. Picture:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{K}_2 & \longrightarrow & \mathcal{F}'_2 & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle j'} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{E}'_2 & \longrightarrow & \mathcal{F} & \longrightarrow & 0
\end{array}
$$

Set $\mathcal{E}'_3 = \mathcal{E}'_2$ but viewed as an $\mathcal{O}_{X'_3}$-module via $\mathcal{O}_{X'_3} \to \mathcal{O}_{X'_2}$. Then $c_{\mathcal{E}'_3} = j \circ c_3$. The proof of Lemma 4.4 constructs $o(\mathcal{F}, \mathcal{K}_3, c_3)$ as the boundary of the class of the extension of $\mathcal{O}_X$-modules

$$0 \to \mathcal{K}/\mathcal{K}_3 \to \mathcal{E}'_3/\mathcal{K}_3 \to \mathcal{F} \to 0$$

On the other hand, note that $\mathcal{F}'_1 = \mathcal{F}'_2/\mathcal{K}_3$ hence the class $\xi_{\mathcal{F}'_1}$ is the class of the extension

$$0 \to \mathcal{K}_2/\mathcal{K}_3 \to \mathcal{F}'_2/\mathcal{K}_3 \to \mathcal{F} \to 0$$

seen as a sequence of $\mathcal{O}_X$-modules using $\pi^\sharp$ where $\pi : (X'_1, \mathcal{O}_{X'_1}) \to (X, \mathcal{O}_X)$ is the canonical splitting. Thus finally, the claim follows from the fact that we have a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{K}_2/\mathcal{K}_3 & \longrightarrow & \mathcal{F}'_2/\mathcal{K}_3 & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{K}/\mathcal{K}_3 & \longrightarrow & \mathcal{E}'_3/\mathcal{K}_3 & \longrightarrow & \mathcal{F} & \longrightarrow & 0
\end{array}
$$

which is $\mathcal{O}_X$-linear (with the $\mathcal{O}_X$-module structures given above).

5. Infinitesimal deformations of modules on ringed spaces

08LG Let $i : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ be a first order thickening of ringed spaces. We freely use the notation introduced in Section 3. Let $\mathcal{F}'$ be an $\mathcal{O}_{X'}$-module and set $\mathcal{F} = i^* \mathcal{F}'$. In this situation we have a short exact sequence

$$0 \to \mathcal{I}\mathcal{F}' \to \mathcal{F}' \to \mathcal{F} \to 0$$

of $\mathcal{O}_{X'}$-modules. Since $\mathcal{I}^2 = 0$ the $\mathcal{O}_{X'}$-module structure on $\mathcal{I}\mathcal{F}'$ comes from a unique $\mathcal{O}_X$-module structure. Thus the sequence above is an extension as in (4.0.1). As a special case, if $\mathcal{F}' = \mathcal{O}_{X'}$ we have $i^* \mathcal{O}_{X'} = \mathcal{O}_X$ and $\mathcal{I}\mathcal{O}_{X'} = \mathcal{I}$ and we recover the sequence of structure sheaves

$$0 \to \mathcal{I} \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0$$

08LH **Lemma 5.1.** *Let $i : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ be a first order thickening of ringed spaces. Let $\mathcal{F}'$, $\mathcal{G}'$ be $\mathcal{O}_{X'}$-modules. Set $\mathcal{F} = i^* \mathcal{F}'$ and $\mathcal{G} = i^* \mathcal{G}'$. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be an $\mathcal{O}_X$-linear map. The set of lifts of φ to an $\mathcal{O}_{X'}$-linear map $\varphi' : \mathcal{F}' \to \mathcal{G}'$ is, if nonempty, a principal homogeneous space under $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{I}\mathcal{G}')$.*

Proof. This is a special case of Lemma 4.1 but we also give a direct proof. We have short exact sequences of modules

$$0 \to \mathcal{I} \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0 \quad \text{and} \quad 0 \to \mathcal{I}\mathcal{G}' \to \mathcal{G}' \to \mathcal{G} \to 0$$

and similarly for $\mathcal{F}'$. Since $\mathcal{I}$ has square zero the $\mathcal{O}_{X'}$-module structure on $\mathcal{I}$ and $\mathcal{I}\mathcal{G}'$ comes from a unique $\mathcal{O}_X$-module structure. It follows that

$$\text{Hom}_{\mathcal{O}_{X'}}(\mathcal{F}', \mathcal{I}\mathcal{G}') = \text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{I}\mathcal{G}') \quad \text{and} \quad \text{Hom}_{\mathcal{O}_{X'}}(\mathcal{F}', \mathcal{G}) = \text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$$

The lemma now follows from the exact sequence

$$0 \to \text{Hom}_{\mathcal{O}_{X'}}(\mathcal{F}', \mathcal{I}\mathcal{G}') \to \text{Hom}_{\mathcal{O}_{X'}}(\mathcal{F}', \mathcal{G}') \to \text{Hom}_{\mathcal{O}_{X'}}(\mathcal{F}', \mathcal{G})$$

see Homology, Lemma 5.8. $\square$

08LI **Lemma 5.2.** *Let* (f, f') *be a morphism of first order thickenings of ringed spaces as in Situation 3.1. Let* $\mathcal{F}'$ *be an* $\mathcal{O}_{X'}$*-module and set* $\mathcal{F} = i^*\mathcal{F}'$*. Assume that* $\mathcal{F}$ *is flat over* S *and that* (f, f') *is a strict morphism of thickenings (Definition 3.2). Then the following are equivalent*

(1) $\mathcal{F}'$ *is flat over* S'*, and*
(2) *the canonical map* $f^*\mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{I}\mathcal{F}'$ *is an isomorphism.*

Moreover, in this case the maps

$$f^*\mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{I}\mathcal{F}'$$

are isomorphisms.

Proof. The map $f^*\mathcal{J} \to \mathcal{I}$ is surjective as (f, f') is a strict morphism of thickenings. Hence the final statement is a consequence of (2).

Proof of the equivalence of (1) and (2). We may check these conditions at stalks. Let $x \in X \subset X'$ be a point with image $s = f(x) \in S \subset S'$. Set $A' = \mathcal{O}_{S',s}$, $B' = \mathcal{O}_{X',x}$, $A = \mathcal{O}_{S,s}$, and $B = \mathcal{O}_{X,x}$. Then $A = A'/J$ and $B = B'/I$ for some square zero ideals. Since (f, f') is a strict morphism of thickenings we have $I = JB'$. Let $M' = \mathcal{F}'_x$ and $M = \mathcal{F}_x$. Then M' is a B'-module and M is a B-module. Since $\mathcal{F} = i^*\mathcal{F}'$ we see that the kernel of the surjection $M' \to M$ is $IM' = JM'$. Thus we have a short exact sequence

$$0 \to JM' \to M' \to M \to 0$$

Using Sheaves, Lemma 26.4 and Modules, Lemma 16.1 to identify stalks of pullbacks and tensor products we see that the stalk at x of the canonical map of the lemma is the map

$$(J \otimes_A B) \otimes_B M = J \otimes_A M = J \otimes_{A'} M' \longrightarrow JM'$$

The assumption that $\mathcal{F}$ is flat over S signifies that M is a flat A-module.

Assume (1). Flatness implies $\text{Tor}_1^{A'}(M', A) = 0$ by Algebra, Lemma 75.8. This means $J \otimes_{A'} M' \to M'$ is injective by Algebra, Remark 75.9. Hence $J \otimes_A M \to JM'$ is an isomorphism.

Assume (2). Then $J \otimes_{A'} M' \to M'$ is injective. Hence $\text{Tor}_1^{A'}(M', A) = 0$ by Algebra, Remark 75.9. Hence M' is flat over A' by Algebra, Lemma 99.8. $\square$

08LJ **Lemma 5.3.** *Let (f, f') be a morphism of first order thickenings as in Situation 3.1. Let $\mathcal{F}'$, $\mathcal{G}'$ be $\mathcal{O}_{X'}$-modules and set $\mathcal{F} = i^*\mathcal{F}'$ and $\mathcal{G} = i^*\mathcal{G}'$. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be an $\mathcal{O}_X$-linear map. Assume that $\mathcal{G}'$ is flat over S' and that (f, f') is a strict morphism of thickenings. The set of lifts of φ to an $\mathcal{O}_{X'}$-linear map $\varphi' : \mathcal{F}' \to \mathcal{G}'$ is, if nonempty, a principal homogeneous space under*

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{J})$$

Proof. Combine Lemmas 5.1 and 5.2. $\square$

08LK **Lemma 5.4.** *Let $i : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ be a first order thickening of ringed spaces. Let $\mathcal{F}'$, $\mathcal{G}'$ be $\mathcal{O}_{X'}$-modules and set $\mathcal{F} = i^*\mathcal{F}'$ and $\mathcal{G} = i^*\mathcal{G}'$. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be an $\mathcal{O}_X$-linear map. There exists an element*

$$o(\varphi) \in \mathrm{Ext}^1_{\mathcal{O}_X}(Li^*\mathcal{F}', \mathcal{I}\mathcal{G}')$$

whose vanishing is a necessary and sufficient condition for the existence of a lift of φ to an $\mathcal{O}_{X'}$-linear map $\varphi' : \mathcal{F}' \to \mathcal{G}'$.

Proof. It is clear from the proof of Lemma 5.1 that the vanishing of the boundary of φ via the map

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) = \mathrm{Hom}_{\mathcal{O}_{X'}}(\mathcal{F}', \mathcal{G}) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_{X'}}(\mathcal{F}', \mathcal{I}\mathcal{G}')$$

is a necessary and sufficient condition for the existence of a lift. We conclude as

$$\mathrm{Ext}^1_{\mathcal{O}_{X'}}(\mathcal{F}', \mathcal{I}\mathcal{G}') = \mathrm{Ext}^1_{\mathcal{O}_X}(Li^*\mathcal{F}', \mathcal{I}\mathcal{G}')$$

the adjointness of $i_* = Ri_*$ and Li^* on the derived category (Cohomology, Lemma 28.1). $\square$

08LL **Lemma 5.5.** *Let (f, f') be a morphism of first order thickenings as in Situation 3.1. Let $\mathcal{F}'$, $\mathcal{G}'$ be $\mathcal{O}_{X'}$-modules and set $\mathcal{F} = i^*\mathcal{F}'$ and $\mathcal{G} = i^*\mathcal{G}'$. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be an $\mathcal{O}_X$-linear map. Assume that $\mathcal{F}'$ and $\mathcal{G}'$ are flat over S' and that (f, f') is a strict morphism of thickenings. There exists an element*

$$o(\varphi) \in \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{J})$$

whose vanishing is a necessary and sufficient condition for the existence of a lift of φ to an $\mathcal{O}_{X'}$-linear map $\varphi' : \mathcal{F}' \to \mathcal{G}'$.

First proof. This follows from Lemma 5.4 as we claim that under the assumptions of the lemma we have

$$\mathrm{Ext}^1_{\mathcal{O}_X}(Li^*\mathcal{F}', \mathcal{I}\mathcal{G}') = \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{J})$$

Namely, we have $\mathcal{I}\mathcal{G}' = \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{J}$ by Lemma 5.2. On the other hand, observe that

$$H^{-1}(Li^*\mathcal{F}') = \mathrm{Tor}_1^{\mathcal{O}_{X'}}(\mathcal{F}', \mathcal{O}_X)$$

(local computation omitted). Using the short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0$$

we see that this Tor_1 is computed by the kernel of the map $\mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{I}\mathcal{F}'$ which is zero by the final assertion of Lemma 5.2. Thus $\tau_{\geq -1}Li^*\mathcal{F}' = \mathcal{F}$. On the other hand, we have

$$\mathrm{Ext}^1_{\mathcal{O}_X}(Li^*\mathcal{F}', \mathcal{I}\mathcal{G}') = \mathrm{Ext}^1_{\mathcal{O}_X}(\tau_{\geq -1}Li^*\mathcal{F}', \mathcal{I}\mathcal{G}')$$

by the dual of Derived Categories, Lemma 16.1. $\square$

Second proof. We can apply Lemma 4.2 as follows. Note that $\mathcal{K} = \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F}$ and $\mathcal{L} = \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{G}$ by Lemma 5.2, that $c_{\mathcal{F}'} = 1 \otimes 1$ and $c_{\mathcal{G}'} = 1 \otimes 1$ and taking $\psi = 1 \otimes \varphi$ the diagram of the lemma commutes. Thus $o(\varphi) = o(\varphi, 1 \otimes \varphi)$ works. $\square$

08LM **Lemma 5.6.** *Let (f, f') be a morphism of first order thickenings as in Situation 3.1. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Assume (f, f') is a strict morphism of thickenings and $\mathcal{F}$ flat over S. If there exists a pair $(\mathcal{F}', \alpha)$ consisting of an $\mathcal{O}_{X'}$-module $\mathcal{F}'$ flat over S' and an isomorphism $\alpha : i^*\mathcal{F}' \to \mathcal{F}$, then the set of isomorphism classes of such pairs is principal homogeneous under $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F})$.*

Proof. If we assume there exists one such module, then the canonical map

$$f^*\mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F}$$

is an isomorphism by Lemma 5.2. Apply Lemma 4.3 with $\mathcal{K} = \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F}$ and $c = 1$. By Lemma 5.2 the corresponding extensions $\mathcal{F}'$ are all flat over S'. $\square$

08LN **Lemma 5.7.** *Let (f, f') be a morphism of first order thickenings as in Situation 3.1. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Assume (f, f') is a strict morphism of thickenings and $\mathcal{F}$ flat over S. There exists an $\mathcal{O}_{X'}$-module $\mathcal{F}'$ flat over S' with $i^*\mathcal{F}' \cong \mathcal{F}$, if and only if*

(1) *the canonical map $f^*\mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F}$ is an isomorphism, and*
(2) *the class $o(\mathcal{F}, \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F}, 1) \in \mathrm{Ext}^2_{\mathcal{O}_X}(\mathcal{F}, \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F})$ of Lemma 4.4 is zero.*

Proof. This follows immediately from the characterization of $\mathcal{O}_{X'}$-modules flat over S' of Lemma 5.2 and Lemma 4.4. $\square$

6. Application to flat modules on flat thickenings of ringed spaces

08VQ Consider a commutative diagram

$$
\begin{array}{ccc}
(X, \mathcal{O}_X) & \xrightarrow{\ i\ } & (X', \mathcal{O}_{X'}) \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\
(S, \mathcal{O}_S) & \xrightarrow{\ t\ } & (S', \mathcal{O}_{S'})
\end{array}
$$

of ringed spaces whose horizontal arrows are first order thickenings as in Situation 3.1. Set $\mathcal{I} = \mathrm{Ker}(i^\sharp) \subset \mathcal{O}_{X'}$ and $\mathcal{J} = \mathrm{Ker}(t^\sharp) \subset \mathcal{O}_{S'}$. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Assume that

(1) (f, f') is a strict morphism of thickenings,
(2) f' is flat, and
(3) $\mathcal{F}$ is flat over S.

Note that (1) + (2) imply that $\mathcal{I} = f^*\mathcal{J}$ (apply Lemma 5.2 to $\mathcal{O}_{X'}$). The theory of the preceding section is especially nice under these assumptions. We summarize the results already obtained in the following lemma.

08VR **Lemma 6.1.** *In the situation above.*

(1) *There exists an $\mathcal{O}_{X'}$-module $\mathcal{F}'$ flat over S' with $i^*\mathcal{F}' \cong \mathcal{F}$, if and only if the class $o(\mathcal{F}, f^*\mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{F}, 1) \in \mathrm{Ext}^2_{\mathcal{O}_X}(\mathcal{F}, f^*\mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{F})$ of Lemma 4.4 is zero.*
(2) *If such a module exists, then the set of isomorphism classes of lifts is principal homogeneous under $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, f^*\mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{F})$.*

(3) *Given a lift $\mathcal{F}'$, the set of automorphisms of $\mathcal{F}'$ which pull back to $\mathrm{id}_{\mathcal{F}}$ is canonically isomorphic to $\mathrm{Ext}^0_{\mathcal{O}_X}(\mathcal{F}, f^*\mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{F})$.*

Proof. Part (1) follows from Lemma 5.7 as we have seen above that $\mathcal{I} = f^*\mathcal{J}$. Part (2) follows from Lemma 5.6. Part (3) follows from Lemma 5.3. $\square$

08VS **Situation 6.2.** Let $f : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$ be a morphism of ringed spaces. Consider a commutative diagram

$$
\begin{array}{ccccc}
(X'_1, \mathcal{O}'_1) & \xrightarrow{\ h\ } & (X'_2, \mathcal{O}'_2) & \longrightarrow & (X'_3, \mathcal{O}'_3) \\
{\scriptstyle f'_1}\big\downarrow & & {\scriptstyle f'_2}\big\downarrow & & {\scriptstyle f'_3}\big\downarrow \\
(S'_1, \mathcal{O}_{S'_1}) & \longrightarrow & (S'_2, \mathcal{O}_{S'_2}) & \longrightarrow & (S'_3, \mathcal{O}_{S'_3})
\end{array}
$$

where (a) the top row is a short exact sequence of first order thickenings of X, (b) the lower row is a short exact sequence of first order thickenings of S, (c) each f'_i restricts to f, (d) each pair (f, f'_i) is a strict morphism of thickenings, and (e) each f'_i is flat. Finally, let $\mathcal{F}'_2$ be an $\mathcal{O}'_2$-module flat over S'_2 and set $\mathcal{F} = \mathcal{F}'_2|_X$. Let $\pi : X'_1 \to X$ be the canonical splitting (Remark 4.9).

08VT **Lemma 6.3.** *In Situation 6.2 the modules $\pi^*\mathcal{F}$ and $h^*\mathcal{F}'_2$ are $\mathcal{O}'_1$-modules flat over S'_1 restricting to $\mathcal{F}$ on X. Their difference (Lemma 6.1) is an element θ of $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, f^*\mathcal{J}_1 \otimes_{\mathcal{O}_X} \mathcal{F})$ whose boundary in $\mathrm{Ext}^2_{\mathcal{O}_X}(\mathcal{F}, f^*\mathcal{J}_3 \otimes_{\mathcal{O}_X} \mathcal{F})$ equals the obstruction (Lemma 6.1) to lifting $\mathcal{F}$ to an $\mathcal{O}'_3$-module flat over S'_3.*

Proof. Note that both $\pi^*\mathcal{F}$ and $h^*\mathcal{F}'_2$ restrict to $\mathcal{F}$ on X and that the kernels of $\pi^*\mathcal{F} \to \mathcal{F}$ and $h^*\mathcal{F}'_2 \to \mathcal{F}$ are given by $f^*\mathcal{J}_1 \otimes_{\mathcal{O}_X} \mathcal{F}$. Hence flatness by Lemma 5.2. Taking the boundary makes sense as the sequence of modules

$$0 \to f^*\mathcal{J}_3 \otimes_{\mathcal{O}_X} \mathcal{F} \to f^*\mathcal{J}_2 \otimes_{\mathcal{O}_X} \mathcal{F} \to f^*\mathcal{J}_1 \otimes_{\mathcal{O}_X} \mathcal{F} \to 0$$

is short exact due to the assumptions in Situation 6.2 and the fact that $\mathcal{F}$ is flat over S. The statement on the obstruction class is a direct translation of the result of Remark 4.10 to this particular situation. $\square$
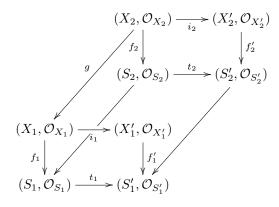
7. Deformations of ringed spaces and the naive cotangent complex

08U6 In this section we use the naive cotangent complex to do a little bit of deformation theory. We start with a first order thickening $t : (S, \mathcal{O}_S) \to (S', \mathcal{O}_{S'})$ of ringed spaces. We denote $\mathcal{J} = \mathrm{Ker}(t^\sharp)$ and we identify the underlying topological spaces of S and S'. Moreover we assume given a morphism of ringed spaces $f : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$, an $\mathcal{O}_X$-module $\mathcal{G}$, and an f-map $c : \mathcal{J} \to \mathcal{G}$ of sheaves of modules (Sheaves, Definition 21.7 and Section 26). In this section we ask ourselves whether we can find the question mark fitting into the following diagram
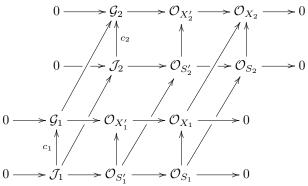
08U7 (7.0.1)
$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{G} & \longrightarrow & ? & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 \\
& & {\scriptstyle c}\big\uparrow & & \big\uparrow & & \big\uparrow & & \\
0 & \longrightarrow & \mathcal{J} & \longrightarrow & \mathcal{O}_{S'} & \longrightarrow & \mathcal{O}_S & \longrightarrow & 0
\end{array}
$$

(where the vertical arrows are f-maps) and moreover how unique the solution is (if it exists). More precisely, we look for a first order thickening $i : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ and a morphism of thickenings (f, f') as in (3.1.1) where $\mathrm{Ker}(i^\sharp)$ is identified with $\mathcal{G}$ such that $(f')^\sharp$ induces the given map c. We will say X' is a *solution* to (7.0.1).

08U8 **Lemma 7.1.** *Assume given a commutative diagram of morphisms of ringed spaces*

$$
\begin{array}{ccc}
(X_2, \mathcal{O}_{X_2}) & \xrightarrow{\ i_2\ } & (X'_2, \mathcal{O}_{X'_2}) \\
\downarrow f_2 & & \downarrow f'_2 \\
(S_2, \mathcal{O}_{S_2}) & \xrightarrow{\ t_2\ } & (S'_2, \mathcal{O}_{S'_2})
\end{array}
$$

08U9 (7.1.1)

$$
\begin{array}{ccc}
(X_1, \mathcal{O}_{X_1}) & \xrightarrow{\ i_1\ } & (X'_1, \mathcal{O}_{X'_1}) \\
\downarrow f_1 & & \downarrow f'_1 \\
(S_1, \mathcal{O}_{S_1}) & \xrightarrow{\ t_1\ } & (S'_1, \mathcal{O}_{S'_1})
\end{array}
$$

whose horizontal arrows are first order thickenings. Set $\mathcal{G}_j = \mathrm{Ker}(i_j^\sharp)$ and assume given a g-map $\nu : \mathcal{G}_1 \to \mathcal{G}_2$ of modules giving rise to the commutative diagram

08UA (7.1.2)

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{G}_2 & \longrightarrow & \mathcal{O}_{X'_2} & \longrightarrow & \mathcal{O}_{X_2} & \longrightarrow & 0 \\
& & \uparrow c_2 & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{J}_2 & \longrightarrow & \mathcal{O}_{S'_2} & \longrightarrow & \mathcal{O}_{S_2} & \longrightarrow & 0 \\
& & & & & & & & \\
0 & \longrightarrow & \mathcal{G}_1 & \longrightarrow & \mathcal{O}_{X'_1} & \longrightarrow & \mathcal{O}_{X_1} & \longrightarrow & 0 \\
& & \uparrow c_1 & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{J}_1 & \longrightarrow & \mathcal{O}_{S'_1} & \longrightarrow & \mathcal{O}_{S_1} & \longrightarrow & 0
\end{array}
$$

with front and back solutions to (7.0.1).

(1) *There exist a canonical element in $\mathrm{Ext}^1_{\mathcal{O}_{X_2}}(Lg^* NL_{X_1/S_1}, \mathcal{G}_2)$ whose vanishing is a necessary and sufficient condition for the existence of a morphism of ringed spaces $X'_2 \to X'_1$ fitting into (7.1.1) compatibly with ν.*

(2) *If there exists a morphism $X'_2 \to X'_1$ fitting into (7.1.1) compatibly with ν the set of all such morphisms is a principal homogeneous space under*

$$
\mathrm{Hom}_{\mathcal{O}_{X_1}}(\Omega_{X_1/S_1}, g_* \mathcal{G}_2) = \mathrm{Hom}_{\mathcal{O}_{X_2}}(g^* \Omega_{X_1/S_1}, \mathcal{G}_2) = \mathrm{Ext}^0_{\mathcal{O}_{X_2}}(Lg^* NL_{X_1/S_1}, \mathcal{G}_2).
$$

Proof. The naive cotangent complex NL_{X_1/S_1} is defined in Modules, Definition 31.6. The equalities in the last statement of the lemma follow from the fact that g^* is adjoint to g_*, the fact that $H^0(NL_{X_1/S_1}) = \Omega_{X_1/S_1}$ (by construction of the naive cotangent complex) and the fact that Lg^* is the left derived functor of g^*. Thus we will work with the groups $\mathrm{Ext}^k_{\mathcal{O}_{X_2}}(Lg^* NL_{X_1/S_1}, \mathcal{G}_2)$, $k = 0, 1$ in the rest of the proof. We first argue that we can reduce to the case where the underlying topological spaces of all ringed spaces in the lemma is the same.

To do this, observe that $g^{-1} NL_{X_1/S_1}$ is equal to the naive cotangent complex of the homomorphism of sheaves of rings $g^{-1} f_1^{-1} \mathcal{O}_{S_1} \to g^{-1} \mathcal{O}_{X_1}$, see Modules, Lemma 31.3. Moreover, the degree 0 term of NL_{X_1/S_1} is a flat $\mathcal{O}_{X_1}$-module, hence the canonical map

$$
Lg^* NL_{X_1/S_1} \longrightarrow g^{-1} NL_{X_1/S_1} \otimes_{g^{-1} \mathcal{O}_{X_1}} \mathcal{O}_{X_2}
$$

induces an isomorphism on cohomology sheaves in degrees 0 and -1. Thus we may replace the Ext groups of the lemma with

$$\mathrm{Ext}^k_{g^{-1}\mathcal{O}_{X_1}}(g^{-1}\,NL_{X_1/S_1}, \mathcal{G}_2) = \mathrm{Ext}^k_{g^{-1}\mathcal{O}_{X_1}}(NL_{g^{-1}\mathcal{O}_{X_1}/g^{-1}f_1^{-1}\mathcal{O}_{S_1}}, \mathcal{G}_2)$$

The set of morphism of ringed spaces $X_2' \to X_1'$ fitting into (7.1.1) compatibly with ν is in one-to-one bijection with the set of homomorphisms of $g^{-1}f_1^{-1}\mathcal{O}_{S_1'}$-algebras $g^{-1}\mathcal{O}_{X_1'} \to \mathcal{O}_{X_2'}$ which are compatible with $f^\sharp$ and ν. In this way we see that we may assume we have a diagram (7.1.2) of sheaves on X and we are looking to find a homomorphism of sheaves of rings $\mathcal{O}_{X_1'} \to \mathcal{O}_{X_2'}$ fitting into it.

In the rest of the proof of the lemma we assume all underlying topological spaces are the same, i.e., we have a diagram (7.1.2) of sheaves on a space X and we are looking for homomorphisms of sheaves of rings $\mathcal{O}_{X_1'} \to \mathcal{O}_{X_2'}$ fitting into it. As ext groups we will use $\mathrm{Ext}^k_{\mathcal{O}_{X_1}}(NL_{\mathcal{O}_{X_1}/\mathcal{O}_{S_1}}, \mathcal{G}_2)$, $k = 0, 1$.

Step 1. Construction of the obstruction class. Consider the sheaf of sets

$$\mathcal{E} = \mathcal{O}_{X_1'} \times_{\mathcal{O}_{X_2}} \mathcal{O}_{X_2'}$$

This comes with a surjective map $\alpha : \mathcal{E} \to \mathcal{O}_{X_1}$ and hence we can use $NL(\alpha)$ instead of $NL_{\mathcal{O}_{X_1}/\mathcal{O}_{S_1}}$, see Modules, Lemma 31.2. Set

$$\mathcal{I}' = \mathrm{Ker}(\mathcal{O}_{S_1'}[\mathcal{E}] \to \mathcal{O}_{X_1}) \quad \text{and} \quad \mathcal{I} = \mathrm{Ker}(\mathcal{O}_{S_1}[\mathcal{E}] \to \mathcal{O}_{X_1})$$

There is a surjection $\mathcal{I}' \to \mathcal{I}$ whose kernel is $\mathcal{J}_1\mathcal{O}_{S_1'}[\mathcal{E}]$. We obtain two homomorphisms of $\mathcal{O}_{S_2'}$-algebras

$$a : \mathcal{O}_{S_1'}[\mathcal{E}] \to \mathcal{O}_{X_1'} \quad \text{and} \quad b : \mathcal{O}_{S_1'}[\mathcal{E}] \to \mathcal{O}_{X_2'}$$

which induce maps $a|_{\mathcal{I}'} : \mathcal{I}' \to \mathcal{G}_1$ and $b|_{\mathcal{I}'} : \mathcal{I}' \to \mathcal{G}_2$. Both a and b annihilate $(\mathcal{I}')^2$. Moreover a and b agree on $\mathcal{J}_1\mathcal{O}_{S_1'}[\mathcal{E}]$ as maps into $\mathcal{G}_2$ because the left hand square of (7.1.2) is commutative. Thus the difference $b|_{\mathcal{I}'} - \nu \circ a|_{\mathcal{I}'}$ induces a well defined $\mathcal{O}_{X_1}$-linear map

$$\xi : \mathcal{I}/\mathcal{I}^2 \longrightarrow \mathcal{G}_2$$

which sends the class of a local section f of $\mathcal{I}$ to $a(f') - \nu(b(f'))$ where f' is a lift of f to a local section of $\mathcal{I}'$. We let $[\xi] \in \mathrm{Ext}^1_{\mathcal{O}_{X_1}}(NL(\alpha), \mathcal{G}_2)$ be the image (see below).

Step 2. Vanishing of $[\xi]$ is necessary. Let us write $\Omega = \Omega_{\mathcal{O}_{S_1}[\mathcal{E}]/\mathcal{O}_{S_1}} \otimes_{\mathcal{O}_{S_1}[\mathcal{E}]} \mathcal{O}_{X_1}$. Observe that $NL(\alpha) = (\mathcal{I}/\mathcal{I}^2 \to \Omega)$ fits into a distinguished triangle

$$\Omega[0] \to NL(\alpha) \to \mathcal{I}/\mathcal{I}^2[1] \to \Omega[1]$$

Thus we see that $[\xi]$ is zero if and only if ξ is a composition $\mathcal{I}/\mathcal{I}^2 \to \Omega \to \mathcal{G}_2$ for some map $\Omega \to \mathcal{G}_2$. Suppose there exists a homomorphisms of sheaves of rings $\varphi : \mathcal{O}_{X_1'} \to \mathcal{O}_{X_2'}$ fitting into (7.1.2). In this case consider the map $\mathcal{O}_{S_1'}[\mathcal{E}] \to \mathcal{G}_2$, $f' \mapsto b(f') - \varphi(a(f'))$. A calculation shows this annihilates $\mathcal{J}_1\mathcal{O}_{S_1'}[\mathcal{E}]$ and induces a derivation $\mathcal{O}_{S_1}[\mathcal{E}] \to \mathcal{G}_2$. The resulting linear map $\Omega \to \mathcal{G}_2$ witnesses the fact that $[\xi] = 0$ in this case.

Step 3. Vanishing of $[\xi]$ is sufficient. Let $\theta : \Omega \to \mathcal{G}_2$ be a $\mathcal{O}_{X_1}$-linear map such that ξ is equal to $\theta \circ (\mathcal{I}/\mathcal{I}^2 \to \Omega)$. Then a calculation shows that

$$b + \theta \circ d : \mathcal{O}_{S_1'}[\mathcal{E}] \to \mathcal{O}_{X_2'}$$

annihilates $\mathcal{I}'$ and hence defines a map $\mathcal{O}_{X_1'} \to \mathcal{O}_{X_2'}$ fitting into (7.1.2).

Proof of (2) in the special case above. Omitted. Hint: This is exactly the same as the proof of (2) of Lemma 2.1. $\square$

08UB **Lemma 7.2.** *Let X be a topological space. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings. Let $\mathcal{G}$ be a $\mathcal{B}$-module. Let $\xi \in \mathrm{Ext}^1_{\mathcal{B}}(NL_{\mathcal{B}/\mathcal{A}}, \mathcal{G})$. There exists a map of sheaves of sets $\alpha : \mathcal{E} \to \mathcal{B}$ such that $\xi \in \mathrm{Ext}^1_{\mathcal{B}}(NL(\alpha), \mathcal{G})$ is the class of a map $\mathcal{I}/\mathcal{I}^2 \to \mathcal{G}$ (see proof for notation).*

Proof. Recall that given $\alpha : \mathcal{E} \to \mathcal{B}$ such that $\mathcal{A}[\mathcal{E}] \to \mathcal{B}$ is surjective with kernel $\mathcal{I}$ the complex $NL(\alpha) = (\mathcal{I}/\mathcal{I}^2 \to \Omega_{\mathcal{A}[\mathcal{E}]/\mathcal{A}} \otimes_{\mathcal{A}[\mathcal{E}]} \mathcal{B})$ is canonically isomorphic to $NL_{\mathcal{B}/\mathcal{A}}$, see Modules, Lemma 31.2. Observe moreover, that $\Omega = \Omega_{\mathcal{A}[\mathcal{E}]/\mathcal{A}} \otimes_{\mathcal{A}[\mathcal{E}]} \mathcal{B}$ is the sheaf associated to the presheaf $U \mapsto \bigoplus_{e \in \mathcal{E}(U)} \mathcal{B}(U)$. In other words, Ω is the free $\mathcal{B}$-module on the sheaf of sets $\mathcal{E}$ and in particular there is a canonical map $\mathcal{E} \to \Omega$.

Having said this, pick some $\mathcal{E}$ (for example $\mathcal{E} = \mathcal{B}$ as in the definition of the naive cotangent complex). The obstruction to writing ξ as the class of a map $\mathcal{I}/\mathcal{I}^2 \to \mathcal{G}$ is an element in $\mathrm{Ext}^1_{\mathcal{B}}(\Omega, \mathcal{G})$. Say this is represented by the extension $0 \to \mathcal{G} \to \mathcal{H} \to \Omega \to 0$ of $\mathcal{B}$-modules. Consider the sheaf of sets $\mathcal{E}' = \mathcal{E} \times_{\Omega} \mathcal{H}$ which comes with an induced map $\alpha' : \mathcal{E}' \to \mathcal{B}$. Let $\mathcal{I}' = \mathrm{Ker}(\mathcal{A}[\mathcal{E}'] \to \mathcal{B})$ and $\Omega' = \Omega_{\mathcal{A}[\mathcal{E}']/\mathcal{A}} \otimes_{\mathcal{A}[\mathcal{E}']} \mathcal{B}$. The pullback of ξ under the quasi-isomorphism $NL(\alpha') \to NL(\alpha)$ maps to zero in $\mathrm{Ext}^1_{\mathcal{B}}(\Omega', \mathcal{G})$ because the pullback of the extension $\mathcal{H}$ by the map $\Omega' \to \Omega$ is split as Ω' is the free $\mathcal{B}$-module on the sheaf of sets $\mathcal{E}'$ and since by construction there is a commutative diagram

$$\begin{array}{ccc} \mathcal{E}' & \longrightarrow & \mathcal{E} \\ \downarrow & & \downarrow \\ \mathcal{H} & \longrightarrow & \Omega \end{array}$$
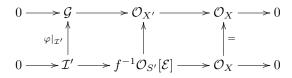
This finishes the proof. $\square$

08UC **Lemma 7.3.** *If there exists a solution to (7.0.1), then the set of isomorphism classes of solutions is principal homogeneous under $\mathrm{Ext}^1_{\mathcal{O}_X}(NL_{X/S}, \mathcal{G})$.*

Proof. We observe right away that given two solutions X'_1 and X'_2 to (7.0.1) we obtain by Lemma 7.1 an obstruction element $o(X'_1, X'_2) \in \mathrm{Ext}^1_{\mathcal{O}_X}(NL_{X/S}, \mathcal{G})$ to the existence of a map $X'_1 \to X'_2$. Clearly, this element is the obstruction to the existence of an isomorphism, hence separates the isomorphism classes. To finish the proof it therefore suffices to show that given a solution X' and an element $\xi \in \mathrm{Ext}^1_{\mathcal{O}_X}(NL_{X/S}, \mathcal{G})$ we can find a second solution X'_ξ such that $o(X', X'_\xi) = \xi$.

Pick $\alpha : \mathcal{E} \to \mathcal{O}_X$ as in Lemma 7.2 for the class ξ. Consider the surjection $f^{-1}\mathcal{O}_S[\mathcal{E}] \to \mathcal{O}_X$ with kernel $\mathcal{I}$ and corresponding naive cotangent complex $NL(\alpha) = (\mathcal{I}/\mathcal{I}^2 \to \Omega_{f^{-1}\mathcal{O}_S[\mathcal{E}]/f^{-1}\mathcal{O}_S} \otimes_{f^{-1}\mathcal{O}_S[\mathcal{E}]} \mathcal{O}_X)$. By the lemma ξ is the class of a morphism $\delta : \mathcal{I}/\mathcal{I}^2 \to \mathcal{G}$. After replacing $\mathcal{E}$ by $\mathcal{E} \times_{\mathcal{O}_X} \mathcal{O}_{X'}$ we may also assume that α factors through a map $\alpha' : \mathcal{E} \to \mathcal{O}_{X'}$.

These choices determine an $f^{-1}\mathcal{O}_{S'}$-algebra map $\varphi : \mathcal{O}_{S'}[\mathcal{E}] \to \mathcal{O}_{X'}$. Let $\mathcal{I}' = \mathrm{Ker}(\varphi)$. Observe that φ induces a map $\varphi|_{\mathcal{I}'} : \mathcal{I}' \to \mathcal{G}$ and that $\mathcal{O}_{X'}$ is the pushout,

as in the following diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{O}_{X'} & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 \\
& & \uparrow{\scriptstyle \varphi|_{\mathcal{I}'}} & & \uparrow & & \uparrow{\scriptstyle =} & & \\
0 & \longrightarrow & \mathcal{I}' & \longrightarrow & f^{-1}\mathcal{O}_{S'}[\mathcal{E}] & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0
\end{array}
$$

Let $\psi : \mathcal{I}' \to \mathcal{G}$ be the sum of the map $\varphi|_{\mathcal{I}'}$ and the composition

$$
\mathcal{I}' \to \mathcal{I}'/(\mathcal{I}')^2 \to \mathcal{I}/\mathcal{I}^2 \xrightarrow{\delta} \mathcal{G}.
$$

Then the pushout along ψ is an other ring extension $\mathcal{O}_{X'_\xi}$ fitting into a diagram as above. A calculation (omitted) shows that $o(X', X'_\xi) = \xi$ as desired. $\qquad \square$

0GPZ **Lemma 7.4.** *Let $f : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$ be a morphism of ringed spaces. Let $\mathcal{G}$ be a $\mathcal{O}_X$-module. The set of isomorphism classes of extensions of $f^{-1}\mathcal{O}_S$-algebras*

$$
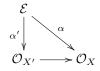0 \to \mathcal{G} \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0
$$

where $\mathcal{G}$ is an ideal of square zero[1] *is canonically bijective to $\operatorname{Ext}^1_{\mathcal{O}_X}(NL_{X/S}, \mathcal{G})$.*

Proof. To prove this we apply the previous results to the case where (7.0.1) is given by the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{G} & \longrightarrow & ? & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 \\
& & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & \mathcal{O}_S & \xrightarrow{\text{id}} & \mathcal{O}_S & \longrightarrow & 0
\end{array}
$$

Thus our lemma follows from Lemma 7.3 and the fact that there exists a solution, namely $\mathcal{G} \oplus \mathcal{O}_X$. (See remark below for a direct construction of the bijection.) $\quad \square$

0GQ0 **Remark 7.5.** Let $f : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$ and $\mathcal{G}$ be as in Lemma 7.4. Consider an extension $0 \to \mathcal{G} \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0$ as in the lemma. We can choose a sheaf of sets $\mathcal{E}$ and a commutative diagram

$$
\begin{array}{ccc}
& \mathcal{E} & \\
{\scriptstyle \alpha'}\downarrow & \searrow{\scriptstyle \alpha} & \\
\mathcal{O}_{X'} & \longrightarrow & \mathcal{O}_X
\end{array}
$$

such that $f^{-1}\mathcal{O}_S[\mathcal{E}] \to \mathcal{O}_X$ is surjective with kernel $\mathcal{J}$. (For example you can take any sheaf of sets surjecting onto $\mathcal{O}_{X'}$.) Then

$$
NL_{X/S} \cong NL(\alpha) = \left(\mathcal{J}/\mathcal{J}^2 \longrightarrow \Omega_{f^{-1}\mathcal{O}_S[\mathcal{E}]/f^{-1}\mathcal{O}_S} \otimes_{f^{-1}\mathcal{O}_S[\mathcal{E}]} \mathcal{O}_X \right)
$$

See Modules, Section 31 and in particular Lemma 31.2. Of course α' determines a map $f^{-1}\mathcal{O}_S[\mathcal{E}] \to \mathcal{O}_{X'}$ which in turn determines a map

$$
\mathcal{J}/\mathcal{J}^2 \longrightarrow \mathcal{G}
$$

which in turn determines the element of $\operatorname{Ext}^1_{\mathcal{O}_X}(NL(\alpha), \mathcal{G}) = \operatorname{Ext}^1_{\mathcal{O}_X}(NL_{X/S}, \mathcal{G})$ corresponding to $\mathcal{O}_{X'}$ by the bijection of the lemma.

---

[1]In other words, the set of isomorphism classes of first order thickenings $i : X \to X'$ over S endowed with an isomorphism $\mathcal{G} \to \operatorname{Ker}(i^\sharp)$ of $\mathcal{O}_X$-modules.

0GQ1 **Lemma 7.6.** *Let $f : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$ and $g : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ be morphisms of ringed spaces. Let $\mathcal{F}$ be a $\mathcal{O}_X$-module. Let $\mathcal{G}$ be a $\mathcal{O}_Y$-module. Let $c : \mathcal{F} \to \mathcal{G}$ be a g-map. Finally, consider*

(a) *$0 \to \mathcal{F} \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0$ an extension of $f^{-1}\mathcal{O}_S$-algebras corresponding to $\xi \in \mathrm{Ext}^1_{\mathcal{O}_X}(NL_{X/S}, \mathcal{F})$, and*

(b) *$0 \to \mathcal{G} \to \mathcal{O}_{Y'} \to \mathcal{O}_Y \to 0$ an extension of $g^{-1}f^{-1}\mathcal{O}_S$-algebras corresponding to $\zeta \in \mathrm{Ext}^1_{\mathcal{O}_Y}(NL_{Y/S}, \mathcal{G})$.*

See Lemma 7.4. Then there is an S-morphism $g' : Y' \to X'$ compatible with g and c if and only if ξ and ζ map to the same element of $\mathrm{Ext}^1_{\mathcal{O}_Y}(Lg^* NL_{X/S}, \mathcal{G})$.

Proof. The stament makes sense as we have the maps

$$\mathrm{Ext}^1_{\mathcal{O}_X}(NL_{X/S}, \mathcal{F}) \to \mathrm{Ext}^1_{\mathcal{O}_Y}(Lg^* NL_{X/S}, Lg^*\mathcal{F}) \to \mathrm{Ext}^1_{\mathcal{O}_Y}(Lg^* NL_{X/S}, \mathcal{G})$$

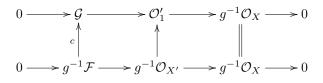using the map $Lg^*\mathcal{F} \to g^*\mathcal{F} \xrightarrow{c} \mathcal{G}$ and

$$\mathrm{Ext}^1_{\mathcal{O}_Y}(NL_{Y/S}, \mathcal{G}) \to \mathrm{Ext}^1_{\mathcal{O}_Y}(Lg^* NL_{X/S}, \mathcal{G})$$

using the map $Lg^* NL_{X/S} \to NL_{Y/S}$. The statement of the lemma can be deduced from Lemma 7.1 applied to the diagram
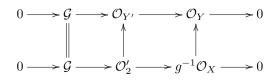

and a compatibility between the constructions in the proofs of Lemmas 7.4 and 7.1 whose statement and proof we omit. (See remark below for a direct argument.) $\square$

0GQ2 **Remark 7.7.** Let $f : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$, $g : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$, $\mathcal{F}$, $\mathcal{G}$, $c : \mathcal{F} \to \mathcal{G}$, $0 \to \mathcal{F} \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0$, $\xi \in \mathrm{Ext}^1_{\mathcal{O}_X}(NL_{X/S}, \mathcal{F})$, $0 \to \mathcal{G} \to \mathcal{O}_{Y'} \to \mathcal{O}_Y \to 0$, and $\zeta \in \mathrm{Ext}^1_{\mathcal{O}_Y}(NL_{Y/S}, \mathcal{G})$ be as in Lemma 7.6. Using pushout along $c : g^{-1}\mathcal{F} \to \mathcal{G}$ we can construct an extension

Using pullback along $g^\sharp : g^{-1}\mathcal{O}_X \to \mathcal{O}_Y$ we can construct an extension

A diagram chase tells us that there exists an S-morphism $Y' \to X'$ compatible with g and c if and only if $\mathcal{O}'_1$ is isomorphic to $\mathcal{O}'_2$ as $g^{-1}f^{-1}\mathcal{O}_S$-algebra extensions of $g^{-1}\mathcal{O}_X$ by $\mathcal{G}$. By Lemma 7.4 these extensions are classified by the LHS of

$$\mathrm{Ext}^1_{g^{-1}\mathcal{O}_X}(NL_{g^{-1}\mathcal{O}_X/g^{-1}f^{-1}\mathcal{O}_S}, \mathcal{G}) = \mathrm{Ext}^1_{\mathcal{O}_Y}(Lg^* NL_{X/S}, \mathcal{G})$$
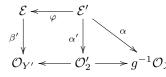
Here the equality comes from tensor-hom adjunction and the equalities

$$NL_{g^{-1}\mathcal{O}_X/g^{-1}f^{-1}\mathcal{O}_S} = g^{-1} NL_{X/S} \quad \text{and} \quad Lg^* NL_{X/S} = g^{-1} NL_{X/S} \otimes^{\mathbf{L}}_{g^{-1}\mathcal{O}_X} \mathcal{O}_Y$$

For the first of these see Modules, Lemma 31.3; the second follows from the definition of derived pullback. Thus, in order to see that Lemma 7.6 is true, it suffices to show that $\mathcal{O}'_1$ corresponds to the image of ξ and that $\mathcal{O}'_2$ correspond to the image of ζ. The correspondence between ξ and $\mathcal{O}'_1$ is immediate from the construction of the class ξ in Remark 7.5. For the correspondence between ζ and $\mathcal{O}'_2$, we first choose a commutative diagram

$$\begin{array}{ccc}
 & \mathcal{E} & \\
\beta' \downarrow & & \searrow \beta \\
\mathcal{O}_{Y'} & \longrightarrow & \mathcal{O}_Y
\end{array}$$

such that $g^{-1}f^{-1}\mathcal{O}_S[\mathcal{E}] \to \mathcal{O}_Y$ is surjective with kernel $\mathcal{K}$. Next choose a commutative diagram

$$\begin{array}{ccccc}
\mathcal{E} & \xleftarrow{\varphi} & \mathcal{E}' & & \\
\beta' \downarrow & & \alpha' \downarrow & \searrow \alpha & \\
\mathcal{O}_{Y'} & \longleftarrow & \mathcal{O}'_2 & \longrightarrow & g^{-1}\mathcal{O}_X
\end{array}$$

such that $g^{-1}f^{-1}\mathcal{O}_S[\mathcal{E}'] \to g^{-1}\mathcal{O}_X$ is surjective with kernel $\mathcal{J}$. (For example just take $\mathcal{E}' = \mathcal{E} \amalg \mathcal{O}'_2$ as a sheaf of sets.) The map φ induces a map of complexes $NL(\alpha) \to NL(\beta)$ (notation as in Modules, Section 31) and in particular $\bar{\varphi} : \mathcal{J}/\mathcal{J}^2 \to \mathcal{K}/\mathcal{K}^2$. Then $NL(\alpha) \cong NL_{Y/S}$ and $NL(\beta) \cong NL_{g^{-1}\mathcal{O}_X/g^{-1}f^{-1}\mathcal{O}_S}$ and the map of complexes $NL(\alpha) \to NL(\beta)$ represents the map $Lg^* NL_{X/S} \to NL_{Y/S}$ used in the statement of Lemma 7.6 (see first part of its proof). Now ζ corresponds to the class of the map $\mathcal{K}/\mathcal{K}^2 \to \mathcal{G}$ induced by β', see Remark 7.5. Similarly, the extension $\mathcal{O}'_2$ corresponds to the map $\mathcal{J}/\mathcal{J}^2 \to \mathcal{G}$ induced by α'. The commutative diagram above shows that this map is the composition of the map $\mathcal{K}/\mathcal{K}^2 \to \mathcal{G}$ induced by β' with the map $\bar{\varphi} : \mathcal{J}/\mathcal{J}^2 \to \mathcal{K}/\mathcal{K}^2$. This proves the compatibility we were looking for.

0GQ3 **Lemma 7.8.** *Let $t : (S, \mathcal{O}_S) \to (S', \mathcal{O}_{S'})$, $\mathcal{J} = \mathrm{Ker}(t^\sharp)$, $f : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$, $\mathcal{G}$, and $c : \mathcal{J} \to \mathcal{G}$ be as in (7.0.1). Denote $\xi \in \mathrm{Ext}^1_{\mathcal{O}_S}(NL_{S/S'}, \mathcal{J})$ the element corresponding to the extension $\mathcal{O}_{S'}$ of $\mathcal{O}_S$ by $\mathcal{J}$ via Lemma 7.4. The set of isomorphism classes of solutions is canonically bijective to the fibre of*

$$\mathrm{Ext}^1_{\mathcal{O}_X}(NL_{X/S'}, \mathcal{G}) \to \mathrm{Ext}^1_{\mathcal{O}_X}(Lf^* NL_{S/S'}, \mathcal{G})$$

over the image of ξ.

Proof. By Lemma 7.4 applied to $X \to S'$ and the $\mathcal{O}_X$-module $\mathcal{G}$ we see that elements ζ of $\mathrm{Ext}^1_{\mathcal{O}_X}(NL_{X/S'}, \mathcal{G})$ parametrize extensions $0 \to \mathcal{G} \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0$ of $f^{-1}\mathcal{O}_{S'}$-algebras. By Lemma 7.6 applied to $X \to S \to S'$ and $c : \mathcal{J} \to \mathcal{G}$ we see that there is an S'-morphism $X' \to S'$ compatible with c and $f : X \to S$ if and

only if ζ maps to ξ. Of course this is the same thing as saying $\mathcal{O}_{X'}$ is a solution of (7.0.1). $\square$

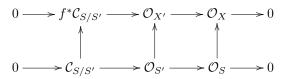0GQ4 **Remark 7.9.** In the situation of Lemma 7.8 we have maps of complexes

$$Lf^* NL_{S'/S} \to NL_{X/S'} \to NL_{X/S}$$

These maps are closed to forming a distinguished triangle, see Modules, Lemma 31.7. If it were a distinguished triangle we would conclude that the image of ξ in $\mathrm{Ext}^2_{\mathcal{O}_X}(NL_{X/S}, \mathcal{G})$ would be the obstruction to the existence of a solution to (7.0.1).

8. Deformations of schemes

0D13 In this section we spell out what the results in Section 7 mean for deformations of schemes.

0D14 **Lemma 8.1.** *Let $S \subset S'$ be a first order thickening of schemes. Let $f : X \to S$ be a flat morphism of schemes. If there exists a flat morphism $f' : X' \to S'$ of schemes and an isomorphism $a : X \to X' \times_{S'} S$ over S, then*

(1) *the set of isomorphism classes of pairs $(f' : X' \to S', a)$ is principal homogeneous under $\mathrm{Ext}^1_{\mathcal{O}_X}(NL_{X/S}, f^*\mathcal{C}_{S/S'})$, and*
(2) *the set of automorphisms of $\varphi : X' \to X'$ over S' which reduce to the identity on $X' \times_{S'} S$ is $\mathrm{Ext}^0_{\mathcal{O}_X}(NL_{X/S}, f^*\mathcal{C}_{S/S'})$.*

Proof. First we observe that thickenings of schemes as defined in More on Morphisms, Section 2 are the same things as morphisms of schemes which are thickenings in the sense of Section 3. We may think of X as a closed subscheme of X' so that $(f, f') : (X \subset X') \to (S \subset S')$ is a morphism of first order thickenings. Then we see from More on Morphisms, Lemma 10.1 (or from the more general Lemma 5.2) that the ideal sheaf of X in X' is equal to $f^*\mathcal{C}_{S/S'}$. Hence we have a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & f^*\mathcal{C}_{S/S'} & \longrightarrow & \mathcal{O}_{X'} & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 \\
& & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{C}_{S/S'} & \longrightarrow & \mathcal{O}_{S'} & \longrightarrow & \mathcal{O}_S & \longrightarrow & 0
\end{array}
$$

where the vertical arrows are f-maps; please compare with (7.0.1). Thus part (1) follows from Lemma 7.3 and part (2) from part (2) of Lemma 7.1. (Note that $NL_{X/S}$ as defined for a morphism of schemes in More on Morphisms, Section 13 agrees with $NL_{X/S}$ as used in Section 7.) $\square$

9. Thickenings of ringed topoi

08M6 This section is the analogue of Section 3 for ringed topoi. In the following few sections we will use the following notions:

(1) A sheaf of ideals $\mathcal{I} \subset \mathcal{O}'$ on a ringed topos $(Sh(\mathcal{D}), \mathcal{O}')$ is *locally nilpotent* if any local section of $\mathcal{I}$ is locally nilpotent.
(2) A *thickening* of ringed topoi is a morphism $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ of ringed topoi such that
  (a) i_* is an equivalence $Sh(\mathcal{C}) \to Sh(\mathcal{D})$,
  (b) the map $i^\sharp : \mathcal{O}' \to i_*\mathcal{O}$ is surjective, and
  (c) the kernel of $i^\sharp$ is a locally nilpotent sheaf of ideals.

(3) A *first order thickening* of ringed topoi is a thickening $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ of ringed topoi such that $\mathrm{Ker}(i^\sharp)$ has square zero.

(4) It is clear how to define *morphisms of thickenings of ringed topoi, morphisms of thickenings of ringed topoi over a base ringed topos*, etc.

If $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ is a thickening of ringed topoi then we identify the underlying topoi and think of $\mathcal{O}$, $\mathcal{O}'$, and $\mathcal{I} = \mathrm{Ker}(i^\sharp)$ as sheaves on $\mathcal{C}$. We obtain a short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}' \to \mathcal{O} \to 0$$

of $\mathcal{O}'$-modules. By Modules on Sites, Lemma 25.1 the category of $\mathcal{O}$-modules is equivalent to the category of $\mathcal{O}'$-modules annihilated by $\mathcal{I}$. In particular, if i is a first order thickening, then $\mathcal{I}$ is a $\mathcal{O}$-module.

08M7 **Situation 9.1.** A morphism of thickenings of ringed topoi (f, f') is given by a commutative diagram

08M8 (9.1.1)
$$
\begin{array}{ccc}
(Sh(\mathcal{C}), \mathcal{O}) & \xrightarrow{\ i\ } & (Sh(\mathcal{D}), \mathcal{O}') \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\
(Sh(\mathcal{B}), \mathcal{O}_{\mathcal{B}}) & \xrightarrow{\ t\ } & (Sh(\mathcal{B}'), \mathcal{O}_{\mathcal{B}'})
\end{array}
$$

of ringed topoi whose horizontal arrows are thickenings. In this situation we set $\mathcal{I} = \mathrm{Ker}(i^\sharp) \subset \mathcal{O}'$ and $\mathcal{J} = \mathrm{Ker}(t^\sharp) \subset \mathcal{O}_{\mathcal{B}'}$. As $f = f'$ on underlying topoi we will identify the pullback functors f^{-1} and $(f')^{-1}$. Observe that $(f')^\sharp : f^{-1}\mathcal{O}_{\mathcal{B}'} \to \mathcal{O}'$ induces in particular a map $f^{-1}\mathcal{J} \to \mathcal{I}$ and therefore a map of $\mathcal{O}'$-modules

$$(f')^* \mathcal{J} \longrightarrow \mathcal{I}$$

If i and t are first order thickenings, then $(f')^* \mathcal{J} = f^* \mathcal{J}$ and the map above becomes a map $f^* \mathcal{J} \to \mathcal{I}$.

08M9 **Definition 9.2.** In Situation 9.1 we say that (f, f') is a *strict morphism of thickenings* if the map $(f')^* \mathcal{J} \longrightarrow \mathcal{I}$ is surjective.

10. Modules on first order thickenings of ringed topoi

08MA In this section we discuss some preliminaries to the deformation theory of modules. Let $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ be a first order thickening of ringed topoi. We will freely use the notation introduced in Section 9, in particular we will identify the underlying topological topoi. In this section we consider short exact sequences

08MB (10.0.1)
$$0 \to \mathcal{K} \to \mathcal{F}' \to \mathcal{F} \to 0$$

of $\mathcal{O}'$-modules, where $\mathcal{F}$, $\mathcal{K}$ are $\mathcal{O}$-modules and $\mathcal{F}'$ is an $\mathcal{O}'$-module. In this situation we have a canonical $\mathcal{O}$-module map

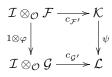$$c_{\mathcal{F}'} : \mathcal{I} \otimes_{\mathcal{O}} \mathcal{F} \longrightarrow \mathcal{K}$$

where $\mathcal{I} = \mathrm{Ker}(i^\sharp)$. Namely, given local sections f of $\mathcal{I}$ and s of $\mathcal{F}$ we set $c_{\mathcal{F}'}(f \otimes s) = fs'$ where s' is a local section of $\mathcal{F}'$ lifting s.

08MC **Lemma 10.1.** *Let $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ be a first order thickening of ringed topoi. Assume given extensions*

$$0 \to \mathcal{K} \to \mathcal{F}' \to \mathcal{F} \to 0 \quad and \quad 0 \to \mathcal{L} \to \mathcal{G}' \to \mathcal{G} \to 0$$

as in (10.0.1) and maps $\varphi : \mathcal{F} \to \mathcal{G}$ and $\psi : \mathcal{K} \to \mathcal{L}$.

(1) *If there exists an $\mathcal{O}'$-module map $\varphi' : \mathcal{F}' \to \mathcal{G}'$ compatible with φ and ψ, then the diagram*

$$\begin{array}{ccc} \mathcal{I} \otimes_{\mathcal{O}} \mathcal{F} & \xrightarrow{\ c_{\mathcal{F}'}\ } & \mathcal{K} \\ {\scriptstyle 1 \otimes \varphi} \downarrow & & \downarrow {\scriptstyle \psi} \\ \mathcal{I} \otimes_{\mathcal{O}} \mathcal{G} & \xrightarrow{\ c_{\mathcal{G}'}\ } & \mathcal{L} \end{array}$$

is commutative.

(2) *The set of $\mathcal{O}'$-module maps $\varphi' : \mathcal{F}' \to \mathcal{G}'$ compatible with φ and ψ is, if nonempty, a principal homogeneous space under $\mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{L})$.*

Proof. Part (1) is immediate from the description of the maps. For (2), if φ' and φ'' are two maps $\mathcal{F}' \to \mathcal{G}'$ compatible with φ and ψ, then $\varphi' - \varphi''$ factors as

$$\mathcal{F}' \to \mathcal{F} \to \mathcal{L} \to \mathcal{G}'$$

The map in the middle comes from a unique element of $\mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{L})$ by Modules on Sites, Lemma 25.1. Conversely, given an element α of this group we can add the composition (as displayed above with α in the middle) to φ'. Some details omitted. $\qquad\square$

08MD **Lemma 10.2.** *Let $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ be a first order thickening of ringed topoi. Assume given extensions*

$$0 \to \mathcal{K} \to \mathcal{F}' \to \mathcal{F} \to 0 \quad and \quad 0 \to \mathcal{L} \to \mathcal{G}' \to \mathcal{G} \to 0$$

as in (10.0.1) and maps $\varphi : \mathcal{F} \to \mathcal{G}$ and $\psi : \mathcal{K} \to \mathcal{L}$. Assume the diagram

$$\begin{array}{ccc} \mathcal{I} \otimes_{\mathcal{O}} \mathcal{F} & \xrightarrow{\ c_{\mathcal{F}'}\ } & \mathcal{K} \\ {\scriptstyle 1 \otimes \varphi} \downarrow & & \downarrow {\scriptstyle \psi} \\ \mathcal{I} \otimes_{\mathcal{O}} \mathcal{G} & \xrightarrow{\ c_{\mathcal{G}'}\ } & \mathcal{L} \end{array}$$

is commutative. Then there exists an element

$$o(\varphi, \psi) \in \mathrm{Ext}^1_{\mathcal{O}}(\mathcal{F}, \mathcal{L})$$

whose vanishing is a necessary and sufficient condition for the existence of a map $\varphi' : \mathcal{F}' \to \mathcal{G}'$ compatible with φ and ψ.

Proof. We can construct explicitly an extension

$$0 \to \mathcal{L} \to \mathcal{H} \to \mathcal{F} \to 0$$

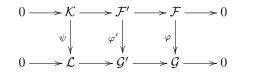by taking $\mathcal{H}$ to be the cohomology of the complex

$$\mathcal{K} \xrightarrow{\ 1, -\psi\ } \mathcal{F}' \oplus \mathcal{G}' \xrightarrow{\ \varphi, 1\ } \mathcal{G}$$

in the middle (with obvious notation). A calculation with local sections using the assumption that the diagram of the lemma commutes shows that $\mathcal{H}$ is annihilated by $\mathcal{I}$. Hence $\mathcal{H}$ defines a class in

$$\mathrm{Ext}^1_{\mathcal{O}}(\mathcal{F}, \mathcal{L}) \subset \mathrm{Ext}^1_{\mathcal{O}'}(\mathcal{F}, \mathcal{L})$$

Finally, the class of $\mathcal{H}$ is the difference of the pushout of the extension $\mathcal{F}'$ via ψ and the pullback of the extension $\mathcal{G}'$ via φ (calculations omitted). Thus the vanishing

of the class of $\mathcal{H}$ is equivalent to the existence of a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\varphi'} & & \downarrow{\scriptstyle\varphi} & & \\
0 & \longrightarrow & \mathcal{L} & \longrightarrow & \mathcal{G}' & \longrightarrow & \mathcal{G} & \longrightarrow & 0
\end{array}
$$

as desired. $\qquad\qquad\square$

08ME   **Lemma 10.3.** *Let $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ be a first order thickening of ringed topoi. Assume given $\mathcal{O}$-modules $\mathcal{F}$, $\mathcal{K}$ and an $\mathcal{O}$-linear map $c : \mathcal{I} \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{K}$. If there exists a sequence (10.0.1) with $c_{\mathcal{F}'} = c$ then the set of isomorphism classes of these extensions is principal homogeneous under $\mathrm{Ext}^1_{\mathcal{O}}(\mathcal{F}, \mathcal{K})$.*

Proof. Assume given extensions

$$0 \to \mathcal{K} \to \mathcal{F}'_1 \to \mathcal{F} \to 0 \quad \text{and} \quad 0 \to \mathcal{K} \to \mathcal{F}'_2 \to \mathcal{F} \to 0$$

with $c_{\mathcal{F}'_1} = c_{\mathcal{F}'_2} = c$. Then the difference (in the extension group, see Homology, Section 6) is an extension

$$0 \to \mathcal{K} \to \mathcal{E} \to \mathcal{F} \to 0$$

where $\mathcal{E}$ is annihilated by $\mathcal{I}$ (local computation omitted). Hence the sequence is an extension of $\mathcal{O}$-modules, see Modules on Sites, Lemma 25.1. Conversely, given such an extension $\mathcal{E}$ we can add the extension $\mathcal{E}$ to the $\mathcal{O}'$-extension $\mathcal{F}'$ without affecting the map $c_{\mathcal{F}'}$. Some details omitted. $\qquad\square$

08MF   **Lemma 10.4.** *Let $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ be a first order thickening of ringed topoi. Assume given $\mathcal{O}$-modules $\mathcal{F}$, $\mathcal{K}$ and an $\mathcal{O}$-linear map $c : \mathcal{I} \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{K}$. Then there exists an element*

$$o(\mathcal{F}, \mathcal{K}, c) \in \mathrm{Ext}^2_{\mathcal{O}}(\mathcal{F}, \mathcal{K})$$

whose vanishing is a necessary and sufficient condition for the existence of a sequence (10.0.1) with $c_{\mathcal{F}'} = c$.
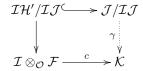
Proof. We first show that if $\mathcal{K}$ is an injective $\mathcal{O}$-module, then there does exist a sequence (10.0.1) with $c_{\mathcal{F}'} = c$. To do this, choose a flat $\mathcal{O}'$-module $\mathcal{H}'$ and a surjection $\mathcal{H}' \to \mathcal{F}$ (Modules on Sites, Lemma 28.8). Let $\mathcal{J} \subset \mathcal{H}'$ be the kernel. Since $\mathcal{H}'$ is flat we have

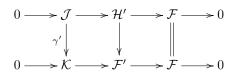$$\mathcal{I} \otimes_{\mathcal{O}'} \mathcal{H}' = \mathcal{I}\mathcal{H}' \subset \mathcal{J} \subset \mathcal{H}'$$

Observe that the map

$$\mathcal{I}\mathcal{H}' = \mathcal{I} \otimes_{\mathcal{O}'} \mathcal{H}' \longrightarrow \mathcal{I} \otimes_{\mathcal{O}'} \mathcal{F} = \mathcal{I} \otimes_{\mathcal{O}} \mathcal{F}$$

annihilates $\mathcal{I}\mathcal{J}$. Namely, if f is a local section of $\mathcal{I}$ and s is a local section of $\mathcal{H}$, then fs is mapped to $f \otimes \bar{s}$ where $\bar{s}$ is the image of s in $\mathcal{F}$. Thus we obtain

$$
\begin{array}{ccc}
\mathcal{I}\mathcal{H}'/\mathcal{I}\mathcal{J} & \lhook\joinrel\longrightarrow & \mathcal{J}/\mathcal{I}\mathcal{J} \\
\downarrow & & \downarrow{\scriptstyle\gamma} \\
\mathcal{I} \otimes_{\mathcal{O}} \mathcal{F} & \xrightarrow{\ c\ } & \mathcal{K}
\end{array}
$$

a diagram of $\mathcal{O}$-modules. If $\mathcal{K}$ is injective as an $\mathcal{O}$-module, then we obtain the dotted arrow. Denote $\gamma' : \mathcal{J} \to \mathcal{K}$ the composition of γ with $\mathcal{J} \to \mathcal{J}/\mathcal{I}\mathcal{J}$. A local calculation shows the pushout

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{J} & \longrightarrow & \mathcal{H}' & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\
& & \gamma' \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{F} & \longrightarrow & 0
\end{array}
$$

is a solution to the problem posed by the lemma.

General case. Choose an embedding $\mathcal{K} \subset \mathcal{K}'$ with $\mathcal{K}'$ an injective $\mathcal{O}$-module. Let $\mathcal{Q}$ be the quotient, so that we have an exact sequence

$$0 \to \mathcal{K} \to \mathcal{K}' \to \mathcal{Q} \to 0$$

Denote $c' : \mathcal{I} \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{K}'$ be the composition. By the paragraph above there exists a sequence

$$0 \to \mathcal{K}' \to \mathcal{E}' \to \mathcal{F} \to 0$$

as in (10.0.1) with $c_{\mathcal{E}'} = c'$. Note that c' composed with the map $\mathcal{K}' \to \mathcal{Q}$ is zero, hence the pushout of $\mathcal{E}'$ by $\mathcal{K}' \to \mathcal{Q}$ is an extension

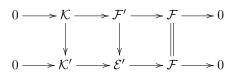$$0 \to \mathcal{Q} \to \mathcal{D}' \to \mathcal{F} \to 0$$

as in (10.0.1) with $c_{\mathcal{D}'} = 0$. This means exactly that $\mathcal{D}'$ is annihilated by $\mathcal{I}$, in other words, the $\mathcal{D}'$ is an extension of $\mathcal{O}$-modules, i.e., defines an element

$$o(\mathcal{F}, \mathcal{K}, c) \in \mathrm{Ext}^1_{\mathcal{O}}(\mathcal{F}, \mathcal{Q}) = \mathrm{Ext}^2_{\mathcal{O}}(\mathcal{F}, \mathcal{K})$$

(the equality holds by the long exact cohomology sequence associated to the exact sequence above and the vanishing of higher ext groups into the injective module $\mathcal{K}'$). If $o(\mathcal{F}, \mathcal{K}, c) = 0$, then we can choose a splitting $s : \mathcal{F} \to \mathcal{D}'$ and we can set

$$\mathcal{F}' = \mathrm{Ker}(\mathcal{E}' \to \mathcal{D}'/s(\mathcal{F}))$$

so that we obtain the following diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{K}' & \longrightarrow & \mathcal{E}' & \longrightarrow & \mathcal{F} & \longrightarrow & 0
\end{array}
$$

with exact rows which shows that $c_{\mathcal{F}'} = c$. Conversely, if $\mathcal{F}'$ exists, then the pushout of $\mathcal{F}'$ by the map $\mathcal{K} \to \mathcal{K}'$ is isomorphic to $\mathcal{E}'$ by Lemma 10.3 and the vanishing of higher ext groups into the injective module $\mathcal{K}'$. This gives a diagram as above, which implies that $\mathcal{D}'$ is split as an extension, i.e., the class $o(\mathcal{F}, \mathcal{K}, c)$ is zero. $\square$

08MG **Remark 10.5.** Let $(Sh(\mathcal{C}), \mathcal{O})$ be a ringed topos. A first order thickening $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ is said to be *trivial* if there exists a morphism of ringed topoi $\pi : (Sh(\mathcal{D}), \mathcal{O}') \to (Sh(\mathcal{C}), \mathcal{O})$ which is a left inverse to i. The choice of such a morphism π is called a *trivialization* of the first order thickening. Given π we obtain a splitting

08MH (10.5.1) $$\mathcal{O}' = \mathcal{O} \oplus \mathcal{I}$$

as sheaves of algebras on $\mathcal{C}$ by using $\pi^\sharp$ to split the surjection $\mathcal{O}' \to \mathcal{O}$. Conversely, such a splitting determines a morphism π. The category of trivialized first order thickenings of $(Sh(\mathcal{C}), \mathcal{O})$ is equivalent to the category of $\mathcal{O}$-modules.

08MI **Remark 10.6.** Let $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ be a trivial first order thickening of ringed topoi and let $\pi : (Sh(\mathcal{D}), \mathcal{O}') \to (Sh(\mathcal{C}), \mathcal{O})$ be a trivialization. Then given any triple $(\mathcal{F}, \mathcal{K}, c)$ consisting of a pair of $\mathcal{O}$-modules and a map $c : \mathcal{I} \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{K}$ we may set
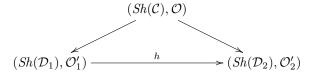
$$\mathcal{F}'_{c,triv} = \mathcal{F} \oplus \mathcal{K}$$

and use the splitting (10.5.1) associated to π and the map c to define the $\mathcal{O}'$-module structure and obtain an extension (10.0.1). We will call $\mathcal{F}'_{c,triv}$ the *trivial extension* of $\mathcal{F}$ by $\mathcal{K}$ corresponding to c and the trivialization π. Given any extension $\mathcal{F}'$ as in (10.0.1) we can use $\pi^\sharp : \mathcal{O} \to \mathcal{O}'$ to think of $\mathcal{F}'$ as an $\mathcal{O}$-module extension, hence a class $\xi_{\mathcal{F}'}$ in $\mathrm{Ext}^1_{\mathcal{O}}(\mathcal{F}, \mathcal{K})$. Lemma 10.3 assures that $\mathcal{F}' \mapsto \xi_{\mathcal{F}'}$ induces a bijection

$$\left\{ \begin{array}{c} \text{isomorphism classes of extensions} \\ \mathcal{F}' \text{ as in (10.0.1) with } c = c_{\mathcal{F}'} \end{array} \right\} \longrightarrow \mathrm{Ext}^1_{\mathcal{O}}(\mathcal{F}, \mathcal{K})$$

Moreover, the trivial extension $\mathcal{F}'_{c,triv}$ maps to the zero class.

08MJ **Remark 10.7.** Let $(Sh(\mathcal{C}), \mathcal{O})$ be a ringed topos. Let $(Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}_i), \mathcal{O}'_i)$, $i = 1, 2$ be first order thickenings with ideal sheaves $\mathcal{I}_i$. Let $h : (Sh(\mathcal{D}_1), \mathcal{O}'_1) \to (Sh(\mathcal{D}_2), \mathcal{O}'_2)$ be a morphism of first order thickenings of $(Sh(\mathcal{C}), \mathcal{O})$. Picture

$$(Sh(\mathcal{C}), \mathcal{O})$$
$$(Sh(\mathcal{D}_1), \mathcal{O}'_1) \xrightarrow{\quad h \quad} (Sh(\mathcal{D}_2), \mathcal{O}'_2)$$

Observe that $h^\sharp : \mathcal{O}'_2 \to \mathcal{O}'_1$ in particular induces an $\mathcal{O}$-module map $\mathcal{I}_2 \to \mathcal{I}_1$. Let $\mathcal{F}$ be an $\mathcal{O}$-module. Let $(\mathcal{K}_i, c_i)$, $i = 1, 2$ be a pair consisting of an $\mathcal{O}$-module $\mathcal{K}_i$ and a map $c_i : \mathcal{I}_i \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{K}_i$. Assume furthermore given a map of $\mathcal{O}$-modules $\mathcal{K}_2 \to \mathcal{K}_1$ such that

$$\begin{array}{ccc} \mathcal{I}_2 \otimes_{\mathcal{O}} \mathcal{F} & \xrightarrow{c_2} & \mathcal{K}_2 \\ \downarrow & & \downarrow \\ \mathcal{I}_1 \otimes_{\mathcal{O}} \mathcal{F} & \xrightarrow{c_1} & \mathcal{K}_1 \end{array}$$

is commutative. Then there is a canonical functoriality

$$\left\{ \begin{array}{c} \mathcal{F}'_2 \text{ as in (10.0.1) with} \\ c_2 = c_{\mathcal{F}'_2} \text{ and } \mathcal{K} = \mathcal{K}_2 \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \mathcal{F}'_1 \text{ as in (10.0.1) with} \\ c_1 = c_{\mathcal{F}'_1} \text{ and } \mathcal{K} = \mathcal{K}_1 \end{array} \right\}$$

Namely, thinking of all sheaves $\mathcal{O}$, $\mathcal{O}'_i$, $\mathcal{F}$, $\mathcal{K}_i$, etc as sheaves on $\mathcal{C}$, we set given $\mathcal{F}'_2$ the sheaf $\mathcal{F}'_1$ equal to the pushout, i.e., fitting into the following diagram of extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{K}_2 & \longrightarrow & \mathcal{F}'_2 & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathcal{K}_1 & \longrightarrow & \mathcal{F}'_1 & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \end{array}$$

We omit the construction of the $\mathcal{O}'_1$-module structure on the pushout (this uses the commutativity of the diagram involving c_1 and c_2).

08MK **Remark 10.8.** Let $(Sh(\mathcal{C}), \mathcal{O})$, $(Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}_i), \mathcal{O}'_i)$, $\mathcal{I}_i$, and $h : (Sh(\mathcal{D}_1), \mathcal{O}'_1) \to (Sh(\mathcal{D}_2), \mathcal{O}'_2)$ be as in Remark 10.7. Assume that we are given trivializations $\pi_i : (Sh(\mathcal{D}_i), \mathcal{O}'_i) \to (Sh(\mathcal{C}), \mathcal{O})$ such that $\pi_1 = h \circ \pi_2$. In other words, assume

h is a morphism of trivialized first order thickenings of $(Sh(\mathcal{C}), \mathcal{O})$. Let $(\mathcal{K}_i, c_i)$, $i = 1, 2$ be a pair consisting of an $\mathcal{O}$-module $\mathcal{K}_i$ and a map $c_i : \mathcal{I}_i \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{K}_i$. Assume furthermore given a map of $\mathcal{O}$-modules $\mathcal{K}_2 \to \mathcal{K}_1$ such that

$$
\begin{array}{ccc}
\mathcal{I}_2 \otimes_{\mathcal{O}} \mathcal{F} & \xrightarrow{\ c_2\ } & \mathcal{K}_2 \\
\downarrow & & \downarrow \\
\mathcal{I}_1 \otimes_{\mathcal{O}} \mathcal{F} & \xrightarrow{\ c_1\ } & \mathcal{K}_1
\end{array}
$$

is commutative. In this situation the construction of Remark 10.6 induces a commutative diagram

$$
\begin{array}{ccc}
\{\mathcal{F}_2' \text{ as in } (10.0.1) \text{ with } c_2 = c_{\mathcal{F}_2'} \text{ and } \mathcal{K} = \mathcal{K}_2\} & \longrightarrow & \mathrm{Ext}^1_{\mathcal{O}}(\mathcal{F}, \mathcal{K}_2) \\
\downarrow & & \downarrow \\
\{\mathcal{F}_1' \text{ as in } (10.0.1) \text{ with } c_1 = c_{\mathcal{F}_1'} \text{ and } \mathcal{K} = \mathcal{K}_1\} & \longrightarrow & \mathrm{Ext}^1_{\mathcal{O}}(\mathcal{F}, \mathcal{K}_1)
\end{array}
$$

where the vertical map on the right is given by functoriality of Ext and the map $\mathcal{K}_2 \to \mathcal{K}_1$ and the vertical map on the left is the one from Remark 10.7.

Remark 10.9. Let $(Sh(\mathcal{C}), \mathcal{O})$, $(Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}_i), \mathcal{O}_i')$, $\mathcal{I}_i$, and $h : (Sh(\mathcal{D}_1), \mathcal{O}_1') \to (Sh(\mathcal{D}_2), \mathcal{O}_2')$ be as in Remark 10.7. Observe that $h^{\sharp} : \mathcal{O}_2' \to \mathcal{O}_1'$ in particular induces an $\mathcal{O}$-module map $\mathcal{I}_2 \to \mathcal{I}_1$. Let $\mathcal{F}$ be an $\mathcal{O}$-module. Let $(\mathcal{K}_i, c_i)$, $i = 1, 2$ be a pair consisting of an $\mathcal{O}$-module $\mathcal{K}_i$ and a map $c_i : \mathcal{I}_i \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{K}_i$. Assume furthermore given a map of $\mathcal{O}$-modules $\mathcal{K}_2 \to \mathcal{K}_1$ such that

$$
\begin{array}{ccc}
\mathcal{I}_2 \otimes_{\mathcal{O}} \mathcal{F} & \xrightarrow{\ c_2\ } & \mathcal{K}_2 \\
\downarrow & & \downarrow \\
\mathcal{I}_1 \otimes_{\mathcal{O}} \mathcal{F} & \xrightarrow{\ c_1\ } & \mathcal{K}_1
\end{array}
$$

is commutative. Then we **claim** the map

$$
\mathrm{Ext}^2_{\mathcal{O}}(\mathcal{F}, \mathcal{K}_2) \longrightarrow \mathrm{Ext}^2_{\mathcal{O}}(\mathcal{F}, \mathcal{K}_1)
$$

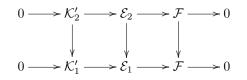sends $o(\mathcal{F}, \mathcal{K}_2, c_2)$ to $o(\mathcal{F}, \mathcal{K}_1, c_1)$.

To prove this claim choose an embedding $j_2 : \mathcal{K}_2 \to \mathcal{K}_2'$ where $\mathcal{K}_2'$ is an injective $\mathcal{O}$-module. As in the proof of Lemma 10.4 we can choose an extension of $\mathcal{O}_2$-modules

$$
0 \to \mathcal{K}_2' \to \mathcal{E}_2 \to \mathcal{F} \to 0
$$

such that $c_{\mathcal{E}_2} = j_2 \circ c_2$. The proof of Lemma 10.4 constructs $o(\mathcal{F}, \mathcal{K}_2, c_2)$ as the Yoneda extension class (in the sense of Derived Categories, Section 27) of the exact sequence of $\mathcal{O}$-modules

$$
0 \to \mathcal{K}_2 \to \mathcal{K}_2' \to \mathcal{E}_2/\mathcal{K}_2 \to \mathcal{F} \to 0
$$

Let $\mathcal{K}_1'$ be the cokernel of $\mathcal{K}_2 \to \mathcal{K}_1 \oplus \mathcal{K}_2'$. There is an injection $j_1 : \mathcal{K}_1 \to \mathcal{K}_1'$ and a map $\mathcal{K}_2' \to \mathcal{K}_1'$ forming a commutative square. We form the pushout:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{K}_2' & \longrightarrow & \mathcal{E}_2 & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{K}_1' & \longrightarrow & \mathcal{E}_1 & \longrightarrow & \mathcal{F} & \longrightarrow & 0
\end{array}
$$

There is a canonical $\mathcal{O}_1$-module structure on $\mathcal{E}_1$ and for this structure we have $c_{\mathcal{E}_1} = j_1 \circ c_1$ (this uses the commutativity of the diagram involving c_1 and c_2 above). The procedure of Lemma 10.4 tells us that $o(\mathcal{F}, \mathcal{K}_1, c_1)$ is the Yoneda extension class of the exact sequence of $\mathcal{O}$-modules

$$0 \to \mathcal{K}_1 \to \mathcal{K}_1' \to \mathcal{E}_1/\mathcal{K}_1 \to \mathcal{F} \to 0$$

Since we have maps of exact sequences

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{K}_2 & \longrightarrow & \mathcal{K}_2' & \longrightarrow & \mathcal{E}_2/\mathcal{K}_2 & \longrightarrow & \mathcal{F} \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \| \\
0 & \longrightarrow & \mathcal{K}_2 & \longrightarrow & \mathcal{K}_2' & \longrightarrow & \mathcal{E}_2/\mathcal{K}_2 & \longrightarrow & \mathcal{F} \longrightarrow 0
\end{array}
$$

we conclude that the claim is true.

08ML   **Remark 10.10.** Let $(Sh(\mathcal{C}), \mathcal{O})$ be a ringed topos. We define a sequence of morphisms of first order thickenings

$$(Sh(\mathcal{D}_1), \mathcal{O}_1') \to (Sh(\mathcal{D}_2), \mathcal{O}_2') \to (Sh(\mathcal{D}_3), \mathcal{O}_3')$$

of $(Sh(\mathcal{C}), \mathcal{O})$ to be a *complex* if the corresponding maps between the ideal sheaves $\mathcal{I}_i$ give a complex of $\mathcal{O}$-modules $\mathcal{I}_3 \to \mathcal{I}_2 \to \mathcal{I}_1$ (i.e., the composition is zero). In this case the composition $(Sh(\mathcal{D}_1), \mathcal{O}_1') \to (Sh(\mathcal{D}_3), \mathcal{O}_3')$ factors through $(Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}_3), \mathcal{O}_3')$, i.e., the first order thickening $(Sh(\mathcal{D}_1), \mathcal{O}_1')$ of $(Sh(\mathcal{C}), \mathcal{O})$ is trivial and comes with a canonical trivialization $\pi : (Sh(\mathcal{D}_1), \mathcal{O}_1') \to (Sh(\mathcal{C}), \mathcal{O})$.

We say a sequence of morphisms of first order thickenings

$$(Sh(\mathcal{D}_1), \mathcal{O}_1') \to (Sh(\mathcal{D}_2), \mathcal{O}_2') \to (Sh(\mathcal{D}_3), \mathcal{O}_3')$$

of $(Sh(\mathcal{C}), \mathcal{O})$ is *a short exact sequence* if the corresponding maps between ideal sheaves is a short exact sequence

$$0 \to \mathcal{I}_3 \to \mathcal{I}_2 \to \mathcal{I}_1 \to 0$$

of $\mathcal{O}$-modules.

08MM   **Remark 10.11.** Let $(Sh(\mathcal{C}), \mathcal{O})$ be a ringed topos. Let $\mathcal{F}$ be an $\mathcal{O}$-module. Let

$$(Sh(\mathcal{D}_1), \mathcal{O}_1') \to (Sh(\mathcal{D}_2), \mathcal{O}_2') \to (Sh(\mathcal{D}_3), \mathcal{O}_3')$$

be a complex first order thickenings of $(Sh(\mathcal{C}), \mathcal{O})$, see Remark 10.10. Let $(\mathcal{K}_i, c_i)$, $i = 1, 2, 3$ be pairs consisting of an $\mathcal{O}$-module $\mathcal{K}_i$ and a map $c_i : \mathcal{I}_i \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{K}_i$. Assume given a short exact sequence of $\mathcal{O}$-modules

$$0 \to \mathcal{K}_3 \to \mathcal{K}_2 \to \mathcal{K}_1 \to 0$$

such that

$$
\begin{array}{ccc}
\mathcal{I}_2 \otimes_{\mathcal{O}} \mathcal{F} \xrightarrow{c_2} \mathcal{K}_2 \\
\downarrow \qquad\qquad \downarrow \\
\mathcal{I}_1 \otimes_{\mathcal{O}} \mathcal{F} \xrightarrow{c_1} \mathcal{K}_1
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
\mathcal{I}_3 \otimes_{\mathcal{O}} \mathcal{F} \xrightarrow{c_3} \mathcal{K}_3 \\
\downarrow \qquad\qquad \downarrow \\
\mathcal{I}_2 \otimes_{\mathcal{O}} \mathcal{F} \xrightarrow{c_2} \mathcal{K}_2
\end{array}
$$

are commutative. Finally, assume given an extension

$$0 \to \mathcal{K}_2 \to \mathcal{F}_2' \to \mathcal{F} \to 0$$

as in (10.0.1) with $\mathcal{K} = \mathcal{K}_2$ of $\mathcal{O}_2'$-modules with $c_{\mathcal{F}_2'} = c_2$. In this situation we can apply the functoriality of Remark 10.7 to obtain an extension $\mathcal{F}_1'$ of $\mathcal{O}_1'$-modules (we'll

describe $\mathcal{F}_1'$ in this special case below). By Remark 10.6 using the canonical splitting $\pi : (Sh(\mathcal{D}_1), \mathcal{O}_1') \to (Sh(\mathcal{C}), \mathcal{O})$ of Remark 10.10 we obtain $\xi_{\mathcal{F}_1'} \in \mathrm{Ext}^1_{\mathcal{O}}(\mathcal{F}, \mathcal{K}_1)$. Finally, we have the obstruction

$$o(\mathcal{F}, \mathcal{K}_3, c_3) \in \mathrm{Ext}^2_{\mathcal{O}}(\mathcal{F}, \mathcal{K}_3)$$

see Lemma 10.4. In this situation we **claim** that the canonical map

$$\partial : \mathrm{Ext}^1_{\mathcal{O}}(\mathcal{F}, \mathcal{K}_1) \longrightarrow \mathrm{Ext}^2_{\mathcal{O}}(\mathcal{F}, \mathcal{K}_3)$$

coming from the short exact sequence $0 \to \mathcal{K}_3 \to \mathcal{K}_2 \to \mathcal{K}_1 \to 0$ sends $\xi_{\mathcal{F}_1'}$ to the obstruction class $o(\mathcal{F}, \mathcal{K}_3, c_3)$.

To prove this claim choose an embedding $j : \mathcal{K}_3 \to \mathcal{K}$ where $\mathcal{K}$ is an injective $\mathcal{O}$-module. We can lift j to a map $j' : \mathcal{K}_2 \to \mathcal{K}$. Set $\mathcal{E}_2' = j_*'\mathcal{F}_2'$ equal to the pushout of $\mathcal{F}_2'$ by j' so that $c_{\mathcal{E}_2'} = j' \circ c_2$. Picture:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{K}_2 & \longrightarrow & \mathcal{F}_2' & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle j'} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{E}_2' & \longrightarrow & \mathcal{F} & \longrightarrow & 0
\end{array}
$$

Set $\mathcal{E}_3' = \mathcal{E}_2'$ but viewed as an $\mathcal{O}_3'$-module via $\mathcal{O}_3' \to \mathcal{O}_2'$. Then $c_{\mathcal{E}_3'} = j \circ c_3$. The proof of Lemma 10.4 constructs $o(\mathcal{F}, \mathcal{K}_3, c_3)$ as the boundary of the class of the extension of $\mathcal{O}$-modules

$$0 \to \mathcal{K}/\mathcal{K}_3 \to \mathcal{E}_3'/\mathcal{K}_3 \to \mathcal{F} \to 0$$

On the other hand, note that $\mathcal{F}_1' = \mathcal{F}_2'/\mathcal{K}_3$ hence the class $\xi_{\mathcal{F}_1'}$ is the class of the extension

$$0 \to \mathcal{K}_2/\mathcal{K}_3 \to \mathcal{F}_2'/\mathcal{K}_3 \to \mathcal{F} \to 0$$

seen as a sequence of $\mathcal{O}$-modules using $\pi^\sharp$ where $\pi : (Sh(\mathcal{D}_1), \mathcal{O}_1') \to (Sh(\mathcal{C}), \mathcal{O})$ is the canonical splitting. Thus finally, the claim follows from the fact that we have a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{K}_2/\mathcal{K}_3 & \longrightarrow & \mathcal{F}_2'/\mathcal{K}_3 & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{K}/\mathcal{K}_3 & \longrightarrow & \mathcal{E}_3'/\mathcal{K}_3 & \longrightarrow & \mathcal{F} & \longrightarrow & 0
\end{array}
$$

which is $\mathcal{O}$-linear (with the $\mathcal{O}$-module structures given above).

11. Infinitesimal deformations of modules on ringed topoi

08MN Let $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ be a first order thickening of ringed topoi. We freely use the notation introduced in Section 9. Let $\mathcal{F}'$ be an $\mathcal{O}'$-module and set $\mathcal{F} = i^*\mathcal{F}'$. In this situation we have a short exact sequence

$$0 \to \mathcal{I}\mathcal{F}' \to \mathcal{F}' \to \mathcal{F} \to 0$$

of $\mathcal{O}'$-modules. Since $\mathcal{I}^2 = 0$ the $\mathcal{O}'$-module structure on $\mathcal{I}\mathcal{F}'$ comes from a unique $\mathcal{O}$-module structure. Thus the sequence above is an extension as in (10.0.1). As a special case, if $\mathcal{F}' = \mathcal{O}'$ we have $i^*\mathcal{O}' = \mathcal{O}$ and $\mathcal{I}\mathcal{O}' = \mathcal{I}$ and we recover the sequence of structure sheaves

$$0 \to \mathcal{I} \to \mathcal{O}' \to \mathcal{O} \to 0$$

08MP **Lemma 11.1.** *Let $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ be a first order thickening of ringed topoi. Let $\mathcal{F}', \mathcal{G}'$ be $\mathcal{O}'$-modules. Set $\mathcal{F} = i^*\mathcal{F}'$ and $\mathcal{G} = i^*\mathcal{G}'$. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be an $\mathcal{O}$-linear map. The set of lifts of φ to an $\mathcal{O}'$-linear map $\varphi' : \mathcal{F}' \to \mathcal{G}'$ is, if nonempty, a principal homogeneous space under $\mathrm{Hom}_\mathcal{O}(\mathcal{F}, \mathcal{I}\mathcal{G}')$.*

Proof. This is a special case of Lemma 10.1 but we also give a direct proof. We have short exact sequences of modules

$$0 \to \mathcal{I} \to \mathcal{O}' \to \mathcal{O} \to 0 \quad \text{and} \quad 0 \to \mathcal{I}\mathcal{G}' \to \mathcal{G}' \to \mathcal{G} \to 0$$

and similarly for $\mathcal{F}'$. Since $\mathcal{I}$ has square zero the $\mathcal{O}'$-module structure on $\mathcal{I}$ and $\mathcal{I}\mathcal{G}'$ comes from a unique $\mathcal{O}$-module structure. It follows that

$$\mathrm{Hom}_{\mathcal{O}'}(\mathcal{F}', \mathcal{I}\mathcal{G}') = \mathrm{Hom}_\mathcal{O}(\mathcal{F}, \mathcal{I}\mathcal{G}') \quad \text{and} \quad \mathrm{Hom}_{\mathcal{O}'}(\mathcal{F}', \mathcal{G}) = \mathrm{Hom}_\mathcal{O}(\mathcal{F}, \mathcal{G})$$

The lemma now follows from the exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{O}'}(\mathcal{F}', \mathcal{I}\mathcal{G}') \to \mathrm{Hom}_{\mathcal{O}'}(\mathcal{F}', \mathcal{G}') \to \mathrm{Hom}_{\mathcal{O}'}(\mathcal{F}', \mathcal{G})$$

see Homology, Lemma 5.8. $\square$

08MQ **Lemma 11.2.** *Let (f, f') be a morphism of first order thickenings of ringed topoi as in Situation 9.1. Let $\mathcal{F}'$ be an $\mathcal{O}'$-module and set $\mathcal{F} = i^*\mathcal{F}'$. Assume that $\mathcal{F}$ is flat over $\mathcal{O}_\mathcal{B}$ and that (f, f') is a strict morphism of thickenings (Definition 9.2). Then the following are equivalent*

(1) *$\mathcal{F}'$ is flat over $\mathcal{O}_{\mathcal{B}'}$, and*
(2) *the canonical map $f^*\mathcal{J} \otimes_\mathcal{O} \mathcal{F} \to \mathcal{I}\mathcal{F}'$ is an isomorphism.*

Moreover, in this case the maps

$$f^*\mathcal{J} \otimes_\mathcal{O} \mathcal{F} \to \mathcal{I} \otimes_\mathcal{O} \mathcal{F} \to \mathcal{I}\mathcal{F}'$$

are isomorphisms.

Proof. The map $f^*\mathcal{J} \to \mathcal{I}$ is surjective as (f, f') is a strict morphism of thickenings. Hence the final statement is a consequence of (2).

Proof of the equivalence of (1) and (2). By definition flatness over $\mathcal{O}_\mathcal{B}$ means flatness over $f^{-1}\mathcal{O}_\mathcal{B}$. Similarly for flatness over $f^{-1}\mathcal{O}_{\mathcal{B}'}$. Note that the strictness of (f, f') and the assumption that $\mathcal{F} = i^*\mathcal{F}'$ imply that

$$\mathcal{F} = \mathcal{F}'/(f^{-1}\mathcal{J})\mathcal{F}'$$

as sheaves on $\mathcal{C}$. Moreover, observe that $f^*\mathcal{J} \otimes_\mathcal{O} \mathcal{F} = f^{-1}\mathcal{J} \otimes_{f^{-1}\mathcal{O}_\mathcal{B}} \mathcal{F}$. Hence the equivalence of (1) and (2) follows from Modules on Sites, Lemma 28.15. $\square$

08VU **Lemma 11.3.** *Let (f, f') be a morphism of first order thickenings of ringed topoi as in Situation 9.1. Let $\mathcal{F}'$ be an $\mathcal{O}'$-module and set $\mathcal{F} = i^*\mathcal{F}'$. Assume that $\mathcal{F}'$ is flat over $\mathcal{O}_{\mathcal{B}'}$ and that (f, f') is a strict morphism of thickenings. Then the following are equivalent*

(1) *$\mathcal{F}'$ is an $\mathcal{O}'$-module of finite presentation, and*
(2) *$\mathcal{F}$ is an $\mathcal{O}$-module of finite presentation.*

Proof. The implication (1) $\Rightarrow$ (2) follows from Modules on Sites, Lemma 23.4. For the converse, assume $\mathcal{F}$ of finite presentation. We may and do assume that $\mathcal{C} = \mathcal{C}'$. By Lemma 11.2 we have a short exact sequence

$$0 \to \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{F}' \to \mathcal{F} \to 0$$

Let U be an object of $\mathcal{C}$ such that $\mathcal{F}|_U$ has a presentation

$$\mathcal{O}_U^{\oplus m} \to \mathcal{O}_U^{\oplus n} \to \mathcal{F}|_U \to 0$$

After replacing U by the members of a covering we may assume the map $\mathcal{O}_U^{\oplus n} \to \mathcal{F}|_U$ lifts to a map $(\mathcal{O}_U')^{\oplus n} \to \mathcal{F}'|_U$. The induced map $\mathcal{I}^{\oplus n} \to \mathcal{I} \otimes \mathcal{F}$ is surjective by right exactness of $\otimes$. Thus after replacing U by the members of a covering we can find a lift $(\mathcal{O}'|_U)^{\oplus m} \to (\mathcal{O}'|_U)^{\oplus n}$ of the given map $\mathcal{O}_U^{\oplus m} \to \mathcal{O}_U^{\oplus n}$ such that

$$(\mathcal{O}_U')^{\oplus m} \to (\mathcal{O}_U')^{\oplus n} \to \mathcal{F}'|_U \to 0$$

is a complex. Using right exactness of $\otimes$ once more it is seen that this complex is exact. $\qquad\square$

08MR **Lemma 11.4.** *Let (f, f') be a morphism of first order thickenings as in Situation 9.1. Let $\mathcal{F}'$, $\mathcal{G}'$ be $\mathcal{O}'$-modules and set $\mathcal{F} = i^*\mathcal{F}'$ and $\mathcal{G} = i^*\mathcal{G}'$. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be an $\mathcal{O}$-linear map. Assume that $\mathcal{G}'$ is flat over $\mathcal{O}_{\mathcal{B}'}$ and that (f, f') is a strict morphism of thickenings. The set of lifts of φ to an $\mathcal{O}'$-linear map $\varphi' : \mathcal{F}' \to \mathcal{G}'$ is, if nonempty, a principal homogeneous space under*

$$\mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{G} \otimes_{\mathcal{O}} f^*\mathcal{J})$$

Proof. Combine Lemmas 11.1 and 11.2. $\qquad\square$

08MS **Lemma 11.5.** *Let $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ be a first order thickening of ringed topoi. Let $\mathcal{F}'$, $\mathcal{G}'$ be $\mathcal{O}'$-modules and set $\mathcal{F} = i^*\mathcal{F}'$ and $\mathcal{G} = i^*\mathcal{G}'$. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be an $\mathcal{O}$-linear map. There exists an element*

$$o(\varphi) \in \mathrm{Ext}^1_{\mathcal{O}}(Li^*\mathcal{F}', \mathcal{I}\mathcal{G}')$$

whose vanishing is a necessary and sufficient condition for the existence of a lift of φ to an $\mathcal{O}'$-linear map $\varphi' : \mathcal{F}' \to \mathcal{G}'$.

Proof. It is clear from the proof of Lemma 11.1 that the vanishing of the boundary of φ via the map

$$\mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{G}) = \mathrm{Hom}_{\mathcal{O}'}(\mathcal{F}', \mathcal{G}) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}'}(\mathcal{F}', \mathcal{I}\mathcal{G}')$$

is a necessary and sufficient condition for the existence of a lift. We conclude as

$$\mathrm{Ext}^1_{\mathcal{O}'}(\mathcal{F}', \mathcal{I}\mathcal{G}') = \mathrm{Ext}^1_{\mathcal{O}}(Li^*\mathcal{F}', \mathcal{I}\mathcal{G}')$$

the adjointness of $i_* = Ri_*$ and Li^* on the derived category (Cohomology on Sites, Lemma 19.1). $\qquad\square$

08MT **Lemma 11.6.** *Let (f, f') be a morphism of first order thickenings as in Situation 9.1. Let $\mathcal{F}'$, $\mathcal{G}'$ be $\mathcal{O}'$-modules and set $\mathcal{F} = i^*\mathcal{F}'$ and $\mathcal{G} = i^*\mathcal{G}'$. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be an $\mathcal{O}$-linear map. Assume that $\mathcal{F}'$ and $\mathcal{G}'$ are flat over $\mathcal{O}_{\mathcal{B}'}$ and that (f, f') is a strict morphism of thickenings. There exists an element*

$$o(\varphi) \in \mathrm{Ext}^1_{\mathcal{O}}(\mathcal{F}, \mathcal{G} \otimes_{\mathcal{O}} f^*\mathcal{J})$$

whose vanishing is a necessary and sufficient condition for the existence of a lift of φ to an $\mathcal{O}'$-linear map $\varphi' : \mathcal{F}' \to \mathcal{G}'$.

First proof. This follows from Lemma 11.5 as we claim that under the assumptions of the lemma we have

$$\mathrm{Ext}^1_{\mathcal{O}}(Li^*\mathcal{F}', \mathcal{I}\mathcal{G}') = \mathrm{Ext}^1_{\mathcal{O}}(\mathcal{F}, \mathcal{G} \otimes_{\mathcal{O}} f^*\mathcal{J})$$

Namely, we have $\mathcal{I}\mathcal{G}' = \mathcal{G} \otimes_{\mathcal{O}} f^*\mathcal{J}$ by Lemma 11.2. On the other hand, observe that

$$H^{-1}(Li^*\mathcal{F}') = \mathrm{Tor}_1^{\mathcal{O}'}(\mathcal{F}', \mathcal{O})$$

(local computation omitted). Using the short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}' \to \mathcal{O} \to 0$$

we see that this Tor_1 is computed by the kernel of the map $\mathcal{I} \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{I}\mathcal{F}'$ which is zero by the final assertion of Lemma 11.2. Thus $\tau_{\geq -1}Li^*\mathcal{F}' = \mathcal{F}$. On the other hand, we have

$$\mathrm{Ext}_{\mathcal{O}}^1(Li^*\mathcal{F}', \mathcal{I}\mathcal{G}') = \mathrm{Ext}_{\mathcal{O}}^1(\tau_{\geq -1}Li^*\mathcal{F}', \mathcal{I}\mathcal{G}')$$

by the dual of Derived Categories, Lemma 16.1. $\qquad\square$

Second proof. We can apply Lemma 10.2 as follows. Note that $\mathcal{K} = \mathcal{I} \otimes_{\mathcal{O}} \mathcal{F}$ and $\mathcal{L} = \mathcal{I} \otimes_{\mathcal{O}} \mathcal{G}$ by Lemma 11.2, that $c_{\mathcal{F}'} = 1 \otimes 1$ and $c_{\mathcal{G}'} = 1 \otimes 1$ and taking $\psi = 1 \otimes \varphi$ the diagram of the lemma commutes. Thus $o(\varphi) = o(\varphi, 1 \otimes \varphi)$ works. $\qquad\square$

08MU **Lemma 11.7.** *Let (f, f') be a morphism of first order thickenings as in Situation 9.1. Let $\mathcal{F}$ be an $\mathcal{O}$-module. Assume (f, f') is a strict morphism of thickenings and $\mathcal{F}$ flat over $\mathcal{O}_{\mathcal{B}}$. If there exists a pair $(\mathcal{F}', \alpha)$ consisting of an $\mathcal{O}'$-module $\mathcal{F}'$ flat over $\mathcal{O}_{\mathcal{B}'}$ and an isomorphism $\alpha : i^*\mathcal{F}' \to \mathcal{F}$, then the set of isomorphism classes of such pairs is principal homogeneous under $\mathrm{Ext}_{\mathcal{O}}^1(\mathcal{F}, \mathcal{I} \otimes_{\mathcal{O}} \mathcal{F})$.*

Proof. If we assume there exists one such module, then the canonical map

$$f^*\mathcal{J} \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{I} \otimes_{\mathcal{O}} \mathcal{F}$$

is an isomorphism by Lemma 11.2. Apply Lemma 10.3 with $\mathcal{K} = \mathcal{I} \otimes_{\mathcal{O}} \mathcal{F}$ and $c = 1$. By Lemma 11.2 the corresponding extensions $\mathcal{F}'$ are all flat over $\mathcal{O}_{\mathcal{B}'}$. $\qquad\square$

08MV **Lemma 11.8.** *Let (f, f') be a morphism of first order thickenings as in Situation 9.1. Let $\mathcal{F}$ be an $\mathcal{O}$-module. Assume (f, f') is a strict morphism of thickenings and $\mathcal{F}$ flat over $\mathcal{O}_{\mathcal{B}}$. There exists an $\mathcal{O}'$-module $\mathcal{F}'$ flat over $\mathcal{O}_{\mathcal{B}'}$ with $i^*\mathcal{F}' \cong \mathcal{F}$, if and only if*

(1) *the canonical map $f^*\mathcal{J} \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{I} \otimes_{\mathcal{O}} \mathcal{F}$ is an isomorphism, and*
(2) *the class $o(\mathcal{F}, \mathcal{I} \otimes_{\mathcal{O}} \mathcal{F}, 1) \in \mathrm{Ext}_{\mathcal{O}}^2(\mathcal{F}, \mathcal{I} \otimes_{\mathcal{O}} \mathcal{F})$ of Lemma 10.4 is zero.*

Proof. This follows immediately from the characterization of $\mathcal{O}'$-modules flat over $\mathcal{O}_{\mathcal{B}'}$ of Lemma 11.2 and Lemma 10.4. $\qquad\square$

12. Application to flat modules on flat thickenings of ringed topoi

08VV Consider a commutative diagram

$$
\begin{array}{ccc}
(Sh(\mathcal{C}), \mathcal{O}) & \xrightarrow{\;\;i\;\;} & (Sh(\mathcal{D}), \mathcal{O}') \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\
(Sh(\mathcal{B}), \mathcal{O}_{\mathcal{B}}) & \xrightarrow{\;\;t\;\;} & (Sh(\mathcal{B}'), \mathcal{O}_{\mathcal{B}'})
\end{array}
$$

of ringed topoi whose horizontal arrows are first order thickenings as in Situation 9.1. Set $\mathcal{I} = \mathrm{Ker}(i^\sharp) \subset \mathcal{O}'$ and $\mathcal{J} = \mathrm{Ker}(t^\sharp) \subset \mathcal{O}_{\mathcal{B}'}$. Let $\mathcal{F}$ be an $\mathcal{O}$-module. Assume that

(1) (f, f') is a strict morphism of thickenings,
(2) f' is flat, and

(3) $\mathcal{F}$ is flat over $\mathcal{O}_\mathcal{B}$.

Note that (1) + (2) imply that $\mathcal{I} = f^*\mathcal{J}$ (apply Lemma 11.2 to $\mathcal{O}'$). The theory of the preceding section is especially nice under these assumptions. We summarize the results already obtained in the following lemma.

08VW **Lemma 12.1.** *In the situation above.*

(1) *There exists an $\mathcal{O}'$-module $\mathcal{F}'$ flat over $\mathcal{O}_{\mathcal{B}'}$ with $i^*\mathcal{F}' \cong \mathcal{F}$, if and only if the class $o(\mathcal{F}, f^*\mathcal{J} \otimes_\mathcal{O} \mathcal{F}, 1) \in \mathrm{Ext}^2_\mathcal{O}(\mathcal{F}, f^*\mathcal{J} \otimes_\mathcal{O} \mathcal{F})$ of Lemma 10.4 is zero.*
(2) *If such a module exists, then the set of isomorphism classes of lifts is principal homogeneous under $\mathrm{Ext}^1_\mathcal{O}(\mathcal{F}, f^*\mathcal{J} \otimes_\mathcal{O} \mathcal{F})$.*
(3) *Given a lift $\mathcal{F}'$, the set of automorphisms of $\mathcal{F}'$ which pull back to $\mathrm{id}_\mathcal{F}$ is canonically isomorphic to $\mathrm{Ext}^0_\mathcal{O}(\mathcal{F}, f^*\mathcal{J} \otimes_\mathcal{O} \mathcal{F})$.*

Proof. Part (1) follows from Lemma 11.8 as we have seen above that $\mathcal{I} = f^*\mathcal{J}$. Part (2) follows from Lemma 11.7. Part (3) follows from Lemma 11.4. $\square$

0CYD **Situation 12.2.** Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{B}), \mathcal{O}_\mathcal{B})$ be a morphism of ringed topoi. Consider a commutative diagram

$$
\begin{array}{ccc}
(Sh(\mathcal{C}'_1), \mathcal{O}'_1) & \xrightarrow{\quad h \quad} & (Sh(\mathcal{C}'_2), \mathcal{O}'_2) \\
{\scriptstyle f'_1} \downarrow & & \downarrow {\scriptstyle f'_2} \\
(Sh(\mathcal{B}'_1), \mathcal{O}_{\mathcal{B}'_1}) & \longrightarrow & (Sh(\mathcal{B}'_2), \mathcal{O}_{\mathcal{B}'_2})
\end{array}
$$

where h is a morphism of first order thickenings of $(Sh(\mathcal{C}), \mathcal{O})$, the lower horizontal arrow is a morphism of first order thickenings of $(Sh(\mathcal{B}), \mathcal{O}_\mathcal{B})$, each f'_i restricts to f, both pairs (f, f'_i) are strict morphisms of thickenings, and both f'_i are flat. Finally, let $\mathcal{F}$ be an $\mathcal{O}$-module flat over $\mathcal{O}_\mathcal{B}$.

0CYE **Lemma 12.3.** *In Situation 12.2 the obstruction class $o(\mathcal{F}, f^*\mathcal{J}_2 \otimes_\mathcal{O} \mathcal{F}, 1)$ maps to the obstruction class $o(\mathcal{F}, f^*\mathcal{J}_1 \otimes_\mathcal{O} \mathcal{F}, 1)$ under the canonical map*

$$\mathrm{Ext}^2_\mathcal{O}(\mathcal{F}, f^*\mathcal{J}_2 \otimes_\mathcal{O} \mathcal{F}) \to \mathrm{Ext}^2_\mathcal{O}(\mathcal{F}, f^*\mathcal{J}_1 \otimes_\mathcal{O} \mathcal{F})$$

Proof. Follows from Remark 10.9. $\square$

08VX **Situation 12.4.** Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{B}), \mathcal{O}_\mathcal{B})$ be a morphism of ringed topoi. Consider a commutative diagram

$$
\begin{array}{ccccc}
(Sh(\mathcal{C}'_1), \mathcal{O}'_1) & \xrightarrow{\quad h \quad} & (Sh(\mathcal{C}'_2), \mathcal{O}'_2) & \longrightarrow & (Sh(\mathcal{C}'_3), \mathcal{O}'_3) \\
{\scriptstyle f'_1} \downarrow & & {\scriptstyle f'_2} \downarrow & & \downarrow {\scriptstyle f'_3} \\
(Sh(\mathcal{B}'_1), \mathcal{O}_{\mathcal{B}'_1}) & \longrightarrow & (Sh(\mathcal{B}'_2), \mathcal{O}_{\mathcal{B}'_2}) & \longrightarrow & (Sh(\mathcal{B}'_3), \mathcal{O}_{\mathcal{B}'_3})
\end{array}
$$

where (a) the top row is a short exact sequence of first order thickenings of $(Sh(\mathcal{C}), \mathcal{O})$, (b) the lower row is a short exact sequence of first order thickenings of $(Sh(\mathcal{B}), \mathcal{O}_\mathcal{B})$, (c) each f'_i restricts to f, (d) each pair (f, f'_i) is a strict morphism of thickenings, and (e) each f'_i is flat. Finally, let $\mathcal{F}'_2$ be an $\mathcal{O}'_2$-module flat over $\mathcal{O}_{\mathcal{B}'_2}$ and set $\mathcal{F} = \mathcal{F}'_2 \otimes \mathcal{O}$. Let $\pi : (Sh(\mathcal{C}'_1), \mathcal{O}'_1) \to (Sh(\mathcal{C}), \mathcal{O})$ be the canonical splitting (Remark 10.10).

08VY   **Lemma 12.5.** *In Situation 12.4 the modules $\pi^*\mathcal{F}$ and $h^*\mathcal{F}_2'$ are $\mathcal{O}_1'$-modules flat over $\mathcal{O}_{\mathcal{B}_1'}$ restricting to $\mathcal{F}$ on $(Sh(\mathcal{C}), \mathcal{O})$. Their difference (Lemma 12.1) is an element θ of $\mathrm{Ext}_{\mathcal{O}}^1(\mathcal{F}, f^*\mathcal{J}_1 \otimes_{\mathcal{O}} \mathcal{F})$ whose boundary in $\mathrm{Ext}_{\mathcal{O}}^2(\mathcal{F}, f^*\mathcal{J}_3 \otimes_{\mathcal{O}} \mathcal{F})$ equals the obstruction (Lemma 12.1) to lifting $\mathcal{F}$ to an $\mathcal{O}_3'$-module flat over $\mathcal{O}_{\mathcal{B}_3'}$.*

Proof. Note that both $\pi^*\mathcal{F}$ and $h^*\mathcal{F}_2'$ restrict to $\mathcal{F}$ on $(Sh(\mathcal{C}), \mathcal{O})$ and that the kernels of $\pi^*\mathcal{F} \to \mathcal{F}$ and $h^*\mathcal{F}_2' \to \mathcal{F}$ are given by $f^*\mathcal{J}_1 \otimes_{\mathcal{O}} \mathcal{F}$. Hence flatness by Lemma 11.2. Taking the boundary makes sense as the sequence of modules

$$0 \to f^*\mathcal{J}_3 \otimes_{\mathcal{O}} \mathcal{F} \to f^*\mathcal{J}_2 \otimes_{\mathcal{O}} \mathcal{F} \to f^*\mathcal{J}_1 \otimes_{\mathcal{O}} \mathcal{F} \to 0$$

is short exact due to the assumptions in Situation 12.4 and the fact that $\mathcal{F}$ is flat over $\mathcal{O}_{\mathcal{B}}$. The statement on the obstruction class is a direct translation of the result of Remark 10.11 to this particular situation. $\qquad\square$

13. Deformations of ringed topoi and the naive cotangent complex

08UE   In this section we use the naive cotangent complex to do a little bit of deformation theory. We start with a first order thickening $t : (Sh(\mathcal{B}), \mathcal{O}_{\mathcal{B}}) \to (Sh(\mathcal{B}'), \mathcal{O}_{\mathcal{B}'})$ of ringed topoi. We denote $\mathcal{J} = \mathrm{Ker}(t^\sharp)$ and we identify the underlying topoi of $\mathcal{B}$ and $\mathcal{B}'$. Moreover we assume given a morphism of ringed topoi $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{B}), \mathcal{O}_{\mathcal{B}})$, an $\mathcal{O}$-module $\mathcal{G}$, and a map $f^{-1}\mathcal{J} \to \mathcal{G}$ of sheaves of $f^{-1}\mathcal{O}_{\mathcal{B}}$-modules. In this section we ask ourselves whether we can find the question mark fitting into the following diagram

08UF   (13.0.1)
$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{G} & \longrightarrow & ? & \longrightarrow & \mathcal{O} & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle c} & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & f^{-1}\mathcal{J} & \longrightarrow & f^{-1}\mathcal{O}_{\mathcal{B}'} & \longrightarrow & f^{-1}\mathcal{O}_{\mathcal{B}} & \longrightarrow & 0
\end{array}
$$

and moreover how unique the solution is (if it exists). More precisely, we look for a first order thickening $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ and a morphism of thickenings (f, f') as in (9.1.1) where $\mathrm{Ker}(i^\sharp)$ is identified with $\mathcal{G}$ such that $(f')^\sharp$ induces the given map c. We will say $(Sh(\mathcal{C}'), \mathcal{O}')$ is a *solution* to (13.0.1).
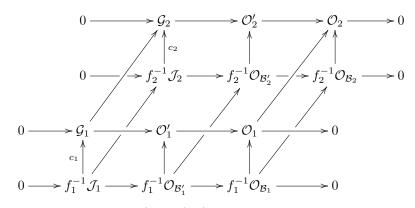
08UG   **Lemma 13.1.** *Assume given a commutative diagram of morphisms ringed topoi*

08UH   (13.1.1)
$$
\begin{array}{ccc}
(Sh(\mathcal{C}_2), \mathcal{O}_2) & \xrightarrow{\;\;i_2\;\;} & (Sh(\mathcal{C}_2'), \mathcal{O}_2') \\
\end{array}
$$

whose horizontal arrows are first order thickenings. Set $\mathcal{G}_j = \mathrm{Ker}(i_j^\sharp)$ and assume given a map of $g^{-1}\mathcal{O}_1$-modules $\nu : g^{-1}\mathcal{G}_1 \to \mathcal{G}_2$ giving rise to the commutative

diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{G}_2 & \longrightarrow & \mathcal{O}_2' & \longrightarrow & \mathcal{O}_2 & \longrightarrow & 0
\end{array}
$$

08UI (13.1.2)

with front and back solutions to (13.0.1). (The north-north-west arrows are maps on $\mathcal{C}_2$ after applying g^{-1} to the source.)

(1) There exist a canonical element in $\mathrm{Ext}^1_{\mathcal{O}_2}(Lg^*\, NL_{\mathcal{O}_1/\mathcal{O}_{\mathcal{B}_1}}, \mathcal{G}_2)$ whose vanishing is a necessary and sufficient condition for the existence of a morphism of ringed topoi $(Sh(\mathcal{C}_2'), \mathcal{O}_2') \to (Sh(\mathcal{C}_1'), \mathcal{O}_1')$ fitting into (13.1.1) compatibly with ν.

(2) If there exists a morphism $(Sh(\mathcal{C}_2'), \mathcal{O}_2') \to (Sh(\mathcal{C}_1'), \mathcal{O}_1')$ fitting into (13.1.1) compatibly with ν the set of all such morphisms is a principal homogeneous space under

$$
\mathrm{Hom}_{\mathcal{O}_1}(\Omega_{\mathcal{O}_1/\mathcal{O}_{\mathcal{B}_1}}, g_*\mathcal{G}_2) = \mathrm{Hom}_{\mathcal{O}_2}(g^*\Omega_{\mathcal{O}_1/\mathcal{O}_{\mathcal{B}_1}}, \mathcal{G}_2) = \mathrm{Ext}^0_{\mathcal{O}_2}(Lg^*\, NL_{\mathcal{O}_1/\mathcal{O}_{\mathcal{B}_1}}, \mathcal{G}_2).
$$

Proof. The proof of this lemma is identical to the proof of Lemma 7.1. We urge the reader to read that proof instead of this one. We will identify the underlying topoi for every thickening in sight (we have already used this convention in the statement). The equalities in the last statement of the lemma are immediate from the definitions. Thus we will work with the groups $\mathrm{Ext}^k_{\mathcal{O}_2}(Lg^*\, NL_{\mathcal{O}_1/\mathcal{O}_{\mathcal{B}_1}}, \mathcal{G}_2)$, $k = 0, 1$ in the rest of the proof. We first argue that we can reduce to the case where the underlying topos of all ringed topoi in the lemma is the same.

To do this, observe that $g^{-1}\, NL_{\mathcal{O}_1/\mathcal{O}_{\mathcal{B}_1}}$ is equal to the naive cotangent complex of the homomorphism of sheaves of rings $g^{-1}f_1^{-1}\mathcal{O}_{\mathcal{B}_1} \to g^{-1}\mathcal{O}_1$, see Modules on Sites, Lemma 33.5. Moreover, the degree 0 term of $NL_{\mathcal{O}_1/\mathcal{O}_{\mathcal{B}_1}}$ is a flat $\mathcal{O}_1$-module, hence the canonical map

$$
Lg^*\, NL_{\mathcal{O}_1/\mathcal{O}_{\mathcal{B}_1}} \longrightarrow g^{-1}\, NL_{\mathcal{O}_1/\mathcal{O}_{\mathcal{B}_1}} \otimes_{g^{-1}\mathcal{O}_1} \mathcal{O}_2
$$

induces an isomorphism on cohomology sheaves in degrees 0 and -1. Thus we may replace the Ext groups of the lemma with

$$
\mathrm{Ext}^k_{g^{-1}\mathcal{O}_1}(g^{-1}\, NL_{\mathcal{O}_1/\mathcal{O}_{\mathcal{B}_1}}, \mathcal{G}_2) = \mathrm{Ext}^k_{g^{-1}\mathcal{O}_1}(NL_{g^{-1}\mathcal{O}_1/g^{-1}f_1^{-1}\mathcal{O}_{\mathcal{B}_1}}, \mathcal{G}_2)
$$

The set of morphism of ringed topoi $(Sh(\mathcal{C}_2'), \mathcal{O}_2') \to (Sh(\mathcal{C}_1'), \mathcal{O}_1')$ fitting into (13.1.1) compatibly with ν is in one-to-one bijection with the set of homomorphisms of $g^{-1}f_1^{-1}\mathcal{O}_{\mathcal{B}_1}$-algebras $g^{-1}\mathcal{O}_1' \to \mathcal{O}_2'$ which are compatible with $f^\sharp$ and ν. In this way we see that we may assume we have a diagram (13.1.2) of sheaves on a site $\mathcal{C}$ (with $f_1 = f_2 = \mathrm{id}$ on underlying topoi) and we are looking to find a homomorphism of sheaves of rings $\mathcal{O}_1' \to \mathcal{O}_2'$ fitting into it.

In the rest of the proof of the lemma we assume all underlying topological spaces are the same, i.e., we have a diagram (13.1.2) of sheaves on a site $\mathcal{C}$ (with $f_1 = f_2 = \mathrm{id}$ on underlying topoi) and we are looking for homomorphisms of sheaves of rings $\mathcal{O}'_1 \to \mathcal{O}'_2$ fitting into it. As ext groups we will use $\mathrm{Ext}^k_{\mathcal{O}_1}(NL_{\mathcal{O}_1/\mathcal{O}_{\mathcal{B}_1}}, \mathcal{G}_2)$, $k = 0, 1$.

Step 1. Construction of the obstruction class. Consider the sheaf of sets

$$\mathcal{E} = \mathcal{O}'_1 \times_{\mathcal{O}_2} \mathcal{O}'_2$$

This comes with a surjective map $\alpha : \mathcal{E} \to \mathcal{O}_1$ and hence we can use $NL(\alpha)$ instead of $NL_{\mathcal{O}_1/\mathcal{O}_{\mathcal{B}_1}}$, see Modules on Sites, Lemma 35.2. Set

$$\mathcal{I}' = \mathrm{Ker}(\mathcal{O}_{\mathcal{B}'_1}[\mathcal{E}] \to \mathcal{O}_1) \quad \text{and} \quad \mathcal{I} = \mathrm{Ker}(\mathcal{O}_{\mathcal{B}_1}[\mathcal{E}] \to \mathcal{O}_1)$$

There is a surjection $\mathcal{I}' \to \mathcal{I}$ whose kernel is $\mathcal{J}_1 \mathcal{O}_{\mathcal{B}'_1}[\mathcal{E}]$. We obtain two homomorphisms of $\mathcal{O}_{\mathcal{B}'_2}$-algebras

$$a : \mathcal{O}_{\mathcal{B}'_1}[\mathcal{E}] \to \mathcal{O}'_1 \quad \text{and} \quad b : \mathcal{O}_{\mathcal{B}'_1}[\mathcal{E}] \to \mathcal{O}'_2$$

which induce maps $a|_{\mathcal{I}'} : \mathcal{I}' \to \mathcal{G}_1$ and $b|_{\mathcal{I}'} : \mathcal{I}' \to \mathcal{G}_2$. Both a and b annihilate $(\mathcal{I}')^2$. Moreover a and b agree on $\mathcal{J}_1 \mathcal{O}_{\mathcal{B}'_1}[\mathcal{E}]$ as maps into $\mathcal{G}_2$ because the left hand square of (13.1.2) is commutative. Thus the difference $b|_{\mathcal{I}'} - \nu \circ a|_{\mathcal{I}'}$ induces a well defined $\mathcal{O}_1$-linear map

$$\xi : \mathcal{I}/\mathcal{I}^2 \longrightarrow \mathcal{G}_2$$

which sends the class of a local section f of $\mathcal{I}$ to $a(f') - \nu(b(f'))$ where f' is a lift of f to a local section of $\mathcal{I}'$. We let $[\xi] \in \mathrm{Ext}^1_{\mathcal{O}_1}(NL(\alpha), \mathcal{G}_2)$ be the image (see below).

Step 2. Vanishing of $[\xi]$ is necessary. Let us write $\Omega = \Omega_{\mathcal{O}_{\mathcal{B}_1}[\mathcal{E}]/\mathcal{O}_{\mathcal{B}_1}} \otimes_{\mathcal{O}_{\mathcal{B}_1}[\mathcal{E}]} \mathcal{O}_1$. Observe that $NL(\alpha) = (\mathcal{I}/\mathcal{I}^2 \to \Omega)$ fits into a distinguished triangle

$$\Omega[0] \to NL(\alpha) \to \mathcal{I}/\mathcal{I}^2[1] \to \Omega[1]$$

Thus we see that $[\xi]$ is zero if and only if ξ is a composition $\mathcal{I}/\mathcal{I}^2 \to \Omega \to \mathcal{G}_2$ for some map $\Omega \to \mathcal{G}_2$. Suppose there exists a homomorphisms of sheaves of rings $\varphi : \mathcal{O}'_1 \to \mathcal{O}'_2$ fitting into (13.1.2). In this case consider the map $\mathcal{O}'_1[\mathcal{E}] \to \mathcal{G}_2$, $f' \mapsto b(f') - \varphi(a(f'))$. A calculation shows this annihilates $\mathcal{J}_1 \mathcal{O}_{\mathcal{B}'_1}[\mathcal{E}]$ and induces a derivation $\mathcal{O}_{\mathcal{B}_1}[\mathcal{E}] \to \mathcal{G}_2$. The resulting linear map $\Omega \to \mathcal{G}_2$ witnesses the fact that $[\xi] = 0$ in this case.

Step 3. Vanishing of $[\xi]$ is sufficient. Let $\theta : \Omega \to \mathcal{G}_2$ be a $\mathcal{O}_1$-linear map such that ξ is equal to $\theta \circ (\mathcal{I}/\mathcal{I}^2 \to \Omega)$. Then a calculation shows that

$$b + \theta \circ d : \mathcal{O}_{\mathcal{B}'_1}[\mathcal{E}] \longrightarrow \mathcal{O}'_2$$

annihilates $\mathcal{I}'$ and hence defines a map $\mathcal{O}'_1 \to \mathcal{O}'_2$ fitting into (13.1.2).

Proof of (2) in the special case above. Omitted. Hint: This is exactly the same as the proof of (2) of Lemma 2.1. $\qquad\square$

08UJ   **Lemma 13.2.** *Let $\mathcal{C}$ be a site. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings on $\mathcal{C}$. Let $\mathcal{G}$ be a $\mathcal{B}$-module. Let $\xi \in \mathrm{Ext}^1_{\mathcal{B}}(NL_{\mathcal{B}/\mathcal{A}}, \mathcal{G})$. There exists a map of sheaves of sets $\alpha : \mathcal{E} \to \mathcal{B}$ such that $\xi \in \mathrm{Ext}^1_{\mathcal{B}}(NL(\alpha), \mathcal{G})$ is the class of a map $\mathcal{I}/\mathcal{I}^2 \to \mathcal{G}$ (see proof for notation).*

Proof. Recall that given $\alpha : \mathcal{E} \to \mathcal{B}$ such that $\mathcal{A}[\mathcal{E}] \to \mathcal{B}$ is surjective with kernel $\mathcal{I}$ the complex $NL(\alpha) = (\mathcal{I}/\mathcal{I}^2 \to \Omega_{\mathcal{A}[\mathcal{E}]/\mathcal{A}} \otimes_{\mathcal{A}[\mathcal{E}]} \mathcal{B})$ is canonically isomorphic to $NL_{\mathcal{B}/\mathcal{A}}$, see Modules on Sites, Lemma 35.2. Observe moreover, that $\Omega = \Omega_{\mathcal{A}[\mathcal{E}]/\mathcal{A}} \otimes_{\mathcal{A}[\mathcal{E}]} \mathcal{B}$ is the sheaf associated to the presheaf $U \mapsto \bigoplus_{e \in \mathcal{E}(U)} \mathcal{B}(U)$. In other words, Ω is the free $\mathcal{B}$-module on the sheaf of sets $\mathcal{E}$ and in particular there is a canonical map $\mathcal{E} \to \Omega$.

Having said this, pick some $\mathcal{E}$ (for example $\mathcal{E} = \mathcal{B}$ as in the definition of the naive cotangent complex). The obstruction to writing ξ as the class of a map $\mathcal{I}/\mathcal{I}^2 \to \mathcal{G}$ is an element in $\mathrm{Ext}^1_{\mathcal{B}}(\Omega, \mathcal{G})$. Say this is represented by the extension $0 \to \mathcal{G} \to \mathcal{H} \to \Omega \to 0$ of $\mathcal{B}$-modules. Consider the sheaf of sets $\mathcal{E}' = \mathcal{E} \times_\Omega \mathcal{H}$ which comes with an induced map $\alpha' : \mathcal{E}' \to \mathcal{B}$. Let $\mathcal{I}' = \mathrm{Ker}(\mathcal{A}[\mathcal{E}'] \to \mathcal{B})$ and $\Omega' = \Omega_{\mathcal{A}[\mathcal{E}']/\mathcal{A}} \otimes_{\mathcal{A}[\mathcal{E}']} \mathcal{B}$. The pullback of ξ under the quasi-isomorphism $NL(\alpha') \to NL(\alpha)$ maps to zero in $\mathrm{Ext}^1_{\mathcal{B}}(\Omega', \mathcal{G})$ because the pullback of the extension $\mathcal{H}$ by the map $\Omega' \to \Omega$ is split as Ω' is the free $\mathcal{B}$-module on the sheaf of sets $\mathcal{E}'$ and since by construction there is a commutative diagram

$$\begin{array}{ccc} \mathcal{E}' & \longrightarrow & \mathcal{E} \\ \downarrow & & \downarrow \\ \mathcal{H} & \longrightarrow & \Omega \end{array}$$

This finishes the proof. $\qquad\square$

08UK **Lemma 13.3.** *If there exists a solution to (13.0.1), then the set of isomorphism classes of solutions is principal homogeneous under $\mathrm{Ext}^1_{\mathcal{O}}(NL_{\mathcal{O}/\mathcal{O}_{\mathcal{B}}}, \mathcal{G})$.*

Proof. We observe right away that given two solutions $\mathcal{O}'_1$ and $\mathcal{O}'_2$ to (13.0.1) we obtain by Lemma 13.1 an obstruction element $o(\mathcal{O}'_1, \mathcal{O}'_2) \in \mathrm{Ext}^1_{\mathcal{O}}(NL_{\mathcal{O}/\mathcal{O}_{\mathcal{B}}}, \mathcal{G})$ to the existence of a map $\mathcal{O}'_1 \to \mathcal{O}'_2$. Clearly, this element is the obstruction to the existence of an isomorphism, hence separates the isomorphism classes. To finish the proof it therefore suffices to show that given a solution $\mathcal{O}'$ and an element $\xi \in \mathrm{Ext}^1_{\mathcal{O}}(NL_{\mathcal{O}/\mathcal{O}_{\mathcal{B}}}, \mathcal{G})$ we can find a second solution $\mathcal{O}'_\xi$ such that $o(\mathcal{O}', \mathcal{O}'_\xi) = \xi$.

Pick $\alpha : \mathcal{E} \to \mathcal{O}$ as in Lemma 13.2 for the class ξ. Consider the surjection $f^{-1}\mathcal{O}_{\mathcal{B}}[\mathcal{E}] \to \mathcal{O}$ with kernel $\mathcal{I}$ and corresponding naive cotangent complex $NL(\alpha) = (\mathcal{I}/\mathcal{I}^2 \to \Omega_{f^{-1}\mathcal{O}_{\mathcal{B}}[\mathcal{E}]/f^{-1}\mathcal{O}_{\mathcal{B}}} \otimes_{f^{-1}\mathcal{O}_{\mathcal{B}}[\mathcal{E}]} \mathcal{O})$. By the lemma ξ is the class of a morphism $\delta : \mathcal{I}/\mathcal{I}^2 \to \mathcal{G}$. After replacing $\mathcal{E}$ by $\mathcal{E} \times_\mathcal{O} \mathcal{O}'$ we may also assume that α factors through a map $\alpha' : \mathcal{E} \to \mathcal{O}'$.

These choices determine an $f^{-1}\mathcal{O}_{\mathcal{B}'}$-algebra map $\varphi : \mathcal{O}_{\mathcal{B}'}[\mathcal{E}] \to \mathcal{O}'$. Let $\mathcal{I}' = \mathrm{Ker}(\varphi)$. Observe that φ induces a map $\varphi|_{\mathcal{I}'} : \mathcal{I}' \to \mathcal{G}$ and that $\mathcal{O}'$ is the pushout, as in the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{O}' & \longrightarrow & \mathcal{O} & \longrightarrow & 0 \\ & & \varphi|_{\mathcal{I}'} \uparrow & & \uparrow & & \| & & \\ 0 & \longrightarrow & \mathcal{I}' & \longrightarrow & f^{-1}\mathcal{O}_{\mathcal{B}'}[\mathcal{E}] & \longrightarrow & \mathcal{O} & \longrightarrow & 0 \end{array}$$

Let $\psi : \mathcal{I}' \to \mathcal{G}$ be the sum of the map $\varphi|_{\mathcal{I}'}$ and the composition

$$\mathcal{I}' \to \mathcal{I}'/(\mathcal{I}')^2 \to \mathcal{I}/\mathcal{I}^2 \xrightarrow{\delta} \mathcal{G}.$$
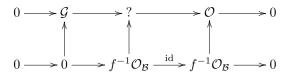
Then the pushout along ψ is an other ring extension $\mathcal{O}'_\xi$ fitting into a diagram as above. A calculation (omitted) shows that $o(\mathcal{O}', \mathcal{O}'_\xi) = \xi$ as desired. $\qquad\square$

0GQ5 **Lemma 13.4.** *Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{B}), \mathcal{O}_\mathcal{B})$ be a morphism of ringed topoi. Let $\mathcal{G}$ be an $\mathcal{O}$-module. The set of isomorphism classes of extensions of $f^{-1}\mathcal{O}_\mathcal{B}$-algebras*

$$0 \to \mathcal{G} \to \mathcal{O}' \to \mathcal{O} \to 0$$

where $\mathcal{G}$ is an ideal of square zero[2] is canonically bijective to $\mathrm{Ext}^1_\mathcal{O}(NL_{\mathcal{O}/\mathcal{O}_\mathcal{B}}, \mathcal{G})$.

Proof. To prove this we apply the previous results to the case where (13.0.1) is given by the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{G} & \longrightarrow & ? & \longrightarrow & \mathcal{O} & \longrightarrow & 0 \\
& & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & f^{-1}\mathcal{O}_\mathcal{B} & \xrightarrow{\ \mathrm{id}\ } & f^{-1}\mathcal{O}_\mathcal{B} & \longrightarrow & 0
\end{array}
$$

Thus our lemma follows from Lemma 13.3 and the fact that there exists a solution, namely $\mathcal{G} \oplus \mathcal{O}$. (See remark below for a direct construction of the bijection.) $\qquad\square$

0GQ6 **Remark 13.5.** Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (\mathcal{B}, \mathcal{O}_\mathcal{B})$ and $\mathcal{G}$ be as in Lemma 13.4. Consider an extension $0 \to \mathcal{G} \to \mathcal{O}' \to \mathcal{O} \to 0$ as in the lemma. We can choose a sheaf of sets $\mathcal{E}$ and a commutative diagram

$$
\begin{array}{ccc}
& \mathcal{E} & \\
{\scriptstyle \alpha'}\downarrow & & \searrow {\scriptstyle \alpha} \\
\mathcal{O}' & \longrightarrow & \mathcal{O}
\end{array}
$$

such that $f^{-1}\mathcal{O}_\mathcal{B}[\mathcal{E}] \to \mathcal{O}$ is surjective with kernel $\mathcal{J}$. (For example you can take any sheaf of sets surjecting onto $\mathcal{O}'$.) Then

$$NL_{\mathcal{O}/\mathcal{O}_\mathcal{B}} \cong NL(\alpha) = \left(\mathcal{J}/\mathcal{J}^2 \longrightarrow \Omega_{f^{-1}\mathcal{O}_\mathcal{B}[\mathcal{E}]/f^{-1}\mathcal{O}_\mathcal{B}} \otimes_{f^{-1}\mathcal{O}_\mathcal{B}[\mathcal{E}]} \mathcal{O} \right)$$

See Modules on Sites, Section 35 and in particular Lemma 35.2. Of course α' determines a map $f^{-1}\mathcal{O}_\mathcal{B}[\mathcal{E}] \to \mathcal{O}'$ which in turn determines a map

$$\mathcal{J}/\mathcal{J}^2 \longrightarrow \mathcal{G}$$

which in turn determines the element of $\mathrm{Ext}^1_\mathcal{O}(NL(\alpha), \mathcal{G}) = \mathrm{Ext}^1_\mathcal{O}(NL_{\mathcal{O}/\mathcal{O}_\mathcal{B}}, \mathcal{G})$ corresponding to $\mathcal{O}'$ by the bijection of the lemma.

0GQ7 **Lemma 13.6.** *Let $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{B}), \mathcal{O}_\mathcal{B})$ and $g : (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D}) \to (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C})$ be morphisms of ringed topoi. Let $\mathcal{F}$ be a $\mathcal{O}_\mathcal{C}$-module. Let $\mathcal{G}$ be a $\mathcal{O}_\mathcal{D}$-module. Let $c : g^*\mathcal{F} \to \mathcal{G}$ be a $\mathcal{O}_\mathcal{D}$-linear map. Finally, consider*

(a) *$0 \to \mathcal{F} \to \mathcal{O}_{\mathcal{C}'} \to \mathcal{O}_\mathcal{C} \to 0$ an extension of $f^{-1}\mathcal{O}_\mathcal{B}$-algebras corresponding to $\xi \in \mathrm{Ext}^1_{\mathcal{O}_\mathcal{C}}(NL_{\mathcal{O}_\mathcal{C}/\mathcal{O}_\mathcal{B}}, \mathcal{F})$, and*

(b) *$0 \to \mathcal{G} \to \mathcal{O}_{\mathcal{D}'} \to \mathcal{O}_\mathcal{D} \to 0$ an extension of $g^{-1}f^{-1}\mathcal{O}_\mathcal{B}$-algebras corresponding to $\zeta \in \mathrm{Ext}^1_{\mathcal{O}_\mathcal{D}}(NL_{\mathcal{O}_\mathcal{D}/\mathcal{O}_\mathcal{B}}, \mathcal{G})$.*

---

[2]In other words, the set of isomorphism classes of first order thickenings $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}), \mathcal{O}')$ over $(Sh(\mathcal{B}), \mathcal{O}_\mathcal{B})$ endowed with an isomorphism $\mathcal{G} \to \mathrm{Ker}(i^\sharp)$ of $\mathcal{O}$-modules.

See Lemma 13.4. Then there is a morphism

$$g' : (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}'}) \longrightarrow (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}'})$$
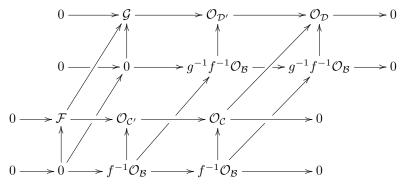
of ringed topoi over $(Sh(\mathcal{B}), \mathcal{O}_{\mathcal{B}})$ compatible with g and c if and only if ξ and ζ map to the same element of $\mathrm{Ext}^1_{\mathcal{O}_{\mathcal{D}}}(Lg^* NL_{\mathcal{O}_{\mathcal{C}}/\mathcal{O}_{\mathcal{B}}}, \mathcal{G})$.

Proof. The stament makes sense as we have the maps

$$\mathrm{Ext}^1_{\mathcal{O}_{\mathcal{C}}}(NL_{\mathcal{O}_{\mathcal{C}}/\mathcal{O}_{\mathcal{B}}}, \mathcal{F}) \to \mathrm{Ext}^1_{\mathcal{O}_{\mathcal{D}}}(Lg^* NL_{\mathcal{O}_{\mathcal{C}}/\mathcal{O}_{\mathcal{B}}}, Lg^* \mathcal{F}) \to \mathrm{Ext}^1_{\mathcal{O}_{\mathcal{D}}}(Lg^* NL_{\mathcal{O}_{\mathcal{C}}/\mathcal{O}_{\mathcal{B}}}, \mathcal{G})$$

using the map $Lg^* \mathcal{F} \to g^* \mathcal{F} \xrightarrow{c} \mathcal{G}$ and

$$\mathrm{Ext}^1_{\mathcal{O}_Y}(NL_{\mathcal{O}_{\mathcal{D}}/\mathcal{O}_{\mathcal{B}}}, \mathcal{G}) \to \mathrm{Ext}^1_{\mathcal{O}_Y}(Lg^* NL_{\mathcal{O}_{\mathcal{C}}/\mathcal{O}_{\mathcal{B}}}, \mathcal{G})$$

using the map $Lg^* NL_{\mathcal{O}_{\mathcal{C}}/\mathcal{O}_{\mathcal{B}}} \to NL_{\mathcal{O}_{\mathcal{D}}/\mathcal{O}_{\mathcal{B}}}$. The statement of the lemma can be deduced from Lemma 13.1 applied to the diagram

and a compatibility between the constructions in the proofs of Lemmas 13.4 and 13.1 whose statement and proof we omit. (See remark below for a direct argument.) $\square$

0GQ8 **Remark 13.7.** Let $f : (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \to (Sh(\mathcal{B}), \mathcal{O}_{\mathcal{B}})$, $g : (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}}) \to (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}})$, $\mathcal{F}$, $\mathcal{G}$, $c : g^* \mathcal{F} \to \mathcal{G}$, $0 \to \mathcal{F} \to \mathcal{O}_{\mathcal{C}'} \to \mathcal{O}_{\mathcal{C}} \to 0$, $\xi \in \mathrm{Ext}^1_{\mathcal{O}_{\mathcal{C}}}(NL_{\mathcal{O}_{\mathcal{C}}/\mathcal{O}_{\mathcal{B}}}, \mathcal{F})$, $0 \to \mathcal{G} \to \mathcal{O}_{\mathcal{D}'} \to \mathcal{O}_{\mathcal{D}} \to 0$, and $\zeta \in \mathrm{Ext}^1_{\mathcal{O}_{\mathcal{D}}}(NL_{\mathcal{O}_{\mathcal{D}}/\mathcal{O}_{\mathcal{B}}}, \mathcal{G})$ be as in Lemma 13.6. Using pushout along $c : g^{-1} \mathcal{F} \to \mathcal{G}$ we can construct an extension

Using pullback along $g^\sharp : g^{-1} \mathcal{O}_{\mathcal{C}} \to \mathcal{O}_{\mathcal{D}}$ we can construct an extension

A diagram chase tells us that there exists a morphism $g' : (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}'}) \to (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}'})$ over $(Sh(\mathcal{B}), \mathcal{O}_{\mathcal{B}})$ compatible with g and c if and only if $\mathcal{O}'_1$ is isomorphic to $\mathcal{O}'_2$ as $g^{-1} f^{-1} \mathcal{O}_{\mathcal{B}}$-algebra extensions of $g^{-1} \mathcal{O}_{\mathcal{C}}$ by $\mathcal{G}$. By Lemma 13.4 these extensions are classified by the LHS of
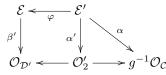
$$\mathrm{Ext}^1_{g^{-1} \mathcal{O}_{\mathcal{C}}}(NL_{g^{-1} \mathcal{O}_{\mathcal{C}}/g^{-1} f^{-1} \mathcal{O}_{\mathcal{B}}}, \mathcal{G}) = \mathrm{Ext}^1_{\mathcal{O}_{\mathcal{D}}}(Lg^* NL_{\mathcal{O}_{\mathcal{C}}/\mathcal{O}_{\mathcal{B}}}, \mathcal{G})$$

Here the equality comes from tensor-hom adjunction and the equalities

$$NL_{g^{-1}\mathcal{O}_\mathcal{C}/g^{-1}f^{-1}\mathcal{O}_\mathcal{B}} = g^{-1}NL_{\mathcal{O}_\mathcal{C}/\mathcal{O}_\mathcal{B}} \quad \text{and} \quad Lg^*NL_{\mathcal{O}_\mathcal{C}/\mathcal{O}_\mathcal{B}} = g^{-1}NL_{\mathcal{O}_\mathcal{C}/\mathcal{O}_\mathcal{B}} \otimes^{\mathbf{L}}_{g^{-1}\mathcal{O}_X} \mathcal{O}_Y$$

For the first of these see Modules on Sites, Lemma 35.3; the second follows from the definition of derived pullback. Thus, in order to see that Lemma 13.6 is true, it suffices to show that $\mathcal{O}'_1$ corresponds to the image of ξ and that $\mathcal{O}'_2$ correspond to the image of ζ. The correspondence between ξ and $\mathcal{O}'_1$ is immediate from the construction of the class ξ in Remark 13.5. For the correspondence between ζ and $\mathcal{O}'_2$, we first choose a commutative diagram

$$\begin{array}{ccc}
\mathcal{E} & & \\
\beta' \downarrow & \searrow^{\beta} & \\
\mathcal{O}_{\mathcal{D}'} & \longrightarrow & \mathcal{O}_\mathcal{D}
\end{array}$$

such that $g^{-1}f^{-1}\mathcal{O}_\mathcal{B}[\mathcal{E}] \to \mathcal{O}_\mathcal{D}$ is surjective with kernel $\mathcal{K}$. Next choose a commutative diagram

$$\begin{array}{ccccc}
\mathcal{E} & \xleftarrow{\varphi} & \mathcal{E}' & & \\
\beta' \downarrow & & \alpha' \downarrow & \searrow^{\alpha} & \\
\mathcal{O}_{\mathcal{D}'} & \longleftarrow & \mathcal{O}'_2 & \longrightarrow & g^{-1}\mathcal{O}_\mathcal{C}
\end{array}$$

such that $g^{-1}f^{-1}\mathcal{O}_\mathcal{B}[\mathcal{E}'] \to g^{-1}\mathcal{O}_\mathcal{C}$ is surjective with kernel $\mathcal{J}$. (For example just take $\mathcal{E}' = \mathcal{E} \amalg \mathcal{O}'_2$ as a sheaf of sets.) The map φ induces a map of complexes $NL(\alpha) \to NL(\beta)$ (notation as in Modules, Section 31) and in particular $\bar\varphi : \mathcal{J}/\mathcal{J}^2 \to \mathcal{K}/\mathcal{K}^2$. Then $NL(\alpha) \cong NL_{\mathcal{O}_\mathcal{D}/\mathcal{O}_\mathcal{B}}$ and $NL(\beta) \cong NL_{g^{-1}\mathcal{O}_\mathcal{C}/g^{-1}f^{-1}\mathcal{O}_\mathcal{B}}$ and the map of complexes $NL(\alpha) \to NL(\beta)$ represents the map $Lg^*NL_{\mathcal{O}_\mathcal{C}/\mathcal{O}_\mathcal{B}} \to NL_{\mathcal{O}_\mathcal{D}/\mathcal{O}_\mathcal{B}}$ used in the statement of Lemma 13.6 (see first part of its proof). Now ζ corresponds to the class of the map $\mathcal{K}/\mathcal{K}^2 \to \mathcal{G}$ induced by β', see Remark 13.5. Similarly, the extension $\mathcal{O}'_2$ corresponds to the map $\mathcal{J}/\mathcal{J}^2 \to \mathcal{G}$ induced by α'. The commutative diagram above shows that this map is the composition of the map $\mathcal{K}/\mathcal{K}^2 \to \mathcal{G}$ induced by β' with the map $\bar\varphi : \mathcal{J}/\mathcal{J}^2 \to \mathcal{K}/\mathcal{K}^2$. This proves the compatibility we were looking for.

0GQ9 **Lemma 13.8.** *Let $t : (Sh(\mathcal{B}), \mathcal{O}_\mathcal{B}) \to (Sh(\mathcal{B}'), \mathcal{O}_{\mathcal{B}'})$, $\mathcal{J} = \mathrm{Ker}(t^\sharp)$, $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{B}), \mathcal{O}_\mathcal{B})$, $\mathcal{G}$, and $c : \mathcal{J} \to \mathcal{G}$ be as in (13.0.1). Denote $\xi \in \mathrm{Ext}^1_{\mathcal{O}_\mathcal{B}}(NL_{\mathcal{O}_\mathcal{B}/\mathcal{O}_{\mathcal{B}'}}, \mathcal{J})$ the element corresponding to the extension $\mathcal{O}_{\mathcal{B}'}$ of $\mathcal{O}_\mathcal{B}$ by $\mathcal{J}$ via Lemma 13.4. The set of isomorphism classes of solutions is canonically bijective to the fibre of*

$$\mathrm{Ext}^1_\mathcal{O}(NL_{\mathcal{O}/\mathcal{O}_{\mathcal{B}'}}, \mathcal{G}) \to \mathrm{Ext}^1_\mathcal{O}(Lf^*NL_{\mathcal{O}_\mathcal{B}/\mathcal{O}_{\mathcal{B}'}}, \mathcal{G})$$

over the image of ξ.

Proof. By Lemma 13.4 applied to $t \circ f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{B}'), \mathcal{O}_{\mathcal{B}'})$ and the $\mathcal{O}$-module $\mathcal{G}$ we see that elements ζ of $\mathrm{Ext}^1_\mathcal{O}(NL_{\mathcal{O}/\mathcal{O}_{\mathcal{B}'}}, \mathcal{G})$ parametrize extensions $0 \to \mathcal{G} \to \mathcal{O}' \to \mathcal{O} \to 0$ of $f^{-1}\mathcal{O}_{\mathcal{B}'}$-algebras. By Lemma 13.6 applied to

$$(Sh(\mathcal{C}), \mathcal{O}) \xrightarrow{f} (Sh(\mathcal{B}), \mathcal{O}_\mathcal{B}) \xrightarrow{t} (Sh(\mathcal{B}'), \mathcal{O}_{\mathcal{B}'})$$

and $c : \mathcal{J} \to \mathcal{G}$ we see that there is an morphism

$$f' : (Sh(\mathcal{C}), \mathcal{O}') \longrightarrow (Sh(\mathcal{B}'), \mathcal{O}_{\mathcal{B}'})$$

over $(Sh(\mathcal{B}'), \mathcal{O}_{\mathcal{B}'})$ compatible with c and f if and only if ζ maps to ξ. Of course this is the same thing as saying $\mathcal{O}'$ is a solution of (13.0.1). $\square$

14. Deformations of algebraic spaces

0D15   In this section we spell out what the results in Section 13 mean for deformations of
algebraic spaces.

0D16   **Lemma 14.1.** *Let S be a scheme. Let $i : Z \to Z'$ be a morphism of algebraic
spaces over S. The following are equivalent*

(1) *i is a thickening of algebraic spaces as defined in More on Morphisms of
Spaces, Section 9, and*
(2) *the associated morphism $i_{small} : (Sh(Z_{étale}), \mathcal{O}_Z) \to (Sh(Z'_{étale}), \mathcal{O}_{Z'})$ of
ringed topoi (Properties of Spaces, Lemma 21.3) is a thickening in the sense
of Section 9.*

Proof. We stress that this is not a triviality.

Assume (1). By More on Morphisms of Spaces, Lemma 9.6 the morphism i induces
an equivalence of small étale sites and in particular of topoi. Of course $i^\sharp$ is surjective
with locally nilpotent kernel by definition of thickenings.

Assume (2). (This direction is less important and more of a curiosity.) For any
étale morphism $Y' \to Z'$ we see that $Y = Z \times_{Z'} Y'$ has the same étale topos as Y'.
In particular, Y' is quasi-compact if and only if Y is quasi-compact because being
quasi-compact is a topos theoretic notion (Sites, Lemma 17.3). Having said this we
see that Y' is quasi-compact and quasi-separated if and only if Y is quasi-compact
and quasi-separated (because you can characterize Y' being quasi-separated by
saying that for all Y'_1, Y'_2 quasi-compact algebraic spaces étale over Y' we have
that $Y'_1 \times_{Y'} Y'_2$ is quasi-compact). Take Y' affine. Then the algebraic space Y
is quasi-compact and quasi-separated. For any quasi-coherent $\mathcal{O}_Y$-module $\mathcal{F}$ we
have $H^q(Y, \mathcal{F}) = H^q(Y', (Y \to Y')_* \mathcal{F})$ because the étale topoi are the same. Then
$H^q(Y', (Y \to Y')_* \mathcal{F}) = 0$ because the pushforward is quasi-coherent (Morphisms
of Spaces, Lemma 11.2) and Y is affine. It follows that Y' is affine by Cohomology
of Spaces, Proposition 16.7 (there surely is a proof of this direction of the lemma
avoiding this reference). Hence i is an affine morphism. In the affine case it follows
easily from the conditions in Section 9 that i is a thickening of algebraic spaces. $\square$

0D17   **Lemma 14.2.** *Let S be a scheme. Let $Y \subset Y'$ be a first order thickening of
algebraic spaces over S. Let $f : X \to Y$ be a flat morphism of algebraic spaces over
S. If there exists a flat morphism $f' : X' \to Y'$ of algebraic spaces over S and an
isomorphsm $a : X \to X' \times_{Y'} Y$ over Y, then*

(1) *the set of isomorphism classes of pairs $(f' : X' \to Y', a)$ is principal homo-
geneous under $\mathrm{Ext}^1_{\mathcal{O}_X}(NL_{X/Y}, f^*\mathcal{C}_{Y/Y'})$, and*
(2) *the set of automorphisms of $\varphi : X' \to X'$ over Y' which reduce to the
identity on $X' \times_{Y'} Y$ is $\mathrm{Ext}^0_{\mathcal{O}_X}(NL_{X/Y}, f^*\mathcal{C}_{Y/Y'})$.*

Proof. We will apply the material on deformations of ringed topoi to the small
étale topoi of the algebraic spaces in the lemma. We may think of X as a closed
subspace of X' so that $(f, f') : (X \subset X') \to (Y \subset Y')$ is a morphism of first
order thickenings. By Lemma 14.1 this translates into a morphism of thickenings
of ringed topoi. Then we see from More on Morphisms of Spaces, Lemma 18.1 (or
from the more general Lemma 11.2) that the ideal sheaf of X in X' is equal to
$f^*\mathcal{C}_{Y'/Y}$ and this is in fact equivalent to flatness of X' over Y'. Hence we have a

commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & f^*\mathcal{C}_{Y/Y'} & \longrightarrow & \mathcal{O}_{X'} & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 \\
& & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & f^{-1}_{small}\mathcal{C}_{Y/Y'} & \longrightarrow & f^{-1}_{small}\mathcal{O}_{Y'} & \longrightarrow & f^{-1}_{small}\mathcal{O}_Y & \longrightarrow & 0
\end{array}
$$

Please compare with (13.0.1). Observe that automorphisms φ as in (2) give automorphisms $\varphi^\sharp : \mathcal{O}_{X'} \to \mathcal{O}_{X'}$ fitting in the diagram above. Conversely, an automorphism $\alpha : \mathcal{O}_{X'} \to \mathcal{O}_{X'}$ fitting into the diagram of sheaves above is equal to $\varphi^\sharp$ for some automorphism φ as in (2) by More on Morphisms of Spaces, Lemma 9.2. Finally, by More on Morphisms of Spaces, Lemma 9.7 if we find another sheaf of rings $\mathcal{A}$ on $X_{\acute{e}tale}$ fitting into the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & f^*\mathcal{C}_{Y/Y'} & \longrightarrow & \mathcal{A} & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 \\
& & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & f^{-1}_{small}\mathcal{C}_{Y/Y'} & \longrightarrow & f^{-1}_{small}\mathcal{O}_{Y'} & \longrightarrow & f^{-1}_{small}\mathcal{O}_Y & \longrightarrow & 0
\end{array}
$$

then there exists a first order thickening $X \subset X''$ with $\mathcal{O}_{X''} = \mathcal{A}$ and applying More on Morphisms of Spaces, Lemma 9.2 once more, we obtain a morphism $(f, f'') : (X \subset X'') \to (Y \subset Y')$ with all the desired properties. Thus part (1) follows from Lemma 13.3 and part (2) from part (2) of Lemma 13.1. (Note that $NL_{X/Y}$ as defined for a morphism of algebraic spaces in More on Morphisms of Spaces, Section 21 agrees with $NL_{X/Y}$ as used in Section 13.) $\qquad\square$

Let S be a scheme. Let $f : X \to B$ be a morphism of algebraic spaces over S. Let $\mathcal{F} \to \mathcal{G}$ be a homomorphism of $\mathcal{O}_X$-modules (not necessarily quasi-coherent). Consider the functor

$$
F : \left\{
\begin{array}{c}
\text{extensions of } f^{-1}\mathcal{O}_B \text{ algebras} \\
0 \to \mathcal{F} \to \mathcal{O}' \to \mathcal{O}_X \to 0 \\
\text{where } \mathcal{F} \text{ is an ideal of square zero}
\end{array}
\right\} \longrightarrow \left\{
\begin{array}{c}
\text{extensions of } f^{-1}\mathcal{O}_B \text{ algebras} \\
0 \to \mathcal{G} \to \mathcal{O}' \to \mathcal{O}_X \to 0 \\
\text{where } \mathcal{G} \text{ is an ideal of square zero}
\end{array}
\right\}
$$

given by pushout.

0D3P **Lemma 14.3.** *In the situation above assume that X is quasi-compact and quasi-separated and that $DQ_X(\mathcal{F}) \to DQ_X(\mathcal{G})$ (Derived Categories of Spaces, Section 19) is an isomorphism. Then the functor F is an equivalence of categories.*

Proof. Recall that $NL_{X/B}$ is an object of $D_{QCoh}(\mathcal{O}_X)$, see More on Morphisms of Spaces, Lemma 21.4. Hence our assumption implies the maps

$$
\mathrm{Ext}^i_X(NL_{X/B}, \mathcal{F}) \longrightarrow \mathrm{Ext}^i_X(NL_{X/B}, \mathcal{G})
$$

are isomorphisms for all i. This implies our functor is fully faithful by Lemma 13.1. On the other hand, the functor is essentially surjective by Lemma 13.3 because we have the solutions $\mathcal{O}_X \oplus \mathcal{F}$ and $\mathcal{O}_X \oplus \mathcal{G}$ in both categories. $\qquad\square$

Let S be a scheme. Let $B \subset B'$ be a first order thickening of algebraic spaces over S with ideal sheaf $\mathcal{J}$ which we view either as a quasi-coherent $\mathcal{O}_B$-module or as a quasi-coherent sheaf of ideals on B', see More on Morphisms of Spaces, Section 9. Let $f : X \to B$ be a morphism of algebraic spaces over S. Let $\mathcal{F} \to \mathcal{G}$ be a homomorphism of $\mathcal{O}_X$-modules (not necessarily quasi-coherent). Let $c : f^{-1}\mathcal{J} \to \mathcal{F}$

be a map of $f^{-1}\mathcal{O}_B$-modules and denote $c' : f^{-1}\mathcal{J} \to \mathcal{G}$ the composition. Consider the functor

$$FT : \{\text{solutions to (13.0.1) for } \mathcal{F} \text{ and } c\} \longrightarrow \{\text{solutions to (13.0.1) for } \mathcal{G} \text{ and } c'\}$$

given by pushout.

0D3Q **Lemma 14.4.** *In the situation above assume that X is quasi-compact and quasi-separated and that $DQ_X(\mathcal{F}) \to DQ_X(\mathcal{G})$ (Derived Categories of Spaces, Section 19) is an isomorphism. Then the functor FT is an equivalence of categories.*

Proof. A solution of (13.0.1) for $\mathcal{F}$ in particular gives an extension of $f^{-1}\mathcal{O}_{B'}$-algebras

$$0 \to \mathcal{F} \to \mathcal{O}' \to \mathcal{O}_X \to 0$$

where $\mathcal{F}$ is an ideal of square zero. Similarly for $\mathcal{G}$. Moreover, given such an extension, we obtain a map $c_{\mathcal{O}'} : f^{-1}\mathcal{J} \to \mathcal{F}$. Thus we are looking at the full subcategory of such extensions of $f^{-1}\mathcal{O}_{B'}$-algebras with $c = c_{\mathcal{O}'}$. Clearly, if $\mathcal{O}'' = F(\mathcal{O}')$ where F is the equivalence of Lemma 14.3 (applied to $X \to B'$ this time), then $c_{\mathcal{O}''}$ is the composition of $c_{\mathcal{O}'}$ and the map $\mathcal{F} \to \mathcal{G}$. This proves the lemma. $\square$

15. Deformations of complexes

0DYQ This section is a warmup for the next one. We will use as much as possible the material in the chapters on commutative algebra.

0DYR **Lemma 15.1.** *Let $R' \to R$ be a surjection of rings whose kernel is an ideal I of square zero. For every $K \in D^-(R)$ there is a canonical map*

$$\omega(K) : K \longrightarrow K \otimes_R^{\mathbf{L}} I[2]$$

in $D(R)$ with the following properties

(1) *$\omega(K) = 0$ if and only if there exists $K' \in D(R')$ with $K' \otimes_{R'}^{\mathbf{L}} R = K$,*
(2) *given $K \to L$ in $D^-(R)$ the diagram*

$$
\begin{array}{ccc}
K & \xrightarrow{\;\;\omega(K)\;\;} & K \otimes_R^{\mathbf{L}} I[2] \\
\downarrow & & \downarrow \\
L & \xrightarrow{\;\;\omega(L)\;\;} & L \otimes_R^{\mathbf{L}} I[2]
\end{array}
$$

commutes, and
(3) *formation of $\omega(K)$ is compatible with ring maps $R' \to S'$ (see proof for a precise statement).*

Proof. Choose a bounded above complex $K^\bullet$ of free R-modules representing K. Then we can choose free R'-modules $(K')^n$ lifting K^n. We can choose R'-module maps $(d')_K^n : (K')^n \to (K')^{n+1}$ lifting the differentials $d_K^n : K^n \to K^{n+1}$ of $K^\bullet$. Although the compositions

$$(d')_K^{n+1} \circ (d')_K^n : (K')^n \to (K')^{n+2}$$

may not be zero, they do factor as

$$(K')^n \to K^n \xrightarrow{\omega_K^n} K^{n+2} \otimes_R I = I(K')^{n+2} \to (K')^{n+2}$$

because $d^{n+1} \circ d^n = 0$. A calculation shows that ω_K^n defines a map of complexes. This map of complexes defines $\omega(K)$.

Let us prove this construction is compatible with a map of complexes $\alpha^\bullet : K^\bullet \to L^\bullet$ of bounded above free R-modules and given choices of lifts $(K')^n, (L')^n, (d')_K^n, (d')_L^n$. Namely, choose $(\alpha')^n : (K')^n \to (L')^n$ lifting the components $\alpha^n : K^n \to L^n$. As before we get a factorization

$$(K')^n \to K^n \xrightarrow{h^n} L^{n+1} \otimes_R I = I(L')^{n+1} \to (L')^{n+2}$$

of $(d')_L^n \circ (\alpha')^n - (\alpha')^{n+1} \circ (d')_K^n$. Then it is an pleasant calculation to show that

$$\omega_L^n \circ \alpha^n = (d_L^{n+1} \otimes \mathrm{id}_I) \circ h^n + h^{n+1} \circ d_K^n + (\alpha^{n+2} \otimes \mathrm{id}_I) \circ \omega_K^n$$

This proves the commutativity of the diagram in (2) of the lemma in this particular case. Using this for two different choices of bounded above free complexes representing K, we find that $\omega(K)$ is well defined! And of course (2) holds in general as well.

If K lifts to K' in $D^-(R')$, then we can represent K' by a bounded above complex of free R'-modules and we see immediately that $\omega(K) = 0$. Conversely, going back to our choices $K^\bullet, (K')^n, (d')_K^n$, if $\omega(K) = 0$, then we can find $g^n : K^n \to K^{n+1} \otimes_R I$ with

$$\omega^n = (d_K^{n+1} \otimes \mathrm{id}_I) \circ g^n + g^{n+1} \circ d_K^n$$

This means that with differentials $(d')_K^n + g^n : (K')^n \to (K')^{n+1}$ we obtain a complex of free R'-modules lifting $K^\bullet$. This proves (1).

Finally, part (3) means the following: Let $R' \to S'$ be a map of rings. Set $S = S' \otimes_{R'} R$ and denote $J = IS' \subset S'$ the square zero kernel of $S' \to S$. Then given $K \in D^-(R)$ the statement is that we get a commutative diagram

$$
\begin{array}{ccc}
K \otimes_R^{\mathbf{L}} S & \xrightarrow{\ \omega(K) \otimes \mathrm{id}\ } & (K \otimes_R^{\mathbf{L}} I[2]) \otimes_R^{\mathbf{L}} S \\
\downarrow & & \downarrow \\
K \otimes_R^{\mathbf{L}} S & \xrightarrow{\ \omega(K \otimes_R^{\mathbf{L}} S)\ } & (K \otimes_R^{\mathbf{L}} S) \otimes_S^{\mathbf{L}} J[2]
\end{array}
$$

Here the right vertical arrow comes from

$$(K \otimes_R^{\mathbf{L}} I[2]) \otimes_R^{\mathbf{L}} S = (K \otimes_R^{\mathbf{L}} S) \otimes_S^{\mathbf{L}} (I \otimes_R^{\mathbf{L}} S)[2] \longrightarrow (K \otimes_R^{\mathbf{L}} S) \otimes_S^{\mathbf{L}} J[2]$$

Choose $K^\bullet, (K')^n$, and $(d')_K^n$ as above. Then we can use $K^\bullet \otimes_R S, (K')^n \otimes_{R'} S'$, and $(d')_K^n \otimes \mathrm{id}_{S'}$ for the construction of $\omega(K \otimes_R^{\mathbf{L}} S)$. With these choices commutativity is immediately verified on the level of maps of complexes. $\square$

16. Deformations of complexes on ringed topoi

0DIS   This material is taken from [Lie06].

The material in this section works in the setting of a first order thickening of ringed topoi as defined in Section 9. However, in order to simplify the notation we will assume the underlying sites $\mathcal{C}$ and $\mathcal{D}$ are the same. Moreover, the surjective homomorphism $\mathcal{O}' \to \mathcal{O}$ of sheaves of rings will be denoted $\mathcal{O} \to \mathcal{O}_0$ as is perhaps more customary in the literature.

0DIT   **Lemma 16.1.** *Let $\mathcal{C}$ be a site. Let $\mathcal{O} \to \mathcal{O}_0$ be a surjection of sheaves of rings. Assume given the following data*

(1) *flat $\mathcal{O}$-modules $\mathcal{G}^n$,*
(2) *maps of $\mathcal{O}$-modules $\mathcal{G}^n \to \mathcal{G}^{n+1}$,*

(3) *a complex $\mathcal{K}_0^\bullet$ of $\mathcal{O}_0$-modules,*
(4) *maps of $\mathcal{O}$-modules $\mathcal{G}^n \to \mathcal{K}_0^n$*

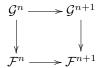*such that*

(a) $H^n(\mathcal{K}_0^\bullet) = 0$ *for $n \gg 0$,*
(b) $\mathcal{G}^n = 0$ *for $n \gg 0$,*
(c) *with $\mathcal{G}_0^n = \mathcal{G}^n \otimes_{\mathcal{O}} \mathcal{O}_0$ the induced maps determine a complex $\mathcal{G}_0^\bullet$ and a map of complexes $\mathcal{G}_0^\bullet \to \mathcal{K}_0^\bullet$.*
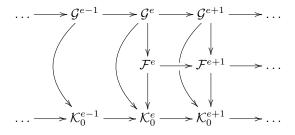
Then there exist

(i) *flat $\mathcal{O}$-modules $\mathcal{F}^n$,*
(ii) *maps of $\mathcal{O}$-modules $\mathcal{F}^n \to \mathcal{F}^{n+1}$,*
(iii) *maps of $\mathcal{O}$-modules $\mathcal{F}^n \to \mathcal{K}_0^n$,*
(iv) *maps of $\mathcal{O}$-modules $\mathcal{G}^n \to \mathcal{F}^n$,*

such that $\mathcal{F}^n = 0$ for $n \gg 0$, such that the diagrams

$$
\begin{array}{ccc}
\mathcal{G}^n & \longrightarrow & \mathcal{G}^{n+1} \\
\downarrow & & \downarrow \\
\mathcal{F}^n & \longrightarrow & \mathcal{F}^{n+1}
\end{array}
$$

commute for all n, such that the composition $\mathcal{G}^n \to \mathcal{F}^n \to \mathcal{K}_0^n$ is the given map $\mathcal{G}^n \to \mathcal{K}_0^n$, and such that with $\mathcal{F}_0^n = \mathcal{F}^n \otimes_{\mathcal{O}} \mathcal{O}_0$ we obtain a complex $\mathcal{F}_0^\bullet$ and map of complexes $\mathcal{F}_0^\bullet \to \mathcal{K}_0^\bullet$ which is a quasi-isomorphism.

Proof. We will prove by descending induction on e that we can find $\mathcal{F}^n$, $\mathcal{G}^n \to \mathcal{F}^n$, and $\mathcal{F}^n \to \mathcal{F}^{n+1}$ for $n \geq e$ fitting into a commutative diagram

$$
\begin{array}{ccccccc}
\cdots \longrightarrow & \mathcal{G}^{e-1} & \longrightarrow & \mathcal{G}^e & \longrightarrow & \mathcal{G}^{e+1} & \longrightarrow \cdots \\
& & & \downarrow & & \downarrow & \\
& & & \mathcal{F}^e & \longrightarrow & \mathcal{F}^{e+1} & \longrightarrow \cdots \\
& & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & \mathcal{K}_0^{e-1} & \longrightarrow & \mathcal{K}_0^e & \longrightarrow & \mathcal{K}_0^{e+1} & \longrightarrow \cdots
\end{array}
$$

such that $\mathcal{F}_0^\bullet$ is a complex, the induced map $\mathcal{F}_0^\bullet \to \mathcal{K}_0^\bullet$ induces an isomorphism on H^n for $n > e$ and a surjection for $n = e$. For $e \gg 0$ this is true because we can take $\mathcal{F}^n = 0$ for $n \geq e$ in that case by assumptions (a) and (b).

Induction step. We have to construct $\mathcal{F}^{e-1}$ and the maps $\mathcal{G}^{e-1} \to \mathcal{F}^{e-1}$, $\mathcal{F}^{e-1} \to \mathcal{F}^e$, and $\mathcal{F}^{e-1} \to \mathcal{K}_0^{e-1}$. We will choose $\mathcal{F}^{e-1} = A \oplus B \oplus C$ as a direct sum of three pieces.

For the first we take $A = \mathcal{G}^{e-1}$ and we choose our map $\mathcal{G}^{e-1} \to \mathcal{F}^{e-1}$ to be the inclusion of the first summand. The maps $A \to \mathcal{K}_0^{e-1}$ and $A \to \mathcal{F}^e$ will be the obvious ones.

To choose B we consider the surjection (by induction hypothesis)

$$
\gamma : \mathrm{Ker}(\mathcal{F}_0^e \to \mathcal{F}_0^{e+1}) \longrightarrow \mathrm{Ker}(\mathcal{K}_0^e \to \mathcal{K}_0^{e+1})/\mathrm{Im}(\mathcal{K}_0^{e-1} \to \mathcal{K}_0^e)
$$

We can choose a set I, for each $i \in I$ an object U_i of $\mathcal{C}$, and sections $s_i \in \mathcal{F}^e(U_i)$, $t_i \in \mathcal{K}_0^{e-1}(U_i)$ such that

(1) s_i maps to a section of $\text{Ker}(\gamma) \subset \text{Ker}(\mathcal{F}_0^e \to \mathcal{F}_0^{e+1})$,
(2) s_i and t_i map to the same section of $\mathcal{K}_0^e$,
(3) the sections s_i generate $\text{Ker}(\gamma)$ as an $\mathcal{O}_0$-module.

We omit giving the full justification for this; one uses that $\mathcal{F}^e \to \mathcal{F}_0^e$ is a surjective maps of sheaves of sets. Then we set to put

$$B = \bigoplus\nolimits_{i \in I} j_{U_i!}\mathcal{O}_{U_i}$$

and define the maps $B \to \mathcal{F}^e$ and $B \to \mathcal{K}_0^{e-1}$ by using s_i and t_i to determine where to send the summand $j_{U_i!}\mathcal{O}_{U_i}$.

With $\mathcal{F}^{e-1} = A \oplus B$ and maps as above, this produces a diagram as above for $e-1$ such that $\mathcal{F}_0^\bullet \to \mathcal{K}_0^\bullet$ induces an isomorphism on H^n for $n \geq e$. To get the map to be surjective on H^{e-1} we choose the summand C as follows. Choose a set J, for each $j \in J$ an object U_j of $\mathcal{C}$ and a section t_j of $\text{Ker}(\mathcal{K}_0^{e-1} \to \mathcal{K}_0^e)$ over U_j such that these sections generate this kernel over $\mathcal{O}_0$. Then we put

$$C = \bigoplus\nolimits_{j \in J} j_{U_j!}\mathcal{O}_{U_j}$$

and the zero map $C \to \mathcal{F}^e$ and the map $C \to \mathcal{K}_0^{e-1}$ by using s_j to determine where to the summand $j_{U_j!}\mathcal{O}_{U_j}$. This finishes the induction step by taking $\mathcal{F}^{e-1} = A \oplus B \oplus C$ and maps as indicated. $\square$

0DIU **Lemma 16.2.** *Let $\mathcal{C}$ be a site. Let $\mathcal{O} \to \mathcal{O}_0$ be a surjection of sheaves of rings whose kernel is an ideal sheaf $\mathcal{I}$ of square zero. For every object K_0 in $D^-(\mathcal{O}_0)$ there is a canonical map*

$$\omega(K_0) : K_0 \longrightarrow K_0 \otimes_{\mathcal{O}_0}^{\mathbf{L}} \mathcal{I}[2]$$

in $D(\mathcal{O}_0)$ such that for any map $K_0 \to L_0$ in $D^-(\mathcal{O}_0)$ the diagram

$$\begin{array}{ccc} K_0 & \xrightarrow{\ \omega(K_0)\ } & (K_0 \otimes_{\mathcal{O}_0}^{\mathbf{L}} \mathcal{I})[2] \\ \downarrow & & \downarrow \\ L_0 & \xrightarrow{\ \omega(L_0)\ } & (L_0 \otimes_{\mathcal{O}_0}^{\mathbf{L}} \mathcal{I})[2] \end{array}$$

commutes.

Proof. Represent K_0 by any complex $\mathcal{K}_0^\bullet$ of $\mathcal{O}_0$-modules. Apply Lemma 16.1 with $\mathcal{G}^n = 0$ for all n. Denote $d : \mathcal{F}^n \to \mathcal{F}^{n+1}$ the maps produced by the lemma. Then we see that $d \circ d : \mathcal{F}^n \to \mathcal{F}^{n+2}$ is zero modulo $\mathcal{I}$. Since $\mathcal{F}^n$ is flat, we see that $\mathcal{I}\mathcal{F}^n = \mathcal{F}^n \otimes_{\mathcal{O}} \mathcal{I} = \mathcal{F}_0^n \otimes_{\mathcal{O}_0} \mathcal{I}$. Hence we obtain a canonical map of complexes

$$d \circ d : \mathcal{F}_0^\bullet \longrightarrow (\mathcal{F}_0^\bullet \otimes_{\mathcal{O}_0} \mathcal{I})[2]$$

Since $\mathcal{F}_0^\bullet$ is a bounded above complex of flat $\mathcal{O}_0$-modules, it is K-flat and may be used to compute derived tensor product. Moreover, the map of complexes $\mathcal{F}_0^\bullet \to \mathcal{K}_0^\bullet$ is a quasi-isomorphism by construction. Therefore the source and target of the map just constructed represent K_0 and $K_0 \otimes_{\mathcal{O}_0}^{\mathbf{L}} \mathcal{I}[2]$ and we obtain our map $\omega(K_0)$.

Let us show that this procedure is compatible with maps of complexes. Namely, let $\mathcal{L}_0^\bullet$ represent another object of $D^-(\mathcal{O}_0)$ and suppose that

$$\mathcal{K}_0^\bullet \longrightarrow \mathcal{L}_0^\bullet$$

is a map of complexes. Apply Lemma 16.1 for the complex $\mathcal{L}_0^\bullet$, the flat modules $\mathcal{F}^n$, the maps $\mathcal{F}^n \to \mathcal{F}^{n+1}$, and the compositions $\mathcal{F}^n \to \mathcal{K}_0^n \to \mathcal{L}_0^n$ (we apologize for the reversal of letters used). We obtain flat modules $\mathcal{G}^n$, maps $\mathcal{F}^n \to \mathcal{G}^n$, maps $\mathcal{G}^n \to \mathcal{G}^{n+1}$, and maps $\mathcal{G}^n \to \mathcal{L}_0^n$ with all properties as in the lemma. Then it is clear that

$$
\begin{array}{ccc}
\mathcal{F}_0^\bullet & \longrightarrow & (\mathcal{F}_0^\bullet \otimes_{\mathcal{O}_0} \mathcal{I})[2] \\
\downarrow & & \downarrow \\
\mathcal{G}_0^\bullet & \longrightarrow & (\mathcal{G}_0^\bullet \otimes_{\mathcal{O}_0} \mathcal{I})[2]
\end{array}
$$

is a commutative diagram of complexes.

To see that $\omega(K_0)$ is well defined, suppose that we have two complexes $\mathcal{K}_0^\bullet$ and $(\mathcal{K}_0')^\bullet$ of $\mathcal{O}_0$-modules representing K_0 and two systems $(\mathcal{F}^n, d : \mathcal{F}^n \to \mathcal{F}^{n+1}, \mathcal{F}^n \to \mathcal{K}_0^n)$ and $((\mathcal{F}')^n, d : (\mathcal{F}')^n \to (\mathcal{F}')^{n+1}, (\mathcal{F}')^n \to \mathcal{K}_0^n)$ as above. Then we can choose a complex $(\mathcal{K}_0'')^\bullet$ and quasi-isomorphisms $\mathcal{K}_0^\bullet \to (\mathcal{K}_0'')^\bullet$ and $(\mathcal{K}_0')^\bullet \to (\mathcal{K}_0'')^\bullet$ realizing the fact that both complexes represent K_0 in the derived category. Next, we apply the result of the previous paragraph to

$$
(\mathcal{K}_0)^\bullet \oplus (\mathcal{K}_0')^\bullet \longrightarrow (\mathcal{K}_0'')^\bullet
$$

This produces a commutative diagram

$$
\begin{array}{ccc}
\mathcal{F}_0^\bullet \oplus (\mathcal{F}_0')^\bullet & \longrightarrow & (\mathcal{F}_0^\bullet \otimes_{\mathcal{O}_0} \mathcal{I})[2] \oplus ((\mathcal{F}_0')^\bullet \otimes_{\mathcal{O}_0} \mathcal{I})[2] \\
\downarrow & & \downarrow \\
\mathcal{G}_0^\bullet & \longrightarrow & (\mathcal{G}_0^\bullet \otimes_{\mathcal{O}_0} \mathcal{I})[2]
\end{array}
$$

Since the vertical arrows give quasi-isomorphisms on the summands we conclude the desired commutativity in $D(\mathcal{O}_0)$.

Having established well-definedness, the statement on compatibility with maps is a consequence of the result in the second paragraph. $\square$

0DIV **Lemma 16.3.** *Let* $(\mathcal{C}, \mathcal{O})$ *be a ringed site. Let* $\alpha : K \to L$ *be a map of* $D^-(\mathcal{O})$. *Let* $\mathcal{F}$ *be a sheaf of* $\mathcal{O}$-modules. *Let* $n \in \mathbf{Z}$.

(1) *If* $H^i(\alpha)$ *is an isomorphism for* $i \geq n$, *then* $H^i(\alpha \otimes_{\mathcal{O}}^{\mathbf{L}} id_{\mathcal{F}})$ *is an isomorphism for* $i \geq n$.

(2) *If* $H^i(\alpha)$ *is an isomorphism for* $i > n$ *and surjective for* $i = n$, *then* $H^i(\alpha \otimes_{\mathcal{O}}^{\mathbf{L}} id_{\mathcal{F}})$ *is an isomorphism for* $i > n$ *and surjective for* $i = n$.

Proof. Choose a distinguished triangle

$$
K \to L \to C \to K[1]
$$

In case (2) we see that $H^i(C) = 0$ for $i \geq n$. Hence $H^i(C \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{F}) = 0$ for $i \geq n$ by (the dual of) Derived Categories, Lemma 16.1. This in turn shows that $H^i(\alpha \otimes_{\mathcal{O}}^{\mathbf{L}} id_{\mathcal{F}})$ is an isomorphism for $i > n$ and surjective for $i = n$. In case (1) we moreover see that $H^{n-1}(L) \to H^{n-1}(C)$ is surjective. Considering the diagram

$$
\begin{array}{ccc}
H^{n-1}(L) \otimes_{\mathcal{O}} \mathcal{F} & \longrightarrow & H^{n-1}(C) \otimes_{\mathcal{O}} \mathcal{F} \\
\downarrow & & \| \\
H^{n-1}(L \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{F}) & \longrightarrow & H^{n-1}(C \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{F})
\end{array}
$$

we conclude the lower horizontal arrow is surjective. Combined with what was said before this implies that $H^n(\alpha \otimes^{\mathbf{L}}_{\mathcal{O}} \mathrm{id}_{\mathcal{F}})$ is an isomorphism. $\square$

0DIW   **Lemma 16.4.** *Let $\mathcal{C}$ be a site. Let $\mathcal{O} \to \mathcal{O}_0$ be a surjection of sheaves of rings whose kernel is an ideal sheaf $\mathcal{I}$ of square zero. For every object K_0 in $D^-(\mathcal{O}_0)$ the following are equivalent*

(1) *the class $\omega(K_0) \in \mathrm{Ext}^2_{\mathcal{O}_0}(K_0, K_0 \otimes_{\mathcal{O}_0} \mathcal{I})$ constructed in Lemma 16.2 is zero,*
(2) *there exists $K \in D^-(\mathcal{O})$ with $K \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{O}_0 = K_0$ in $D(\mathcal{O}_0)$.*

Proof. Let K be as in (2). Then we can represent K by a bounded above complex $\mathcal{F}^\bullet$ of flat $\mathcal{O}$-modules. Then $\mathcal{F}^\bullet_0 = \mathcal{F}^\bullet \otimes_{\mathcal{O}} \mathcal{O}_0$ represents K_0 in $D(\mathcal{O}_0)$. Since $d_{\mathcal{F}^\bullet} \circ d_{\mathcal{F}^\bullet} = 0$ as $\mathcal{F}^\bullet$ is a complex, we see from the very construction of $\omega(K_0)$ that it is zero.

Assume (1). Let $\mathcal{F}^n$, $d : \mathcal{F}^n \to \mathcal{F}^{n+1}$ be as in the construction of $\omega(K_0)$. The nullity of $\omega(K_0)$ implies that the map

$$\omega = d \circ d : \mathcal{F}^\bullet_0 \longrightarrow (\mathcal{F}^\bullet_0 \otimes_{\mathcal{O}_0} \mathcal{I})[2]$$

is zero in $D(\mathcal{O}_0)$. By definition of the derived category as the localization of the homotopy category of complexes of $\mathcal{O}_0$-modules, there exists a quasi-isomorphism $\alpha : \mathcal{G}^\bullet_0 \to \mathcal{F}^\bullet_0$ such that there exist $\mathcal{O}_0$-modules maps $h^n : \mathcal{G}^n_0 \to \mathcal{F}^{n+1}_0 \otimes_{\mathcal{O}} \mathcal{I}$ with

$$\omega \circ \alpha = d_{\mathcal{F}^\bullet_0 \otimes \mathcal{I}} \circ h + h \circ d_{\mathcal{G}^\bullet_0}$$

We set

$$\mathcal{H}^n = \mathcal{F}^n \times_{\mathcal{F}^n_0} \mathcal{G}^n_0$$

and we define

$$d' : \mathcal{H}^n \longrightarrow \mathcal{H}^{n+1}, \quad (f^n, g^n_0) \longmapsto (d(f^n) - h^n(g^n_0), d(g^n_0))$$

with obvious notation using that $\mathcal{F}^{n+1}_0 \otimes_{\mathcal{O}_0} \mathcal{I} = \mathcal{F}^{n+1} \otimes_{\mathcal{O}} \mathcal{I} = \mathcal{I}\mathcal{F}^{n+1} \subset \mathcal{F}^{n+1}$. Then one checks $d' \circ d' = 0$ by our choice of h^n and definition of ω. Hence $\mathcal{H}^\bullet$ defines an object in $D(\mathcal{O})$. On the other hand, there is a short exact sequence of complexes of $\mathcal{O}$-modules

$$0 \to \mathcal{F}^\bullet_0 \otimes_{\mathcal{O}_0} \mathcal{I} \to \mathcal{H}^\bullet \to \mathcal{G}^\bullet_0 \to 0$$

We still have to show that $\mathcal{H}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{O}_0$ is isomorphic to K_0. Choose a quasi-isomorphism $\mathcal{E}^\bullet \to \mathcal{H}^\bullet$ where $\mathcal{E}^\bullet$ is a bounded above complex of flat $\mathcal{O}$-modules. We obtain a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{E}^\bullet \otimes_{\mathcal{O}} \mathcal{I} & \longrightarrow & \mathcal{E}^\bullet & \longrightarrow & \mathcal{E}^\bullet_0 & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \delta} & & \\
0 & \longrightarrow & \mathcal{F}^\bullet_0 \otimes_{\mathcal{O}_0} \mathcal{I} & \longrightarrow & \mathcal{H}^\bullet & \longrightarrow & \mathcal{G}^\bullet_0 & \longrightarrow & 0
\end{array}
$$

We claim that δ is a quasi-isomorphism. Since $H^i(\delta)$ is an isomorphism for $i \gg 0$, we can use descending induction on n such that $H^i(\delta)$ is an isomorphism for $i \geq n$. Observe that $\mathcal{E}^\bullet \otimes_{\mathcal{O}} \mathcal{I}$ represents $\mathcal{E}^\bullet_0 \otimes^{\mathbf{L}}_{\mathcal{O}_0} \mathcal{I}$, that $\mathcal{F}^\bullet_0 \otimes_{\mathcal{O}_0} \mathcal{I}$ represents $\mathcal{G}^\bullet_0 \otimes^{\mathbf{L}}_{\mathcal{O}_0} \mathcal{I}$, and that $\beta = \delta \otimes^{\mathbf{L}}_{\mathcal{O}_0} \mathrm{id}_{\mathcal{I}}$ as maps in $D(\mathcal{O}_0)$. This is true because $\beta = (\alpha \otimes \mathrm{id}_{\mathcal{I}}) \circ (\delta \otimes \mathrm{id}_{\mathcal{I}})$.

Suppose that $H^i(\delta)$ is an isomorphism in degrees $\geq n$. Then the same is true for β by what we just said and Lemma 16.3. Then we can look at the diagram

$$
\begin{array}{ccccccccc}
H^{n-1}(\mathcal{E}^\bullet \otimes_\mathcal{O} \mathcal{I}) & \longrightarrow & H^{n-1}(\mathcal{E}^\bullet) & \longrightarrow & H^{n-1}(\mathcal{E}_0^\bullet) & \longrightarrow & H^n(\mathcal{E}^\bullet \otimes_\mathcal{O} \mathcal{I}) & \longrightarrow & H^n(\mathcal{E}^\bullet) \\
\downarrow{\scriptstyle H^{n-1}(\beta)} & & \downarrow & & \downarrow{\scriptstyle H^{n-1}(\delta)} & & \downarrow{\scriptstyle H^n(\beta)} & & \downarrow \\
H^{n-1}(\mathcal{F}_0^\bullet \otimes_\mathcal{O} \mathcal{I}) & \longrightarrow & H^{n-1}(\mathcal{H}^\bullet) & \longrightarrow & H^{n-1}(\mathcal{G}_0^\bullet) & \longrightarrow & H^n(\mathcal{F}_0^\bullet \otimes_\mathcal{O} \mathcal{I}) & \longrightarrow & H^n(\mathcal{H}^\bullet)
\end{array}
$$

Using Homology, Lemma 5.19 we see that $H^{n-1}(\delta)$ is surjective. This in turn implies that $H^{n-1}(\beta)$ is surjective by Lemma 16.3. Using Homology, Lemma 5.19 again we see that $H^{n-1}(\delta)$ is an isomorphism. The claim holds by induction, so δ is a quasi-isomorphism which is what we wanted to show. $\qquad\square$

0DIX **Lemma 16.5.** *Let $\mathcal{C}$ be a site. Let $\mathcal{O} \to \mathcal{O}_0$ be a surjection of sheaves of rings. Assume given the following data*

(1) *a complex of $\mathcal{O}$-modules $\mathcal{F}^\bullet$,*
(2) *a complex $\mathcal{K}_0^\bullet$ of $\mathcal{O}_0$-modules,*
(3) *a quasi-isomorphism $\mathcal{K}_0^\bullet \to \mathcal{F}^\bullet \otimes_\mathcal{O} \mathcal{O}_0$,*

Then there exist a quasi-isomorphism $\mathcal{G}^\bullet \to \mathcal{F}^\bullet$ such that the map of complexes $\mathcal{G}^\bullet \otimes_\mathcal{O} \mathcal{O}_0 \to \mathcal{F}^\bullet \otimes_\mathcal{O} \mathcal{O}_0$ factors through $\mathcal{K}_0^\bullet$ in the homotopy category of complexes of $\mathcal{O}_0$-modules.

Proof. Set $\mathcal{F}_0^\bullet = \mathcal{F}^\bullet \otimes_\mathcal{O} \mathcal{O}_0$. By Derived Categories, Lemma 9.8 there exists a factorization

$$
\mathcal{K}_0^\bullet \to \mathcal{L}_0^\bullet \to \mathcal{F}_0^\bullet
$$

of the given map such that the first arrow has an inverse up to homotopy and the second arrow is termwise split surjective. Hence we may assume that $\mathcal{K}_0^\bullet \to \mathcal{F}_0^\bullet$ is termwise surjective. In that case we take

$$
\mathcal{G}^n = \mathcal{F}^n \times_{\mathcal{F}_0^n} \mathcal{K}_0^n
$$

and everything is clear. $\qquad\square$

0DIY **Lemma 16.6.** *Let $\mathcal{C}$ be a site. Let $\mathcal{O} \to \mathcal{O}_0$ be a surjection of sheaves of rings whose kernel is an ideal sheaf $\mathcal{I}$ of square zero. Let $K, L \in D^-(\mathcal{O})$. Set $K_0 = K \otimes_\mathcal{O}^\mathbf{L} \mathcal{O}_0$ and $L_0 = L \otimes_\mathcal{O}^\mathbf{L} \mathcal{O}_0$ in $D^-(\mathcal{O}_0)$. Given $\alpha_0 : K_0 \to L_0$ in $D(\mathcal{O}_0)$ there is a canonical element*

$$
o(\alpha_0) \in \mathrm{Ext}^1_{\mathcal{O}_0}(K_0, L_0 \otimes_{\mathcal{O}_0}^\mathbf{L} \mathcal{I})
$$

whose vanishing is necessary and sufficient for the existence of a map $\alpha : K \to L$ in $D(\mathcal{O})$ with $\alpha_0 = \alpha \otimes_\mathcal{O}^\mathbf{L} \mathrm{id}$.

Proof. Finding $\alpha : K \to L$ lifing α_0 is the same as finding $\alpha : K \to L$ such that the composition $K \xrightarrow{\alpha} L \to L_0$ is equal to the composition $K \to K_0 \xrightarrow{\alpha_0} L_0$. The short exact sequence $0 \to \mathcal{I} \to \mathcal{O} \to \mathcal{O}_0 \to 0$ gives rise to a canonical distinguished triangle

$$
L \otimes_\mathcal{O}^\mathbf{L} \mathcal{I} \to L \to L_0 \to (L \otimes_\mathcal{O}^\mathbf{L} \mathcal{I})[1]
$$

in $D(\mathcal{O})$. By Derived Categories, Lemma 4.2 the composition

$$
K \to K_0 \xrightarrow{\alpha_0} L_0 \to (L \otimes_\mathcal{O}^\mathbf{L} \mathcal{I})[1]
$$

is zero if and only if we can find $\alpha : K \to L$ lifting α_0. The composition is an element in

$$\mathrm{Hom}_{D(\mathcal{O})}(K, (L \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{I})[1]) = \mathrm{Hom}_{D(\mathcal{O}_0)}(K_0, (L \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{I})[1]) = \mathrm{Ext}_{\mathcal{O}_0}^1(K_0, L_0 \otimes_{\mathcal{O}_0}^{\mathbf{L}} \mathcal{I})$$

by adjunction. $\qquad\square$

0DIZ **Lemma 16.7.** *Let $\mathcal{C}$ be a site. Let $\mathcal{O} \to \mathcal{O}_0$ be a surjection of sheaves of rings whose kernel is an ideal sheaf $\mathcal{I}$ of square zero. Let $K_0 \in D^-(\mathcal{O})$. A lift of K_0 is a pair (K, α_0) consisting of an object K in $D^-(\mathcal{O})$ and an isomorphism $\alpha_0 : K \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{O}_0 \to K_0$ in $D(\mathcal{O}_0)$.*

(1) *Given a lift (K, α) the group of automorphism of the pair is canonically the cokernel of a map*

$$\mathrm{Ext}_{\mathcal{O}_0}^{-1}(K_0, K_0) \longrightarrow \mathrm{Hom}_{\mathcal{O}_0}(K_0, K_0 \otimes_{\mathcal{O}_0}^{\mathbf{L}} \mathcal{I})$$

(2) *If there is a lift, then the set of isomorphism classes of lifts is principal homogenenous under $\mathrm{Ext}_{\mathcal{O}_0}^1(K_0, K_0 \otimes_{\mathcal{O}_0}^{\mathbf{L}} \mathcal{I})$.*

Proof. An automorphism of (K, α) is a map $\varphi : K \to K$ in $D(\mathcal{O})$ with $\varphi \otimes_{\mathcal{O}} \mathrm{id}_{\mathcal{O}_0} = \mathrm{id}$. This is the same thing as saying that

$$K \xrightarrow{\varphi - \mathrm{id}} K \to K \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{O}_0$$

is zero. We conclude the group of automorphisms is the cokernel of a map

$$\mathrm{Hom}_{\mathcal{O}}(K, K_0[-1]) \longrightarrow \mathrm{Hom}_{\mathcal{O}}(K, K_0 \otimes_{\mathcal{O}_0}^{\mathbf{L}} \mathcal{I})$$

by the distinguished triangle

$$K \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{I} \to K \to K \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{O}_0 \to (K \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{I})[1]$$

in $D(\mathcal{O})$ and Derived Categories, Lemma 4.2. To translate into the groups in the lemma use adjunction of the restriction functor $D(\mathcal{O}_0) \to D(\mathcal{O})$ and $- \otimes_{\mathcal{O}} \mathcal{O}_0 : D(\mathcal{O}) \to D(\mathcal{O}_0)$. This proves (1).

Proof of (2). Assume that $K_0 = K \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{O}_0$ in $D(\mathcal{O})$. By Lemma 16.6 the map sending a lift (K', α_0) to the obstruction $o(\alpha_0)$ to lifting α_0 defines a canonical injective map from the set of isomomorphism classes of pairs to $\mathrm{Ext}_{\mathcal{O}_0}^1(K_0, K_0 \otimes_{\mathcal{O}_0}^{\mathbf{L}} \mathcal{I})$. To finish the proof we show that it is surjective. Pick $\xi : K_0 \to (K_0 \otimes_{\mathcal{O}_0}^{\mathbf{L}} \mathcal{I})[1]$ in the Ext^1 of the lemma. Choose a bounded above complex $\mathcal{F}^\bullet$ of flat $\mathcal{O}$-modules representing K. The map ξ can be represented as $t \circ s^{-1}$ where $s : \mathcal{K}_0^\bullet \to \mathcal{F}_0^\bullet$ is a quasi-isomorphism and $t : \mathcal{K}_0^\bullet \to \mathcal{F}_0^\bullet \otimes_{\mathcal{O}_0} \mathcal{I}[1]$ is a map of complexes. By Lemma 16.5 we can assume there exists a quasi-isomorphism $\mathcal{G}^\bullet \to \mathcal{F}^\bullet$ of complexes of $\mathcal{O}$-modules such that $\mathcal{G}_0^\bullet \to \mathcal{F}_0^\bullet$ factors through s up to homotopy. We may and do replace $\mathcal{G}^\bullet$ by a bounded above complex of flat $\mathcal{O}$-modules (by picking a qis from such to $\mathcal{G}^\bullet$ and replacing). Then we see that ξ is represented by a map of complexes $t : \mathcal{G}_0^\bullet \to \mathcal{F}_0^\bullet \otimes_{\mathcal{O}_0} \mathcal{I}[1]$ and the quasi-isomorphism $\mathcal{G}_0^\bullet \to \mathcal{F}_0^\bullet$. Set

$$\mathcal{H}^n = \mathcal{F}^n \times_{\mathcal{F}_0^n} \mathcal{G}_0^n$$

with differentials

$$\mathcal{H}^n \to \mathcal{H}^{n+1}, \quad (f^n, g_0^n) \mapsto (d(f^n) + t(g_0^n), d(g_0^n))$$

This makes sense as $\mathcal{F}_0^{n+1} \otimes_{\mathcal{O}_0} \mathcal{I} = \mathcal{F}^{n+1} \otimes_{\mathcal{O}} \mathcal{I} = \mathcal{I}\mathcal{F}^{n+1} \subset \mathcal{F}^{n+1}$. We omit the computation that shows that $\mathcal{H}^\bullet$ is a complex of $\mathcal{O}$-modules. By construction there is a short exact sequence

$$0 \to \mathcal{F}_0^\bullet \otimes_{\mathcal{O}_0} \mathcal{I} \to \mathcal{H}^\bullet \to \mathcal{G}_0^\bullet \to 0$$

of complexes of $\mathcal{O}$-modules. Exactly as in the proof of Lemma 16.4 one shows that this sequence induces an isomorphism $\alpha_0 : \mathcal{H}^\bullet \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{O}_0 \to \mathcal{G}_0^\bullet$ in $D(\mathcal{O}_0)$. In other words, we have produced a pair $(\mathcal{H}^\bullet, \alpha_0)$. We omit the verification that $o(\alpha_0) = \xi$; hint: $o(\alpha_0)$ can be computed explitly in this case as we have maps $\mathcal{H}^n \to \mathcal{F}^n$ (not compatible with differentials) lifting the components of α_0. This finishes the proof. $\square$

17. Other chapters

Preliminaries

Schemes

Topics in Scheme Theory

Algebraic Spaces

References

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